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ANISOTROPIC PLATES

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S. G. Lekhnitskiy

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ABSTRACT

In the present book three main topics are contained: 1) the generalized plane stressed state of anisotropic plates; 2) the bending of anisotropic plates, and 3) the stability of anisotropic plates. The majority of the solutions set forth in the book (particularly in the chapters devoted to the plane problem) is due to the author himself. All problems discussed are only concerned with small elastic strains of plates. The problems connected with the plastic plate deformations, with the behavior of the plates after the stability has been lost, with temperature and other stresses in plates, etc., art not treated in the book. These problems are still waiting for their investigators. The material on the problem of transverse vibrations of anisotropic plates which is known to the author is collected in a special chapter. In view of the rather great material and the small volume of the book the author endeavored to set forth things as concisely as possible. The main attention was paid to the practical side of the solutions presented; formulas and conclusions having theoretical interest only were mostly given without derivation, with indication of the literature where interested readers may find detailed discussion and proofs. In those cases where this was possible and interesting for the practice the results are brought into the form of theoretical formulas, diagrams, and tables.

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FROM THE PREFACE TO THE FIRST EDITION

Present-day technology makes use of anisotropic plates, i.e., plates with different resistance to mechanical actions in different directions, as constructional elements. To such plates belong plates made of aviation plywood, delta wood, texolite, and a number of other materials. The experimental investigations of such a material as plywood show the great difference between the moduli of elasticity and the flexural rigidities for the principal directions – along the grains of the casings (the external plywood layers) and across the grains. Obviously, it is not correct to calculate plywood plates for the sake of simplicity with the help of formulas derived for the isotropic body; it is necessary to derive special formulas on the basis of the theory of elasticity of the anisotropic body for the calculations. Also those plates in which the difference between the flexural rigidities for different directions has been created artificially may be regarded as anisotropic ones, as corrugated plates or plates reinforced by corrugation, plates reinforced by tightly located stiffening ribs, etc. Not only the constructor, but also the physicist who works with plates cut out of crystals, e.g., with quartz plates, must encounter on the calculation of stresses and strains in anisotropic plates.

The author of this work set himself the task of creating such a boo which would possibly cover most of the present-day investigations of problems concerning the strain of anisotropic plates and which could serve as a means of instruction for engineers, constructors, physicists, and other specialists working with anisotropic plates.

In the present book three main topics are contained: [illegible] generalized plane stressed state of anisotropic plates; [illegible] anisotropic plates, and [illegible]) the stability of anisotropic plates. [illegible] jority of the solutions set forth in the [illegible] chapters devoted to the plane problem [illegible].

All problems [illegible] strains of plates. [illegible] formations, with [illegible] lost, with [illegible] treated in the [illegible] gators. The [illegible] tropic plates which [illegible] chapter.

In view of the [illegible] book the author [illegible] The main attention was [illegible] to the practical side of the solutions [illegible] sented; formulas and conclusions having theoretical interest [illegible] mostly given without derivation, with indication of the literature where interested readers may find detailed discussion and proofs. [illegible] those cases where this was possible and interesting for the practi[illegible] the results are brought into the form of theoretical formulas, diagrams, and tables.

May 1944 S.G. Lekhnitskiy

PREFACE TO THE SECOND EDITION

The theory of strains and stresses in anisotropic plates has been supplemented by numerous new investigations during the time which has elapsed after the day when the first edition of the monograph "Anisotropic plates" (1947) came out. A great part of these investigations carried out, above all, in the USSR and, in particular, by the author himself refers to the plane problem, and a smaller part to the theory of bending and stability plates.

When preparing the second edition the author endeavored to present in the book, if possible, all new results known to him referring to anisotropic plates and being of practical and theoretical interest. As a result the volume of the book has increased compared to the first edition. A new chapter (in the second edition Chapter 8) has been added; it is devoted to an approximate method of studying the stresses in anisotropic plates weakened by holes which are nearly round or elliptic. In particular, the cases of holes similar to an equilateral triangle and to a square with rounded angles, etc., were considered. Approximate formulas for the determination of stresses near such holes in plates strained by arbitrary forces, and, in particular, by stretching forces and bending moments are given; the results of calculating the stresses with different degrees of accuracy in plates with given elastic constants are presented. Nearly all remaining chapters are supplemented by sections setting forth the results of the most recent investigations, as well as of a number of investigations of practical interest which had, however, not entered the first edition. In Chapter 3, e.g., which

is devoted to the bending of beams and [illegible]-

ing are given, and, in particular, [illegible]

in Chapter 4, problems of the elastic [illegible]

gion is limited by a parabola or a [illegible], [illegible]

tribution of stresses in an infinite [illegible]

centrated moment are given; the [illegible]

ing with problems of the elastic [illegible]

another material are added [illegible]

of the bending of an anisotropic [illegible]

in Chapter 11, etc. In accordance with [illegible],

many of which have been replaced by [illegible]

The author endeavored to present [illegible]

in a form which is accessible for engineers [illegible]

ing them where possible into the [illegible]

lustrating them by calculations and [illegible]

constants of three-layer birch plywood. [illegible]

tion were revised: the presentation [illegible],

were corrected, and in a number of [illegible] in which the ma-

terial is set forth was changed. For all special cases of strain in-

vestigated by the author himself and to be found in the first edition

numerous calculations were carried out anew (also for three-layer birch

plywood). As a result the corresponding diagrams of stress distribution

and the diagrams of a number of functions were replaced by new ones,

more illustrative ones, compared to the diagrams of the first edition.

The list of literature used was considerably increased.

Finally, I want to express my gratitude to T.V. Skvortsovaya who has carried all numerical work and helped me in a substantial manner in the preparation of the second edition of the monograph.

December, 1956 S.G. Lekhnitskiy

Chapter 1

THE BASIC EQUATIONS OF THE THEORY OF ELASTICITY OF AN ANISOTROPIC BODY

§1. THE STRESSED STATE OF A CONTINUOUS BODY

When studying the stresses and strains in elastic anisotropic bodies, and, in particular, in plates, we shall consider the elastic body to be a continuous body, a continuous medium, according to the generally accepted model.

As is well known, the stressed state at a given point of a continuous body which is at equilibrium or moves under the action of external forces is entirely determined by the stress components acting on three mutually perpendicular planes passing through this point. Usually the planes are passed perpendicular to the coordinate directions of an orthogonal coordinate system passing through the point in question. In this book we shall only use Cartesian and cylindrical coordinates.

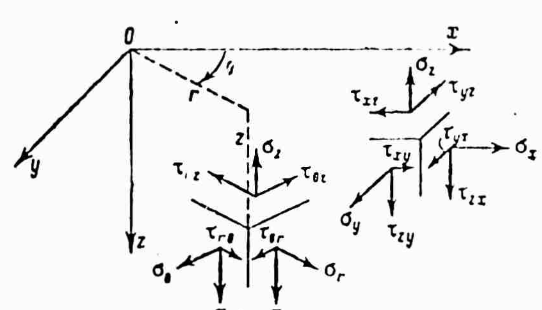

Fig. 1

Referring the body to a Cartesian coordinate system x, y, z we shall choose some point, pass three mutually perpendicular planes normal to the coordinate axes through it, and consider the stress compon-

ents acting on these planes (Fig. 1). It is generally accepted to designate the normal components by the letter σ with a subscript indicating the direction of the normal to the plane (and, therefore, also the direction of the component itself); the tangential components are designated by the letter τ with two subscripts the first of which indicates the direction of the component itself, and the second one the direction of the normal to the plane. On the plane normal to the x axis act the components: σ_x, τ_{yx}, τ_{zx}; on the plane normal to the y axis we have: τ_{xy}, σ_y, τ_{zy}, and on the plane normal to the z axis: τ_{xz}, τ_{yz}, σ_z. As is well known from the mechanics of continuous media, we have $\tau_{zy} = \tau_{yz}$, $\tau_{xz} = \tau_{zx}$, $\tau_{yx} = \tau_{xy}$ and, generally, $\tau_{ji} = \tau_{ij}$, where i and j are mutually perpendicular directions. If we know the stress components on planes normal to the coordinate axes we can always determine the stress on any oblique plane with a normal n passing through the same point. For this purpose serve the formulas

$$\left.\begin{aligned} X_n &= \sigma_x \cos(n, x) + \tau_{xy}\cos(n, y) + \tau_{xz}\cos(n, z), \\ Y_n &= \tau_{xy}\cos(n, x) + \sigma_y \cos(n, y) + \tau_{yz}\cos(n, z), \\ Z_n &= \tau_{xz}\cos(n, x) + \tau_{yz}\cos(n, y) + \sigma_z \cos(n, z), \end{aligned}\right\} \qquad (1.1)$$

where X_n, Y_n, Z_n are the projections on the coordinate axes of the stress acting on the oblique surface. Having determined X_n, Y_n, Z_n we shall easily find (by projecting) the normal and the tangential components of the stress on the plane with normal n.

Furthermore, we shall refer the body under consideration to a cylindrical coordinate system r, θ, z in which the z axis coincides with the z axis of a Cartesian system, and the angle θ is counted from the x axis chosen as the polar axis. The stress components on planes perpendicular to the directions of the r, θ, z coordinates of the cylindrical system are shown in the same Fig. 1 (at another point); they are designated, respectively, by: σ_r, $\tau_{\theta r}$, τ_{zr}; $\tau_{r\theta}$, σ_θ, $\tau_{z\theta}$; τ_{rz}, $\tau_{\theta z}$, σ_z, where

$\tau_{rz} = \tau_{zr}$, $\tau_{\theta z} = \tau_{z\theta}$, $\tau_{r\theta} = \tau_{\theta r}$. The transition from the stress components referred to Cartesian coordinates to the components in cylindrical coordinates is carried out with the help of the well known formulas:

$$\left.\begin{aligned}
\sigma_r &= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta\cos\theta,\\
\sigma_\theta &= \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy} \sin\theta\cos\theta,\\
\tau_{r\theta} &= (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta),\\
\tau_{rz} &= \tau_{xz}\cos\theta + \tau_{yz}\sin\theta,\\
\tau_{\theta z} &= -\tau_{xz}\sin\theta + \tau_{yz}\cos\theta,\\
\sigma_z &= \sigma_z.
\end{aligned}\right\} \qquad (1.2)$$

The stress components in a continuous body which is at equilibrium satisfy the equilibrium equations which have the following form in the Cartesian coordinate system:

$$\left.\begin{aligned}
\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} + X = 0,\\
\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} + Y = 0,\\
\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} + Z = 0.
\end{aligned}\right\} \qquad (1.3)$$

Analogous equations in a cylindrical coordinate system will be written in the form:

$$\left.\begin{aligned}
\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\partial\tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0,\\
\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} + \Theta = 0,\\
\frac{\partial\tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta} + \frac{\partial\sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + Z = 0.
\end{aligned}\right\} \qquad (1.4)$$

In Eqs. (1.3) and (1.4) X, Y, Z and R, Θ, Z denote the projections of the volume forces referred to the unit volume, on the x, y, z and r, θ, z directions.

The equations of motion of a continuous medium differ from the equilibrium equations only by the inertia terms ρw_x, ρw_y, ρw_z or ρw_r, ρw_θ, ρw_z which must be substituted on the right-hand sides of (1.3) and (1.4) instead of the zeroes (ρ is the density of the material and w with the subscripts are the projections of the acceleration on the coordinate directions.

In the theory of elasticity the projections of the acceleration

are usually expressed in terms of the projections of the displacements of the body particles in the coordinate directions (i.e., they are equal to the second derivatives with respect to time of the displacement projections). We shall designate the projections of the displacement in the x, y, z and r, θ, z directions by u, v, w and u_r, u_θ, u_z.

The strained state in the neighborhood of a given point of a continuous body is characterized by the six strain components: three relative elongations which are designated by the letter ε with the corresponding subscript and the three relative shears designated by the letter γ with two subscripts. In a Cartesian system we have the strain components: ε_x, ε_y, ε_z are the relative elongations of infinitely small sections, which were parallel to the x, y, z axes in the unstrained body, and γ_{yz}, γ_{xz}, γ_{xy} the relative shears, i.e., the angles variations between the mentioned sections. The strain components for a cylindrical coordinate system will be ε_r, ε_θ, ε_z (the relative elongations for the r, θ, z directions) and $\gamma_{\theta z}$, γ_{rz}, $\gamma_{r\theta}$ (the relative shears).

The strain components are expressed in terms of the displacement projections. If there are no restrictions as to the value of the strains the connection between the ε_x, ε_y, ..., γ_{xy} and u, v, w are given by the formulas

$$
\left.
\begin{aligned}
\varepsilon_x &= \sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2} - 1, \\
\varepsilon_y &= \sqrt{1 + 2\frac{\partial v}{\partial y} + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2} - 1, \\
\varepsilon_z &= \sqrt{1 + 2\frac{\partial w}{\partial z} + \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2} - 1, \\
\sin\gamma_{yz} &= \frac{\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\cdot\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\cdot\frac{\partial w}{\partial z}}{(1+\varepsilon_y)(1+\varepsilon_z)}, \\
\sin\gamma_{xz} &= \frac{\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial z}}{(1+\varepsilon_x)(1+\varepsilon_z)}, \\
\sin\gamma_{xy} &= \frac{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}}{(1+\varepsilon_x)(1+\varepsilon_y)}.
\end{aligned}
\right\}
\qquad (1.5)
$$

In the case of small strains where the derivatives of the displacements are small quantities compared to unity these formulas are simpler and assume the form:

$$\left.\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x}, & \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \\ \varepsilon_y &= \frac{\partial v}{\partial y}, & \gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \\ \varepsilon_z &= \frac{\partial w}{\partial z}, & \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \end{aligned}\right\} \qquad (1.6)$$

In the case of small strains ε_r, ε_θ, ..., $\gamma_{r\theta}$ are expressed in terms of u_r, u_θ, u_z in the following way:

$$\left.\begin{aligned} \varepsilon_r &= \frac{\partial u_r}{\partial r}, & \gamma_{\theta z} &= \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}, \\ \varepsilon_\theta &= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, & \gamma_{rz} &= \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}, \\ \varepsilon_z &= \frac{\partial u_z}{\partial z}, & \gamma_{r\theta} &= \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}. \end{aligned}\right\} \qquad (1.7)$$

The formulas and equations presented here are correct for any continuous body, elastic or not; their derivations may be found in courses of the theory of elasticity.*

§2. GENERALIZED HOOKE'S LAW

The equations given in §1 are not sufficient to solve problems of equilibrium, motion, or stability of an elastic body. It is, in addition, necessary to indicate the relationship between the stress components and the strain components, and for this purpose some model reflecting the elastic properties of the body must be chosen. If only small strains are involved usually a continuous body obeying a generalized Hooke's law is chosen to be such a model of an elastic body. In all cases considered in this book we shall assume that a generalized Hooke's law holds for the elastic body, and, in particular, for plates, or, in other words, the strain components are linear functions of the stress components.

An elastic body is called *isotropic* if its elastic properties are identical in all directions, and *anisotropic* if its elastic properties

are different for different directions. *Homogeneous* we shall call an elastic body with which the elastic properties are identical for all parallel directions passing through any points, or, in other words, all identical elements having the form of rectangular parallelepipeds with parallel faces, chosen at different points of the body, have identical elastic properties.

Let us first consider a homogeneous elastic body having an anisotropy of general form where some elements of elastic symmetry are missing. Referring the body to an arbitrarily chosen orthogonal coordinate system x, y, z we may write the equations expressing the generalized Hooke's law in the following form:

$$\left.\begin{aligned} e_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z + a_{14}\tau_{yz} + a_{15}\tau_{xz} + a_{16}\tau_{xy}, \\ e_y &= a_{12}\sigma_x + a_{22}\sigma_y + \cdots\cdots + a_{26}\tau_{xy}, \\ &\cdots\cdots\cdots\cdots\cdots \\ \gamma_{xy} &= a_{16}\sigma_x + a_{26}\sigma_y + \cdots\cdots + a_{66}\tau_{xy}. \end{aligned}\right\} \qquad (2.1)$$

a_{11}, a_{12}, ..., a_{66} are here the elastic constants (strain coefficients); in general, the number of different constants is equal to 21.

When strained the body stores a reserve of potential energy. The expression of the potential energy of strain referred to unit volume (of elastic potential) may most simply be written in the so-called bilinear form:

$$\bar{V} = \frac{1}{2}(\sigma_x e_x + \sigma_y e_y + \sigma_z e_z + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz} + \tau_{xy}\gamma_{xy}). \qquad (2.2)$$

If we pass over to the stress components on the basis of Eqs. (2.1) this expression assumes the following form in the general case:

$$\left.\begin{aligned} \bar{V} = \frac{1}{2}a_{11}\sigma_x^2 + a_{12}\sigma_x\sigma_y + a_{13}\sigma_x\sigma_z + a_{14}\sigma_x\tau_{yz} + a_{15}\sigma_x\tau_{xz} + a_{16}\sigma_x\tau_{xy} + \\ + \frac{1}{2}a_{22}\sigma_y^2 + a_{23}\sigma_y\sigma_z + a_{24}\sigma_y\tau_{yz} + a_{25}\sigma_y\tau_{xz} + a_{26}\sigma_y\tau_{xy} + \\ + \frac{1}{2}a_{33}\sigma_z^2 + a_{34}\sigma_z\tau_{yz} + a_{35}\sigma_z\tau_{xz} + a_{36}\sigma_z\tau_{xy} + \\ + \frac{1}{2}a_{44}\tau_{yz}^2 + a_{45}\tau_{yz}\tau_{xz} + a_{46}\tau_{yz}\tau_{xy} + \\ + \frac{1}{2}a_{55}\tau_{xz}^2 + a_{56}\tau_{xz}\tau_{xy} + \\ + \frac{1}{2}a_{66}\tau_{xy}^2. \end{aligned}\right\} \qquad (2.3)$$

The potential energy of strain of the whole body is found by integrating over the entire volume of the body ω:

$$V = \iiint_{\omega} \bar{V}\, d\omega. \qquad (2.4)$$

If the internal structure of the body is symmetric symmetry is also observed in its elastic properties. This elastic symmetry as it is usually called appears in the following way: at each point of the body symmetric directions are detected for which the elastic properties are identical (equivalent directions). Crystals have an elastic symmetry; all naturally occurring crystals are subdivided into nine classes according to the character of the elastic symmetry. Elastic symmetry is also observed in samples produced of natural wood, delta-wood, plywood, and other anisotropic materials. If there is elastic symmetry the equations of the generalized Hooke's law and the equation for the elastic potential simplify; some of the constants a_{ij} prove to be equal to zero, and the rest is connected by relations.

We shall not deal with all possible cases of elastic symmetry, but consider only themost important of them.*

1. The plane of elastic symmetry. Let us assume that through each point of the body passes a plane with the property that any two directions symmetric with respect to this plane have equivalent elastic properties (in a homogeneous body all these planes passing through various points are parallel). If the z axis is passed perpendicularly to the plane of elastic symmetry the equations of the generalized Hooke's law will be written in the following form:

$$\left.\begin{aligned}
\varepsilon_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z + a_{16}\tau_{xy},\\
\varepsilon_y &= a_{12}\sigma_x + a_{22}\sigma_y + a_{23}\sigma_z + a_{26}\tau_{xy},\\
\varepsilon_z &= a_{13}\sigma_x + a_{23}\sigma_y + a_{33}\sigma_z + a_{36}\tau_{xy},\\
\gamma_{yz} &= a_{44}\tau_{yz} + a_{45}\tau_{xz},\\
\gamma_{xz} &= a_{45}\tau_{yz} + a_{55}\tau_{xz},\\
\gamma_{xy} &= a_{16}\sigma_x + a_{26}\sigma_y + a_{36}\sigma_z + a_{66}\tau_{xy};
\end{aligned}\right\} \qquad (2.5)$$

The number of independent elastic constants is reduced to 13.

The properties of the body with a plane of elastic symmetry may be illustrated more plausibly with the help of the following example. Let us consider an element of the body in the form of a rectangular parallelepiped with which two faces are parallel to the plane of symmetry, and assume that stretching or compressive normal stresses σ_z are applied to these faces (Fig. 2). The strain of the element will be characterized by the relative elongations and shears which we shall find from Eqs. (2.5):

$$\left.\begin{aligned} \varepsilon_x &= a_{13}\sigma_z, & \gamma_{yz} &= 0, \\ \varepsilon_y &= a_{23}\sigma_z, & \gamma_{xz} &= 0, \\ \varepsilon_z &= a_{33}\sigma_z, & \gamma_{xy} &= a_{36}\sigma_z. \end{aligned}\right\} \tag{2.6}$$

Hence it is evident that in the case of simple stretching or compression in a direction perpendicular to the plane of elastic symmetry the angles between the sections normal to the plane of elastic symmetry and the sections lying in it are not distorted, but remain rectangular ones. As a result, when strained the chosen element assumes the shape of a straight parallelepiped in which four faces are rectangles, and two

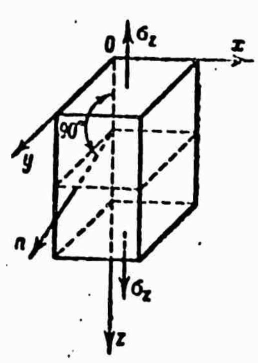

Fig. 2

parallelograms. If, however, there are no planes of elastic symmetry the rectangular parallelepiped which is stretched or compressed in one direction goes over into an oblique parallelepiped. The directions which are normal to the planes of elastic symmetry will be called *principal directions of elasticity* or, briefly, *principal directions*. For the symmetry case under consideration one principal direction passes through each point. The crystals of monoclinal syn-

gony (e.g., feldspar-orthoclase) have this form of elastic symmetry.

2. Three planes of elastic symmetry. If three mutually perpendicular planes of elastic symmetry pass through each point of a homogeneous body the equations of the generalized Hooke's law referred to an x, y, z coordinate system with axes normal to these planes assume the form:

$$\left.\begin{aligned} \varepsilon_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z, & \gamma_{yz} &= a_{44}\tau_{yz}, \\ \varepsilon_y &= a_{12}\sigma_x + a_{22}\sigma_y + a_{23}\sigma_z, & \gamma_{xz} &= a_{55}\tau_{xz}, \\ \varepsilon_z &= a_{13}\sigma_x + a_{23}\sigma_y + a_{33}\sigma_z, & \gamma_{xy} &= a_{66}\tau_{xy}. \end{aligned}\right\} \qquad (2.7)$$

The number of independent elastic constants is equal to nine. Through each point pass three mutually perpendicular principal directions. A homogeneous body with three mutually perpendicular planes of elastic symmetry at each point is called *orthogonally anisotropic*, or, briefly, *orthotropic*.

An element having the shape of a rectangular parallelepiped with faces parallel to the planes of elastic symmetry which is chosen from an orthotropic body remains a rectangular parallelepiped when it is stretched or compressed in one direction (Fig. 2); under strain the rib lengths are varied, but the angles between the faces are not distorted.

Equations (2.7) acquire a higher plausibility if instead of the strain coefficients a_{ij} the so-called technical constants are introduced: the Young's moduli, the Poisson coefficients, and the shear moduli. Let us rewrite (2.7) in the form:

$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{E_1}\sigma_x - \frac{\nu_{21}}{E_2}\sigma_y - \frac{\nu_{31}}{E_3}\sigma_z, & \gamma_{yz} &= \frac{1}{G_{23}}\tau_{yz}, \\ \varepsilon_y &= -\frac{\nu_{12}}{E_1}\sigma_x + \frac{1}{E_2}\sigma_y - \frac{\nu_{32}}{E_3}\sigma_z, & \gamma_{xz} &= \frac{1}{G_{13}}\tau_{xz}, \\ \varepsilon_z &= -\frac{\nu_{13}}{E_1}\sigma_x - \frac{\nu_{23}}{E_2}\sigma_y + \frac{1}{E_3}\sigma_z, & \gamma_{xy} &= \frac{1}{G_{12}}\tau_{xy}. \end{aligned}\right\} \qquad (2.8)$$

E_1, E_2, E_3 are here Young's moduli for stretching (compression) along the principal directions of elasticity x, y, z; ν_{12} is the Poisson coefficient characterizing the contraction in the y direction if stretching in the x direction takes place; ν_{21} is the Poisson coefficient charac-

terizing the contraction the x direction for a stretching in the y direction, etc.; G_{23}, G_{13}, G_{12} are the shear moduli characterizing the variations of the angles between the principal directions y and z, x and z, x and y.* The following relationships hold between the Young's moduli and the Poisson coefficients owing to the symmetry of Eqs. (2.7)

$$E_1\nu_{21} = E_2\nu_{12}, \quad E_2\nu_{32} = E_3\nu_{23}, \quad E_3\nu_{13} = E_1\nu_{31}. \qquad (2.9)$$

The elastic constants of an orthotropic body which enter into the equations of the generalized Hooke's law (2.7) and (2.8) as written for the principal directions of elasticity x, y, z will be called *principal elastic constants* (in constrast to the constants entering into the equations for an arbitrary coordinate system).

The form of elastic symmetry considered is the most important since it occurs most frequently in practice. Such materials as wood with regular annual rings, delta wood and plywood may be considered homogeneous and orthotropic. The crystals of rhombic syngony (e.g., topaz, baryta) are orthotropic.

3. The isotropy plane. If a plane in which all directions are equivalent with respect to the elastic properties passes through each point of a body the equations of the generalized Hooke's law for a coordinate system with a z axis normal to this plane will be written in the following manner:

$$\left.\begin{aligned} e_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z, & \gamma_{yz} &= a_{44}\tau_{yz}, \\ e_y &= a_{12}\sigma_x + a_{11}\sigma_y + a_{13}\sigma_z, & \gamma_{xz} &= a_{44}\tau_{xz}, \\ e_z &= a_{13}(\sigma_x + \sigma_y) + a_{33}\sigma_z, & \gamma_{xy} &= 2(a_{11} - a_{12})\tau_{xy}. \end{aligned}\right\} \qquad (2.10)$$

The number of different elastic constants reduces to five. According to A. Lyav, a body with an anisotropy of this form is called *transversely isotropic*.* The direction normal to the isotropy plane and all directions in this plane are principal ones. Introducing technical constants we shall rewrite Eqs. (2.10) in another form:

$$\left.\begin{aligned} e_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) - \frac{\nu'}{E'}\sigma_z, & \gamma_{yz} &= \frac{1}{G'}\tau_{yz}, \\ e_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x) - \frac{\nu'}{E'}\sigma_z, & \gamma_{xz} &= \frac{1}{G'}\tau_{xz}, \\ e_z &= -\frac{\nu'}{E'}(\sigma_x + \sigma_y) + \frac{1}{E'}\sigma_z, & \gamma_{xy} &= \frac{1}{G}\tau_{xy}. \end{aligned}\right\} \qquad (2.11)$$

Here E is Young's modulus for directions in the isotropy plane; E' is Young's modulus for directions perpendicular to this plane; ν is the Poisson coefficient characterizing the contraction in the isotropy plane for stretching in the same plane; ν' is the Poisson coefficient characterizing the contraction in the isotropy plane for stretching in a direction perpendicular to it; $G = E/2\,(1 + \nu)$ is the shear modulus for the isotropy plane; G' is the shear modulus characterizing the distortion of the angles between the directions in the isotropy plane and the direction perpendicular to it.

The crystals of the hexagonal system (e.g., beryl) are *transversely isotropic*.

4. Full symmetry - isotropic body. In an isotropic body any plane is a plane of elastic symmetry and any direction a principal one. The equations of the generalized Hooke's law for an isotropic body have the form:

$$\left.\begin{aligned} e_x &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)], & \gamma_{yz} &= \frac{1}{G}\tau_{yz}, \\ e_y &= \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)], & \gamma_{xz} &= \frac{1}{G}\tau_{xz}, \\ e_z &= \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)], & \gamma_{xy} &= \frac{1}{G}\tau_{xy}. \end{aligned}\right\} \qquad (2.12)$$

E is here Young's modulus, ν the Poisson coefficient and $G = E/2(1 + \nu)$ the shear modulus. The number of different elastic constants is equal to two.

If in studying the strains of an isotropic body we pass over from the x, y, z coordinate system to any other orthogonal coordinate system x', y', z' the form of Eqs. (2.12) will not change and the elastic constants E and ν will retain their numerical values also in the new system.

Conversely, in the case of an anisotropic body, new elastic constants a'_{ij} which are expressed in terms of the old ones will be obtained on passing over from one coordinate system to another. A number of questions connected with the conversion of elastic constants of anisotropic plates in the transition to new axes will be illustrated in the following.*

§3. CURVILINEAR ANISOTROPY

A homogeneous anisotropic body is, as was shown above, characterized by the fact that parallel directions passing through different points are equivalent in it. Besides this kind of anisotropy which may also be called rectilinear there is another form of anisotropy, the curvilinear one. The latter is characterized by the fact that directions subject to some other regularities rather than parallel ones are equivalent in a body with such a curvilinear anisotropy. If a curvilinear coordinate system is chosen such that at different points of the body the coordinate directions coincide with the equivalent directions then infinitely small elements of the body which are bounded by three pairs of coordinate surfaces will have identical elastic properties. Conversely, the elastic properties of elements having the form of identical rectangular parallelepipeds with mutually parallel faces will no longer be identical. The number of curvilinear anisotropy which in the following will be called *cylindrical anisotropy** occurs most frequently and is most interesting for practical purposes.

The cylindrical anisotropy is characterized by the following. The straight line g, the axis of anisotropy (itmay pass both inside or outside the body), is rigidly connected with the body having cylindrical anisotropy. All directions intersecting the axis of anisotropy under a right angle are equivalent among each other; all directions parallel to the axis of anisotropy and all directions orthogonal to the first

two directions are, respectively, equivalent to each other. All infinitely small elements A_1, A_2, ..., singled out of the body by three pairs of surfaces: a) two planes passing through the axis of anisotropy, b) two parallel planes normal to g, and c) two coaxial cylindrical surfaces with an axis coinciding with g (Fig. 3) have identical elastic properties. When studying problems of equilibrium and motion of such bodies it is most convenient to use a cylindrical coordinate system r, θ, z, passing the z axis parallel to the axis of anisotropy g, and the polar axis x from which the angles θ are counted in an arbitrary manner.

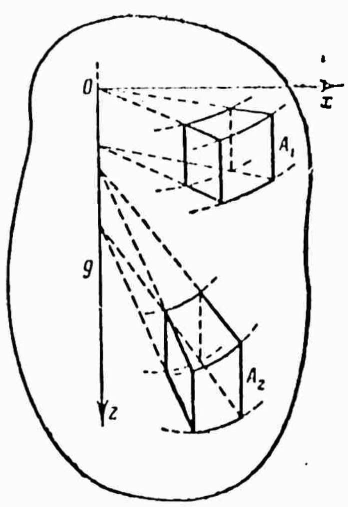

Fig. 3

The equations of the generalized Hooke's law for a body with a cylindrical anisotropy of general form, without any elements of elastic symmetry, will have the following form in the above indicated cylindrical coordinate system:

$$\left.\begin{array}{l} \varepsilon_r = a_{11}\sigma_r + a_{12}\sigma_\theta + a_{13}\sigma_z + a_{14}\tau_{\theta z} + a_{15}\tau_{rz} + a_{16}\tau_{r\theta}, \\ \varepsilon_\theta = a_{12}\sigma_r + a_{22}\sigma_\theta + \cdots \cdots \cdots + a_{26}\tau_{r\theta}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \gamma_{r\theta} = a_{16}\sigma_r + a_{26}\sigma_\theta + \cdots \cdots \cdots + a_{66}\tau_{r\theta}. \end{array}\right\} \quad (3.1)$$

The coefficients a_{ij} are the elastic constants; the number of different elastic constants is equal to 21 in the general case. We note that the equations of the generalized Hooke's law may also be written for an arbitrary Cartesian coordinate system; they will have the form (2.1), but the coefficients a_{ij} will no longer be constant, but vary when passing from one point of the body to another. If, however, for a homogeneous body or, in other words, for a body with rectilinear anisotropy the equations of the generalized Hooke's law are written in an arbitrary cylindrical coordinate system r, θ, z,they will have the form (3.1) in the general case, only the a_{ij} in them will be functions of the angle θ.

In a body with cylindrical anisotropy there may also exist different elements of elastic symmetry. If at each point of the body there is a plane of elastic symmetry normal to the axis of anisotropy g Eqs. (3.1) simplify and assume a form analogous to (2.5) since

$$a_{14} = a_{15} = a_{24} = a_{25} = a_{34} = a_{35} = a_{46} = a_{56} = 0. \tag{3.2}$$

If at each point there are three planes of elastic symmetry one of which is normal to the axis of anisotropy, the other one passes through the axis, and the third one is orthogonal to the first two axes then Eqs. (3.1) assume the same form as Eqs. (2.10) because $a_{16} = a_{26} = a_{36} = a_{46} = 0$. In this case the body may be called an orthotropic body with cylindrical anisotropy. As in the case of a homogeneous orthotropic body it is also here convenient to introduce "technical constants," and then the equations of the generalized Hooke's law will be written in the following form for an orthotropic body with cylindrical anisotropy:

$$\left.\begin{aligned} \varepsilon_r &= \frac{1}{E_r}\sigma_r - \frac{\nu_{\theta r}}{E_\theta}\sigma_\theta - \frac{\nu_{zr}}{E_z}\sigma_z, & \gamma_{\theta z} &= \frac{1}{G_{\theta z}}\tau_{\theta z}, \\ \varepsilon_\theta &= -\frac{\nu_{r\theta}}{E_r}\sigma_r + \frac{1}{E_\theta}\sigma_\theta - \frac{\nu_{z\theta}}{E_z}\sigma_z, & \gamma_{rz} &= \frac{1}{G_{rz}}\tau_{rz}, \\ \varepsilon_z &= -\frac{\nu_{rz}}{E_r}\sigma_r - \frac{\nu_{\theta z}}{E_\theta}\sigma_\theta + \frac{1}{E_z}\sigma_z, & \gamma_{r\theta} &= \frac{1}{G_{r\theta}}\tau_{r\theta}. \end{aligned}\right\} \tag{3.3}$$

Here E_r, E_θ, E_z are Young's moduli for stretching (compression) along the r, θ, z directions, the radial, the tangential and the axial directions (which, at the same time, are also the principal directions of elasticity); $\nu_{r\theta}$ is the Poisson coefficient characterizing the contraction in the θ direction when a stretching in the r direction is applied; etc; $G_{\theta z}$, G_{rz}, $G_{r\theta}$ are the shear moduli characterizing the variations of the angles between the θ and z, r, and z and the r and θ directions.

As an example of a body with cylindrical anisotropy we may use a wooden block with regular cylindrical annual rings. If the inhomogeneity is neglected itmay be considered to be an orthotropic body with cylin-

drical anisotropy.* The pith line plays the role of the axis of anisotropy g (Fig. 4).

Cylindrical anisotropy may appear in metallic details as a result of some technological processes, e.g., in drawing wires, in the production of pipes, etc.

A body with cylindrical anisotropy may be formed artificially, by constructing it from homogeneous (rectilinear-anisotropic) elements with identical elastic properties. Let us, e.g., imagine a great number of homogeneous anisotropic elements ("bricks"), homogeneous in their elastic properties, in which two opposite faces form a small angle. If

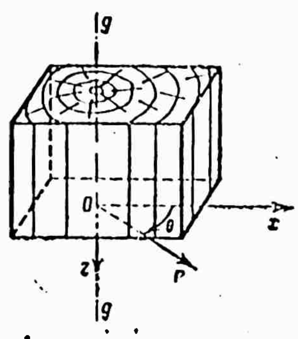

Fig. 4

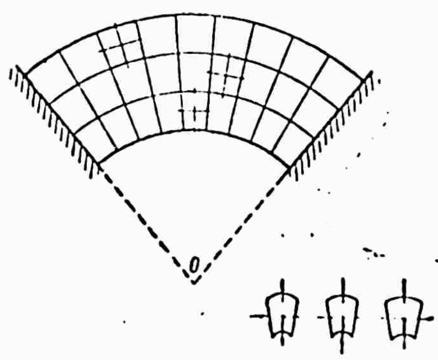

Fig. 5

a value is constructed from these elements, as shown in Fig. 5, it will, as a whole, have the properties of a body with cylindrical anisotropy. The axial directions of the elements in the vault equivalent to each other will be the radial directions.

We shall once again return to this form of anisotropy in Chapters 2, 3, 8, and 9. We shall not consider othercases of curvilinear anisotropy.

§4. BASIC EQUATIONS AND BASIC PROBLEMS OF THE THEORY OF ELASTICITY

The stressed state of an elastic body may be considered given if the stress components acting on three planes normal to the coordinate directions at an arbitrary point of the body (and at an arbitrary in-

stant of time if the motion of the body is considered) are known. The strained state is determined by the components of strain which depend on the three projections of the displacement on the coordinate directions. Consequently, in order to have full information on the stress-strain state of a given elastic body which is acted upon by external loads nine functions must be determined: the six components of the stress and the three projections of displacement. If we use a Cartesian coordinate system σ_x, σ_y, σ_z, τ_{yz}, τ_{xz}, τ_{xy}, u, v, w are these unknown functions of the x, y, z coordinates (and, in the general case, also of time t); we must have nine independent equations to determine them.

Let us concentrate our attention, for the sake of definiteness, on the case of the equilibrium of a homogeneous anisotropic body. Adding Eqs. (2.1) expressing the generalized Hooke's law to the equations of equilibrium of a continuous medium (1.3) we obtain a system of nine equations called the fundamental one:

$$\left.\begin{aligned}
&\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}+X=0,\\
&\cdots\cdots\cdots\cdots\\
&\varepsilon_x=a_{11}\sigma_x+a_{12}\sigma_y+a_{13}\sigma_z+a_{14}\tau_{yz}+a_{15}\tau_{xz}+a_{16}\tau_{xy},\\
&\varepsilon_y=a_{12}\sigma_x+a_{22}\sigma_y+\cdots\cdots\cdots+a_{26}\tau_{xy},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\gamma_{xy}=a_{16}\sigma_x+a_{26}\sigma_y+\cdots\cdots\cdots+a_{66}\tau_{xy}.
\end{aligned}\right\}\qquad(4.1)$$

The connection between the strain components and the displacements of projection is given by Formulas (1.6) in the case of small strains. If the body has elastic symmetry, then, obviously, instead of the general equations (2.1) those corresponding to the given form of elastic symmetry must be taken [e.g., Eqs. (2.7) or (2.8) for an orthotropic body]. The nine unknown functions determining the stress-strain state of the body must be found by integrating the fundamental system (4.1) (or a system equivalent to the fundamental one), taking account of the surface conditions (boundary conditions). According to what is given just

on the surface we distinguish between the first fundamental, the second fundamental, and the mixed problem [which is sometimes called the third fundamental problem of the statics of the elastic body*].

The first fundamental problem. External forces are given on the surface; also the volume forces are given. Designating by X_n, Y_n, Z_n the projections of the external forces referred to unit area, and by n the direction of the normal to the body surface the conditions on the surface may be written in the form

$$\left.\begin{array}{l}\sigma_x \cos(n, x) + \tau_{xy} \cos(n, y) + \tau_{xz} \cos(n, z) = X_n, \\ \tau_{xy} \cos(n, x) + \sigma_y \cos(n, y) + \tau_{yz} \cos(n, z) = Y_n, \\ \tau_{xz} \cos(n, x) + \tau_{yz} \cos(n, y) + \sigma_z \cos(n, z) = Z_n.\end{array}\right\} \qquad (4.2)$$

Instead of the projections of the external forces on the coordinate axes the projections of the forces on the normal n and on two directions perpendicular to n, or, in other words, the normal and tangential stresses may be given.

The second fundamental problem. The displacements are given on the surface; besides, the volume forces are given. In this case the boundary conditions have the form

$$u = u^*, \quad v = v^*, \quad w = w^*. \qquad (4.3)$$

where u^*, v^*, w^* are the given displacement components in the directions of the x, y, z axes.

The mixed problem. On a part of the surface the external forces are given, and on another part the displacements. To the mixed problem, however, also pertain, e.g., such problems where the tangential forces and displacements along the normal or the normal forces and the displacements in a tangent plane are given on the surface, etc.

The uniqueness of the solution of the equilibrium equations of the elastic body for small strains (if the strain components are linear functions of the derivatives of the displacements) is established by

the Kirchhoff theorem.*

Somewhat different are the problems of the stability of elastic bodies having the shape of rods, plates or shells. The main part of the problem boils down to the determination of the critical loads for which the form of equilibrium corresponding to small strains and loads (the principal form) stops being the sole and a stable form of equilibrium.

The fundamental system of the equations of motion of the elastic body has the same form as System (4.1), but the equations of equilibrium of the continuous medium must be replaced by the equations of motion. In other words, on the right-hand sides of the first three equations there will be found inertia terms rather than zeroes:

$$\rho w_x = \rho \frac{\partial^2 u}{\partial t^2}, \quad \rho w_y = \rho \frac{\partial^2 v}{\partial t^2}, \quad \rho w_z = \rho \frac{\partial^2 w}{\partial t^2}.$$

In those cases where it is not possible to obtain an exact solution of the problem of the theory of elasticity (owing to the difficulties due to the determination of functions satisfying the differential equations and the boundary conditions) approximate methods may be used, and an approximate solution of the problem may be constructed with their help. Among these methods the variational methods which are set forth in detail in the book by L.S. Leybenzon play an important part. In the following we shall use a number of approximate methods, among them one variational method whose basis is the principle of virtual displacements and the theorem on the minimum of a certain integral following from it.

Those displacements in an elastic body are understood to be virtual ones with which it remains continuous, but the boundary conditions are satisfied on parts of the surface which are strained in a given way or fixed, i.e., on those where the displacements are given. In other words, displacements permitted by the geometrical connections superimposed on

the elastic body are meant.

Let the body be at equilibrium under the action of the external load. We shall set up the expression for Э equal to the potential energy of strain of the ehole body (expressed in terms of the displacement) minus the work of the external forces, surface and volume ones:

$$Э = \int\int\int \bar{V} d\omega - \int\int (X_n u + Y_n v + Z_n w) ds - \\ - \int\int\int (Xu + Yv + Zw) d\omega \qquad (4.4)$$

(the triple integrals will be taken over the whole volume of the body, and the double one over that part of the surface where the forces are given). Let us consider the expression for Э in which u, v, w are understood to be the virtual displacements (among the virtual displacements, however, there are also real ones which the body experiences when it passes over from the initial state into the state of elastic equilibrium under the action of external forces).

On the basis of the principle of virtual displacements the following theorem may be formulated: *real displacements differ from all virtual ones by the fact that they minimize the expression for* Э.*

The simplest version of an approximate solution based on the use of the above-mentioned theorem will be roughly outlined as follows. Expressions for the displacements are sought in the form of sums with undetermined coefficients by choosing the sum terms such that the displacements satisfy the continuity conditions (on those parts of the surface where they are given). The unknown coefficients are determined by requiring that the expression for Э be a minimum. Ultimately, the problem boils down to determining the minimum of an algebraic integral function of second degree with respect to the coefficients. In the same way, an approximate solution for the elastic body (of finite dimensions)

carrying out simple harmonic vibrations with a frequency p may be obtained, but Э has to be replaced by the expression*

$$Э' = Э - T = Э - \frac{p^2}{2}\int\int\int \rho(u^2 + v^2 + w^2)\,d\omega \qquad (4.5)$$

in this case (T is the kinetic energy of the body).

Manuscript Page No.	[Footnotes]
9	See, e.g., 1) Leybenzon, L.S., Kurs teorii uprugosti [Course on the Theory of Elasticity], Gostekhizdat [State Publishing House of Theoretical and Technical Literature], Moscow-Leningrad, Chapters 1 and 2; 2) Lyav, A., Matematicheskaya teoriya uprugosti [Mathematical Theory of Elasticity], ONTI [Unified Scientific and Technical Publishing House], Moscow-Leningrad, 1935, Chapters 1 and 2.
11	Various cases of elastic symmetry for anisotropic bodies in general and for crystals in particular are considered in our book "The Theory of Elasticity of an Anisotropic Body," Gostekhizdat, Moscow-Leningrad, 1950, Chapter 1. See also: Lyav, A., Matematicheskaya teoriya uprugosti, ONTI, Moscow-Leningrad, 1935, Chapter 6; Bekhterev, P., Analiticheskoye issledovaniye obobshchennogo zakona Hooke'a [Analytic Investigation of the Generalized Hooke's Law, Parts 1 and 2, 1) published by the author (lithographed), Leningrad, 1925; 2) Zhurnal Russkogo fiziko-khimicheskogo obshchestva [Journal of the Russian Physicochemical Society], 57, No. 3-4, 1926, and 58, No. 3, 1926.
14	Sekerzh-Zen'kovich, Ya.I., K raschetu na ustoychivost' lista fanery kak anisotropnoy plastinki [On the Calculation of the Stability of a Plywood Sheet as Anisotropic Plate], Trudy TsAGI [Transactions of the Central Aero'hydrodynamical Institute], No. 76, 1931, page 8. A system of "technical constants" for the general case of anisotropy was proposed by A.L. Rabinovich (see his paper "On the Elastic Constants and the Strength of Anisotropic Materials," Trudy TsAGI, No. 582, 1946).
14	Lyav, A., Matematicheskaya teoriya uprugosti, ONTI, Moscow-Leningrad, 1935, page 172.
16	The general formulas used to transform the elastic constants in the transition to another coordinate system are given in our book "The Theory of Elasticity of an Anisotropic Body", Moscow-Leningrad, 1950, pages 33-45.

16 Already Saint Venant and Voigt have paid attention to this form of anisotropy: 1) B. de Saint Venant, Memoire sur les divers genres d'homogeneite des corps solides [Treatise on the Various Forms of Homogeneity of Solid Bodies], "Journal de Math. pures et appl." [Journal of Pure and Applied Mathematics], (Liouville), Vol. 10, 1865; 2) Voigt. W., Ueber die Elastizitaetsverhaeltnisse cylindridrisch aufgebauter Koerper [On the Elasticity of Cylindrical Bodies], "Nachrichten v.d. Koenigl. Gesellschaft der Wissenschaften und der Georg-Augustin Universitaet zu Goettingen [Bulletin of the Royal Scientific Society and the Georg-Augustin University at Goettingen], 1886, No. 16.

19 Mitinskiy, A.N., Uprugiye postoyannyye drvesiny kak ortotropnogo materiala [The Elastic Constants of Wood as an Orthotropic Material], Trudy Lesotekhnicheskoy akademii im. S.M. Kirova [Transactions of the S.M. Kirov Lumber Technology Academy, No. 63, 1948.

21 See, e.g., the book by N.I. Muskhelishvili, "Nekotoryye osnovnyye zadachi matematicheskoy teorii uprugosti" [Several Basic Problems of the Mathematical Theory of Elasticity], Izd. AN SSSR [Publishing House of the Academy of Sciences of the USSR], Moscow, 1954, pages 65, 71, 72.

22 See, e.g., the above-mentioned course on the theory of elasticity by L.S. Leybenzon, §118, pages 309-311.

22 Leyvebenzon, L.S., Variatsionnyye metody pesheniya zadach teorii uprugosti [Variational Methods of Solving Problems of the Theory of Elasticity], Gostekhizdat, Moscow, 1913.

24 See the mentioned book by L.S. Leybenzon, page 114 and his "Course on the Theory of Elasticity," page 317.

Chapter 2

THE PLANE PROBLEM OF THE THEORY OF ELASTICITY OF AN ANISOTROPIC BODY

§5. THE GENERALIZED PLANE STRESSED STATE OF A HOMOGENEOUS PLATE

Let us consider an elastic homogeneous anisotropic plane plate of constant thickness which is at equilibrium under the action of forces distributed at its boundary, and of volume forces. We shall assume that: 1) at each point of the plate there is a plane of elastic symmetry parallel to the mid plane; 2) the forces applied to the boundary and the volume forces act in planes parallel to the mid plane, are distributed symmetrically with respect to it and vary to a low extent with the thickness; 3) the plate strains are small. The stressed state of the plate working under the conditions mentioned is called *generalized plane stressed state*. The mid plane is not distorted under the strains and remains plane.

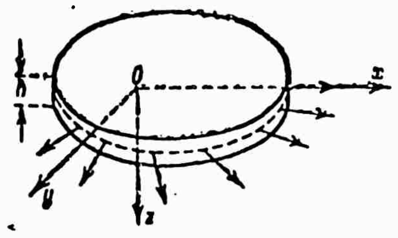

Fig. 6.

Let us choose the mid plane to be the xy coordinate plane, put the origin at an arbitrary point O, and place the x and y axes arbitrarily (Fig. 6). We shall introduce the designations: h is the plate thickness, X_n, Y_n are the projections of the forces distributed along the boundary, per unit area; X, Y are the projections of the volume forces per unit volume ($Z_n = Z = 0$ according to the assumption); a_{11}, a_{12}, ..., a_{66} are the elastic constants of the material in the x, y, z coordinate system.

In studying the generalized plane stressed state the values of the components of stress and the displacement projections averaged over the thickness are considered: $\bar{\sigma}_x$, $\bar{\sigma}_y$, $\bar{\tau}_{xy}$, $\bar{\sigma}_z$, $\bar{u}$, $\bar{v}$, which quantities are determined as integrals of the corresponding stresses and displacements taken over the thickness and divided by the thickness:

$$\left.\begin{aligned} \bar{\sigma}_x &= \frac{1}{h}\int_{-h/2}^{h/2}\sigma_x\,dz, & \bar{\sigma}_y &= \frac{1}{h}\int_{-h/2}^{h/2}\sigma_y\,dz, & \bar{\tau}_{xy} &= \frac{1}{h}\int_{-h/2}^{h/2}\tau_{xy}\,dz,\\ \bar{\sigma}_z &= \frac{1}{h}\int_{-h/2}^{h/2}\sigma_z\,dz, & \bar{u} &= \frac{1}{h}\int_{-h/2}^{h/2}u\,dz, & \bar{v} &= \frac{1}{h}\int_{-h/2}^{h/2}v\,dz. \end{aligned}\right\} \qquad (5.1)$$

The quantity $\bar{\sigma}_z$ will be neglected compared to $\bar{\sigma}_x$, $\bar{\sigma}_y$, and $\bar{\tau}_{xy}$. For the mean stresses and displacements it is easy to obtain five equations according to the number of unknown functions from the fundamental system of equations of equilibrium. Multiplying the first and the second equation of the system (4.1) by dz/h we shall integrate both sides of them over z from $-h/2$ to $h/2$; the same is done with the first, the second and the sixth equations, expressing the generalized Hooke's law [which in our case must be taken in the form (2.5)]. We shall then obtain equations which are satisfied by the mean values:

$$\left.\begin{aligned} \frac{\partial\bar{\sigma}_x}{\partial x} + \frac{\partial\bar{\tau}_{xy}}{\partial y} + \bar{X} &= 0,\\ \frac{\partial\bar{\tau}_{xy}}{\partial x} + \frac{\partial\bar{\sigma}_y}{\partial y} + \bar{Y} &= 0; \end{aligned}\right\} \qquad (5.2)$$

$$\left.\begin{aligned} \bar{\varepsilon}_x &= a_{11}\bar{\sigma}_x + a_{12}\bar{\sigma}_y + a_{16}\bar{\tau}_{xy},\\ \bar{\varepsilon}_y &= a_{12}\bar{\sigma}_x + a_{22}\bar{\sigma}_y + a_{26}\bar{\tau}_{xy},\\ \bar{\gamma}_{xy} &= a_{16}\bar{\sigma}_x + a_{26}\bar{\sigma}_y + a_{66}\bar{\tau}_{xy}. \end{aligned}\right\} \qquad (5.3)$$

Here

$$\bar{X} = \frac{1}{h}\int_{-h/2}^{h/2}X\,dz, \quad \bar{Y} = \frac{1}{h}\int_{-h/2}^{h/2}Y\,dz$$

are the mean (taken over the thickness) values of the volume forces, and $\bar{\varepsilon}_x$, $\bar{\varepsilon}_y$ and $\bar{\gamma}_{xy}$ are the mean (taken over the thickness) values of the strain components equal to:

$$\bar{\varepsilon}_x = \frac{\partial \bar{u}}{\partial x}, \quad \bar{\varepsilon}_y = \frac{\partial \bar{v}}{\partial y}, \quad \bar{\gamma}_{xy} = \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}. \tag{5.4}$$

If the external forces X_n, Y_n are given at the boundary of the plate the boundary conditions will be found from the first two conditions (4.2), by taking the mean of them), i.e., multiplying them by dz/h and integrating over the thickness). Thus we obtain:

$$\left.\begin{aligned} \bar{\sigma}_x \cos(n, x) + \bar{\tau}_{xy} \cos(n, y) = \bar{X}_n, \\ \bar{\tau}_{xy} \cos(n, x) + \bar{\sigma}_y \cos(n, y) = \bar{Y}_n. \end{aligned}\right\} \tag{5.5}$$

Here

$$\bar{X}_n = \frac{1}{h} \int_{-h/2}^{h/2} X_n \, dz, \quad \bar{Y}_n = \frac{1}{h} \int_{-h/2}^{h/2} Y_n \, dz.$$

We shall assume that the volume forces have a potential $\bar{U}(x, y)$ in terms of which they are expressed according to the formulas

$$\bar{X} = -\frac{\partial \bar{U}}{\partial x}, \quad \bar{Y} = -\frac{\partial \bar{U}}{\partial y}. \tag{5.6}$$

The equilibrium equations will be satisfied by introducing the stress function $F(x, y)$ and putting:

$$\bar{\sigma}_x = \frac{\partial^2 F}{\partial y^2} + \bar{U}, \quad \bar{\sigma}_y = \frac{\partial^2 F}{\partial x^2} + \bar{U}, \quad \bar{\tau}_{xy} = -\frac{\partial^2 F}{\partial x \, \partial y}. \tag{5.7}$$

Eliminating the displacements $\bar{u}$ and $\bar{v}$ from Eq. (5.4) by differentiation we obtain the strain compatibility condition

$$\frac{\partial^2 \bar{\varepsilon}_x}{\partial y^2} + \frac{\partial^2 \bar{\varepsilon}_y}{\partial x^2} - \frac{\partial^2 \bar{\gamma}_{xy}}{\partial x \, \partial y} = 0. \tag{5.8}$$

Substituting here the expressions for $\bar{\varepsilon}_x$, $\bar{\varepsilon}_y$, $\bar{\gamma}_{xy}$ from Eqs. (5.3) and expressing the stress components in terms of F, we obtain a differential equation which is satisfied by a function of the stresses

$$a_{22} \frac{\partial^4 F}{\partial x^4} - 2a_{26} \frac{\partial^4 F}{\partial x^3 \partial y} + (2a_{12} + a_{66}) \frac{\partial^4 F}{\partial x^2 \partial y^2} - 2a_{16} \frac{\partial^4 F}{\partial x \, \partial y^3} + a_{11} \frac{\partial^4 F}{\partial y^4} = \tag{5.9}$$

$$= -(a_{12} + a_{22}) \frac{\partial^2 \bar{U}}{\partial x^2} + (a_{16} + a_{26}) \frac{\partial^2 \bar{U}}{\partial x \, \partial y} - (a_{11} + a_{12}) \frac{\partial^2 \bar{U}}{\partial y^2}. \quad (5.9)$$

If there are no volume forces we have instead of (5.9) the homogeneous equation

$$a_{22}\frac{\partial^4 F}{\partial x^4} - 2a_{26}\frac{\partial^4 F}{\partial x^3 \partial y} + (2a_{12} + a_{66})\frac{\partial^4 F}{\partial x^2 \partial y^2} - - 2a_{16}\frac{\partial^4 F}{\partial x \partial y^3} + a_{11}\frac{\partial^4 F}{\partial y^4} = 0. \tag{5.10}$$

In particular, for an orthotropic plate we obtain the following equation if we identify the directions of the x and y axes with the principal directions of elasticity*

$$\frac{1}{E_2}\cdot\frac{\partial^4 F}{\partial x^4} + \left(\frac{1}{G} - \frac{2\nu_1}{E_1}\right)\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{1}{E_1}\cdot\frac{\partial^4 F}{\partial y^4} = 0. \tag{5.11}$$

E_1, E_2 are here Young's moduli for stretching (compression) along the principal directions x and y; $G = G_{12}$ is the shear modulus characterizing the variation of the angles between the principal directions x, y; $\nu_1 = \nu_{12}$ is the Poisson coefficient characterizing the contraction in the y direction for a stretching in the x direction [see Eqs. (2.8)]. In order to study the stresses and strains in an orthotropic plate in which a generalized plane stressed state is realized it is sufficient to know only four of the nine elastic constants: E_1, E_2, G, ν_1. In the following the x and y axes whose directions coincide with the principal directions of elasticity of the orthotropic plate will be called principal axes.

With an isotropic plate $E_1 = E_2 = E$, $G = E/2(1 + \nu)$ and Eq. (5.11) goes over into a biharmonic one**

$$\nabla^2\nabla^2 F = 0, \tag{5.12}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

In detail, this equation reads as follows:

$$\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0. \tag{5.13}$$

As far as the boundary conditions are concerned (which go over into the conditions along the outline of the plane figure S lying on the xy plane) they may be reduced, for given external forces, to giving

the first derivatives of the functions of the stresses. Let the plate region S be limited by an outer contour and one or several inner contours (for the sake of generality we shall assume that the plate has one or several openings, or, in other words, S is a multiply connected region). The contour equations may always be given in parametric form where the arc length s as counted from an initial point (O' on the outer contour, O'' on the inner one) is chosen to be the parameter:

$$x = x(s), \quad y = y(s). \tag{5.14}$$

We shall agree to call the counterclockwise direction of passing along the curve positive [both for the outer and inner one – see Fig. 7*]. Then we obtain:

$$\cos(n, x) = \pm \frac{dy}{ds}, \quad \cos(n, y) = \mp \frac{dx}{ds}.$$

n is the direction of the outer normals to the contours, the outer or the inner one; for the outer contour the upper, and for the inner one the lower signs. Substituting these expressions in the conditions (5.5) and integrating over the arc s from the contour point chosen to be the initial one to a variable point we obtain the boundary conditions for given external forces X_n, Y_n in the form:

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= \int_0^s \left(\mp \overline{Y}_n - \overline{U}\,\frac{dx}{ds}\right) ds + c_1, \\ \frac{\partial F}{\partial y} &= \int_0^s \left(\pm \overline{X}_n - \overline{U}\,\frac{dy}{ds}\right) ds + c_2. \end{aligned}\right\} \tag{5.15}$$

c_1, c_2 are here constants which can be fixed arbitrarily on one of the contours. After the stresses have been found we can find the displacements by integrating Eqs. (5.3).

If the displacements u^*, v^* are given at the plate boundary we obtain the boundary conditions from (4.3) by integrating them:

$$\overline{u} = \overline{u}^*, \quad \overline{v} = \overline{v}^*. \tag{5.16}$$

In the following, we shall omit the dashes over the symbols for

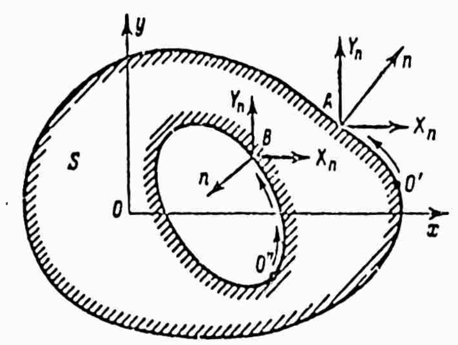

Fig. 7.

stresses and displacements in the consideration of the generalized plane stressed state, in order to simplify the denotation, and understand σ_x, σ_y, τ_{xy}, u, v to be the values of the stresses and displacements obtained by taking the mean over the thickness.

§6. PLANE DEFORMATION IN A HOMOGENEOUS BODY

The problem of plane deformation which is also reduced to the plane problem (i.e., to a two-dimensional problem) has much in common with the problem of the elastic equilibrium of a plate in a generalized plane stressed state. Let us consider a homogeneous anisotropic body having the shape of a long cylinder of arbitrary cross section which is at equilibrium under the action of forces distributed along the side surface, and the volume forces (Fig. 8). We assume that: 1) at each point of the body there is a plane of elastic symmetry which is normal to the generatrix; 2) the forces act in planes normal to the generatrix, and do not vary along the generatrix; 3) the strains are small.

It is obvious that cross sections far from the ends may be considered plane; in this case they are all under the same conditions.

Putting

$$u = u(x, y), \quad v = v(x, y), \quad w = 0, \tag{6.1}$$

we obtain

$$\left.\begin{array}{c} \varepsilon_x = \dfrac{\partial u}{\partial x}, \quad \varepsilon_y = \dfrac{\partial v}{\partial y}, \quad \gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}, \\ \varepsilon_z = \gamma_{yz} = \gamma_{xz} = 0. \end{array}\right\} \tag{6.2}$$

The fundamental system (4.1) assumes the form:

$$\left.\begin{aligned}\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+X=0,\\ \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+Y=0;\end{aligned}\right\} \qquad (6.3)$$

$$\left.\begin{aligned}\varepsilon_x &= \beta_{11}\sigma_x+\beta_{12}\sigma_y+\beta_{16}\tau_{xy},\\ \varepsilon_y &= \beta_{12}\sigma_x+\beta_{22}\sigma_y+\beta_{26}\tau_{xy},\\ \gamma_{xy} &= \beta_{16}\sigma_x+\beta_{26}\sigma_y+\beta_{66}\tau_{xy};\end{aligned}\right\} \qquad (6.4)$$

$$\left.\begin{aligned}\sigma_z &= -\frac{1}{a_{33}}(a_{13}\sigma_x+a_{23}\sigma_y+a_{36}\tau_{xy}),\\ \tau_{yz} &= \tau_{xz}=0.\end{aligned}\right\} \qquad (6.5)$$

β_{ij} are here constants which may be called reduced strain coefficients; they are connected with a_{ij} by the following formulas:

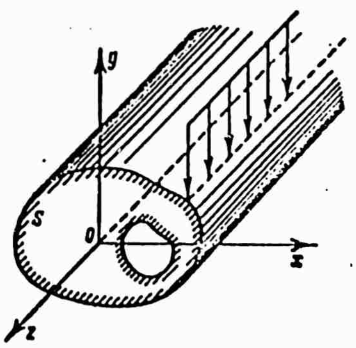

Fig. 8.

$$\beta_{ij}=a_{ij}-\frac{a_{i3}a_{j3}}{a_{33}} \qquad (6.6)$$
$$(i,\ j=1,\ 2,\ 6).$$

Under the assumption that the external forces have a potential, i.e.,

$$X=-\frac{\partial U}{\partial x},\quad Y=-\frac{\partial U}{\partial y}, \qquad (6.7)$$

we obtain formulas which are completely analogous to those obtained in the preceding section*:

$$\sigma_x=\frac{\partial^2 F}{\partial y^2}+U,\quad \sigma_y=\frac{\partial^2 F}{\partial x^2}+U,\quad \tau_{xy}=-\frac{\partial^2 F}{\partial x\,\partial y}; \qquad (6.8)$$

$$\beta_{22}\frac{\partial^4 F}{\partial x^4}-2\beta_{26}\frac{\partial^4 F}{\partial x^3\partial y}+(2\beta_{12}+\beta_{66})\frac{\partial^4 F}{\partial x^2\partial y^2}-2\beta_{16}\frac{\partial^4 F}{\partial x\,\partial y^3}+\beta_{11}\frac{\partial^4 F}{\partial y^4}=$$
$$=-(\beta_{12}+\beta_{22})\frac{\partial^2 U}{\partial x^2}+(\beta_{16}+\beta_{26})\frac{\partial^2 U}{\partial x\,\partial y}-(\beta_{11}+\beta_{12})\frac{\partial^2 U}{\partial y^2}. \qquad (6.9)$$

The boundary conditions reduce to the conditions at the counter of the cross section and coincide formally with conditions (5.15) or (5.16) for a plate in a generalized plane stressed state.

The formulas and equations given here do not take account of the conditions at the ends of the cylinder; strictly speaking, they are only correct for an infinitely long cylinder. In the case of free ends their influence on the distribution of the stresses may be taken into

account approximately, on the basis of the Saint-Venant principle according to which statistically equivalent loads applied to the cylinder ends give rise to identical effects in all of its parts far from the ends. Let the cylinder cross section have finite dimensions. We shall put the origin of coordinates into the center of gravity and place the x and y axes along the principal axes of inertia of the cross section. Determining the stresses σ_x, σ_y, τ_{xy} without taking account of the conditions at the ends we obtain according to Formula (6.5) the normal stress in the cross sections. In special cases it may turn out that it is equal to zero and then the conditions at the free ends will be fulfilled. In the general case, however, the stresses σ_z in each cross section (and, consequently, also at the ends) are reduced to a force P directed along the geometrical axis (z), and to the moment with the components M_1 and M_2 relative to the x and y axes.

In order to remove the "superfluous" forces and moments at the ends we impose a distribution of stresses from force and moments equal to the values of P, M_1 and M_2, but having opposite directions, on the stress distribution for even deformation. In other words, the following correction must be added to the stress σ_z as calculated from Formula (6.5):

$$\Delta\sigma_z = -\frac{P}{S} - \frac{M_1}{I_1}y - \frac{M_2}{I_2}x \tag{6.10}$$

(S is the area of the cross section, I_1 and I_2 are the moments of inertia with respect to the principal axes of inertia x and y). If, however, the cylinder ends are rigidly fixed it is not necessary to add any correction. With the help of the Saint-Venant principle we may verify that the stress distribution in all parts of a cylinder of finite length, except for the zones in the neighborhood of the ends, will be the same as in an infinite cylinder.

In view of the nearly full agreement of the fundamental equations and boundary conditions for the plane at stressed state and for the plane deformation (there is only a difference in the coefficients) both problems are solved by the same methods. Having obtained a solution for the plane stressed state we obtain in the same way also a solution for the corresponding case of plane deformation.

§7. GENERAL EXPRESSIONS FOR THE STRESS FUNCTION

As was shown in the preceding sections the plane problem of the theory of elasticity boils down to determining a stress function $F(x, y)$ satisfying the differential equation of fourth order (5.9) or (6.9) and the boundary conditions on the contours limiting the region, in the region S in the xy plane.

For the sake of definiteness, we shall consider the case of the generalized plane stressed state. If there are no volume forces the function F satisfies the equation

$$a_{22}\frac{\partial^4 F}{\partial x^4} - 2a_{26}\frac{\partial^4 F}{\partial x^3\,\partial y} + (2a_{12} + a_{66})\frac{\partial^4 F}{\partial x^2\,\partial y^2} - 2a_{16}\frac{\partial^4 F}{\partial x\,\partial y^3} + a_{11}\frac{\partial^4 F}{\partial y^4} = 0. \quad (7.1)$$

This equation may be interpreted in a general form, by previously rewriting it symbolically with the help of four linear differential operators of first order in the following manner:

$$D_1 D_2 D_3 D_4 F = 0. \quad (7.2)$$

The symbol D_k $(k = 1, 2, 3, 4)$ designates the operation

$$D_k = \frac{\partial}{\partial y} - \mu_k \frac{\partial}{\partial x}, \quad (7.3)$$

where μ_k are the roots of the characteristic equation

$$a_{11}\mu^4 - 2a_{16}\mu^3 + (2a_{12} + a_{66})\mu^2 - 2a_{26}\mu + a_{22} = 0. \quad (7.4)$$

In the case of an orthotropic plate Eq. (7.4) referred to the principal directions of elasticity assumes the form:

$$\mu^4 + \left(\frac{E_1}{G} - 2\nu_1\right)\mu^2 + \frac{E_1}{E_2} = 0. \qquad (7.5)$$

The author has proved that for any ideal elastic body for which the constants a_{11}, $2a_{12} + a_{66}$, a_{22} are finite and not equal to zero the characteristic equation (7.4) (and the corresponding equation for the plane deformation) may have *either complex or purely imaginary roots*, and cannot have real roots.* Only limiting cases lead to an elimination:

$$1)\ a_{22} = a_{26} = 0;\quad 2)\ a_{22} = a_{26} = 2a_{12} + a_{66} = a_{16} = 0;\quad 3)\ a_{11} = a_{16} = 0;$$
$$4)\ a_{11} = a_{16} = 2a_{12} + a_{66} = a_{26} = 0.$$

In the first case two vanishing roots are obtained, in the second case all four roots are equal to zero, while in the rest of the cases two or all four roots are infinite. In the following, if no special reservation is made, we shall exclude the limiting cases from consideration and always regard the roots μ_k as either complex or purely imaginary numbers; for these roots we shall use the denotations $\mu_1, \mu_2, \bar{\mu}_1, \bar{\mu}_2$.

Two principal cases are possible according to the relationship between the elastic constants:

1) the roots of Eq. (7.4) are all different:

$$\mu_1 = \alpha + \beta i,\quad \mu_2 = \gamma + \delta i,\quad \bar{\mu}_1 = \alpha - \beta i,\quad \bar{\mu}_2 = \gamma - \delta i \qquad (7.6)$$

(α, β, γ, δ are real numbers, $\beta > 0$, $\delta > 0$);

2) the roots of Eq. (7.4) are equal in pairs:

$$\mu_1 = \mu_2 = \alpha + \beta i,\quad \bar{\mu}_1 = \bar{\mu}_2 = \alpha - \beta i \quad (\beta > 0). \qquad (7.7)$$

For an isotropic plate

$$\mu_1 = \mu_2 = i,\quad \bar{\mu}_1 = \bar{\mu}_2 = -i,\quad \alpha = 0,\ \beta = 1, \qquad (7.8)$$

The numbers μ_1 and μ_2 will be called the *complex parameters of the first kind of the plane stressed state* (or, correspondingly, of plane deformation) or simply the *complex parameters*. The complex parameters may be regarded as numbers which to a certain degree characterize the anisotropy in the case of the plane problem; their value can be used to

indicate inhowfar the body deviates from an isotropic one for which always $\mu_1 = \mu_2 = i$, $|\mu_1| = |\mu_2| = 1$.

If the material is orthotropic and the directions of the x and y axes coincide with the principal directions of elasticity then $a_{16} = a_{26} = 0$ and the following three cases of complex parameters are possible (the limiting cases are excluded).

<u>Case 1</u>: $\mu_1 = \beta i$, $\mu_2 = \delta i$ (the complex parameters are purely imaginary and unequal).

<u>Case 2</u>: $\mu_1 = \mu_2 = \beta i$ (the complex parameters are equal).

<u>Case 3</u>: $\mu_1 = \alpha + \beta i$, $\mu_2 = -\alpha + \beta i$.

Having rewritten Eq. (7.1) in the form (7.2) we may reduce its integration to the integration of four first-order partial differential equations.

In fact, if we put

$$D_4 F = g_3, \quad D_3 D_4 F = g_2, \quad D_2 D_3 D_4 F = g_1, \tag{7.9}$$

we obtain the equation

$$D_1 g_1 \equiv \frac{\partial g_1}{\partial y} - \mu_1 \frac{\partial g_1}{\partial x} = 0. \tag{7.10}$$

On integration we find

$$g_1 = f_1(x + \mu_1 y), \tag{7.11}$$

where f_1 is an arbitrary function of the variable $x + \mu_1 y$. Furthermore, from (7.9) obtain the equations:

$$\left.\begin{aligned} \frac{\partial g_2}{\partial y} - \mu_2 \frac{\partial g_2}{\partial x} &= g_1, \\ \frac{\partial g_3}{\partial y} - \bar{\mu}_1 \frac{\partial g_3}{\partial x} &= g_2, \\ \frac{\partial F}{\partial y} - \bar{\mu}_2 \frac{\partial F}{\partial x} &= g_3. \end{aligned}\right\} \tag{7.12}$$

Integrating these inhomogeneous equations one after another we obtain the following expressions for F:

1) in the case of different complex parameters

$$F = F_1(x + \mu_1 y) + F_2(x + \mu_2 y) + F_3(x + \bar{\mu}_1 y) + F_4(x + \bar{\mu}_2 y); \tag{7.13}$$

2) in the case of complex parameters equal in pairs

$$F = F_1(x + \mu_1 y) + (x + \bar{\mu}_1 y) F_2(x + \mu_1 y) + \\ + F_3(x + \bar{\mu}_1 y) + (x + \mu_1 y) F_4(x + \bar{\mu}_1 y) \quad (7.14)$$

(F_1, F_2, F_3, F_4 are arbitrary functions of the variables $x + \mu_k y$ or $x + \bar{\mu}_k y$).

The variables $x + \mu_k y$ are complex, but not of the usual type $x + iy$, but more complicated or general. Introducing the designations

$$z_1 = x + \mu_1 y, \quad z_2 = x + \mu_2 y, \quad \bar{z}_1 = x + \bar{\mu}_1 y, \quad \bar{z}_2 = x + \bar{\mu}_2 y \quad (7.15)$$

for them and bearing in mind that the stress function must be a real function of the variables x and y we shall rewrite Eqs. (7.13) and (7.14) in another form:

1) in the case of different complex parameters

$$F = 2\,\mathrm{Re}\,[F_1(z_1) + F_2(z_2)]; \quad (7.16)$$

2) in the case of complex parameters equal in pairs

$$F = 2\,\mathrm{Re}\,[F_1(z_1) + \bar{z}_1 F_2(z_1)] \quad (7.17)$$

(Re denotes the real part of any complex expression.)

In particular, for an isotropic body $z_1 = x + iy = z$, $\bar{z}_1 = \bar{z}$; changing the designations of the arbitrary functions we obtain the well-known expression*:

$$F = \mathrm{Re}\,[\bar{z}\varphi(z) + \chi(z)]. \quad (7.18)$$

Sometimes it is more convenient to introduce new variables

$$z_1' = z + \lambda_1 \bar{z}, \quad z_2' = z + \lambda_2 \bar{z}, \quad (7.19)$$

where

$$\lambda_1 = \frac{1 + i\mu_1}{1 - i\mu_1}, \quad \lambda_2 = \frac{1 + i\mu_2}{1 - i\mu_2}. \quad (7.20)$$

These variables differ from z_1 and z_2 only by constant factors. The numbers λ_1 and λ_2 which depend only on the elastic constants, all things considered, will be called *complex parameters of the second kind* in contrast to μ_1 and μ_2. For an isotropic body $\lambda_1 = \lambda_2 = 0$; for an anisotropic body they are, in general, complex numbers whose absolute

values do not exceed unity. Designating arbitrary functions of the variables z_1' and z_2' by Θ_1 and Θ_2 we shall rewrite the general expression of the stress function in the case of unequal complex parameters in the form

$$F = 2\,\mathrm{Re}\,[\Theta_1(z_1') + \Theta_2(z_2')]. \qquad (7.21)$$

If volume forces with a potential U act on the body the stress function satisfies, generally speaking, the inhomogeneous equation (5.9) or (6.9). The general expression for this function will be written as the sum of expression (7.16) [or, respectively, (7.17), (7.18), (7.21)], and a special solution of the inhomogeneous equation; usually, it is not very difficult to find this special solution. Also these cases were in spite of the existence of volume forces the function F satisfies the homogeneous equation (7.1) are possible. As an example the problem of the distribution of the proper weight stresses in a homogeneous body may be used; in this case the volume forces have a potential which depends linearly on the coordinates, hence all its second derivatives vanish.

§8. THE CONNECTION OF THE PLANE PROBLEM WITH THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

As shown by Formula (7.16) the stress function in the case of unequal complex parameters is expressed in terms of two arbitrary analytic functions of complex variables $z_1 = x + \mu_1 y$, $z_2 = x + \mu_2 y$ or $z_1' = z + \lambda_1 \bar{z}$, $z_2' = z + \lambda_2 \bar{z}$ (complicated or generalized). In the case of equal parameters we obtain one complex variable z_1 or z_1'.

If we know the expression for the function F it is easy to find the expression for the stress components and then to obtain also the formulas for the displacements by integrating Eqs. (5.3) or (5.4).

Let us focus our attention on the case where the complex parame-

ters are different, but there are no volume forces. Introducing the designations

$$\Phi_1(z_1)=\frac{dF_1}{dz_1},\quad \Phi_2(z_2)=\frac{dF_2}{dz_2},\quad \Phi_1'(z_1)=\frac{d\Phi_1}{dz_1},\quad \Phi_2'(z_2)=\frac{d\Phi_2}{dz_2}, \qquad (8.1)$$

we obtain the following formulas with the help of expression (7.16):

$$\left.\begin{aligned}\sigma_x &= 2\,\mathrm{Re}\,[\mu_1^2\Phi_1'(z_1)+\mu_2^2\Phi_2'(z_2)],\\ \sigma_y &= 2\,\mathrm{Re}\,[\Phi_1'(z_1)+\Phi_2'(z_2)],\\ \tau_{xy} &= -2\,\mathrm{Re}\,[\mu_1\Phi_1'(z_1)+\mu_2\Phi_2'(z_2)];\end{aligned}\right\} \qquad (8.2)$$

$$\left.\begin{aligned}u &= 2\,\mathrm{Re}\,[p_1\Phi_1(z_1)+p_2\Phi_2(z_2)]-\omega y+u_0,\\ v &= 2\,\mathrm{Re}\,[q_1\Phi_1(z_1)+q_2\Phi_2(z_2)]+\omega x+v_0.\end{aligned}\right\} \qquad (8.3)$$

Here we have introduced the denotations

$$\left.\begin{aligned}p_1 &= a_{11}\mu_1^2+a_{12}-a_{16}\mu_1, & p_2 &= a_{11}\mu_2^2+a_{12}-a_{16}\mu_2,\\ q_1 &= a_{12}\mu_1+\frac{a_{22}}{\mu_1}-a_{26}, & q_2 &= a_{12}\mu_2+\frac{a_{22}}{\mu_2}-a_{26};\end{aligned}\right\} \qquad (8.4)$$

ω, u_0, v_0 are arbitrary constants due to the integration which characterize the "rigid" displacement of the plate, i.e., the displacement in the xy plane without deformation (ω characterizes the revolution, and u_0, v_0 the translatory displacement).*

The normal and tangential stress components on a plane with arbitrarily directed normal n will be found from the formulas:

$$\left.\begin{aligned}\sigma_n &= \sigma_x\cos^2(n,x)+\sigma_y\cos^2(n,y)+2\tau_{xy}\cos(n,x)\cos(n,y),\\ \tau_n &= (\sigma_y-\sigma_x)\cos(n,x)\cos(n,y)+\tau_{xy}[\cos^2(n,x)-\cos^2(n,y)].\end{aligned}\right\} \qquad (8.5)$$

Substituting here the expressions for σ_x, σ_y, τ_{xy} we obtain:

$$\left.\begin{aligned}\sigma_n = 2\,\mathrm{Re}\,\{&[\cos(n,y)-\mu_1\cos(n,x)]^2\Phi_1'(z_1)+\\ &+[\cos(n,y)-\mu_2\cos(n,x)]^2\Phi_2'(z_2)\},\\ \tau_n = 2\,\mathrm{Re}\,\{&[\cos(n,y)-\mu_1\cos(n,x)]\times\\ &\times[\cos(n,x)+\mu_1\cos(n,y)]\Phi_1'(z_1)+\\ &+[\cos(n,y)-\mu_2\cos(n,x)]\times\\ &\times[\cos(n,x)+\mu_2\cos(n,y)]\Phi_2'(z_2)\}.\end{aligned}\right\} \qquad (8.6)$$

For the given external forces X_n, Y_n the boundary conditions assume the form [see Formulas (5.15)]:

$$\left.\begin{aligned} 2\,\mathrm{Re}\,[\Phi_1(z_1)+\Phi_2(z_2)] &= \mp\int_0^s Y_n\,ds + c_1, \\ 2\,\mathrm{Re}\,[\mu_1\Phi_1(z_1)+\mu_2\Phi_2(z_2)] &= \pm\int_0^s X_n\,ds + c_2. \end{aligned}\right\} \qquad (8.7)$$

If, however, the displacements are given we obtain the following boundary conditions:

$$\left.\begin{aligned} 2\,\mathrm{Re}\,[p_1\Phi_1(z_1)+p_2\Phi_2(z_2)] &= u^* + \omega y - u_0, \\ 2\,\mathrm{Re}\,[q_1\Phi_1(z_1)+q_2\Phi_2(z_2)] &= v^* - \omega x - v_0. \end{aligned}\right\} \qquad (8.8)$$

The equations given indicate the connection of the stress and displacement components with the functions of the complex variables. Introducing the designations

$$x_1 = x + \alpha y, \quad y_1 = \beta y; \qquad (8.9a)$$

$$x_2 = x + \gamma y, \quad y_2 = \delta y, \qquad (8.9b)$$

then the functions Φ_1 and Φ_2 may be considered to be functions of complex variables of the ordinary type

$$z_1 = x_1 + i y_1, \quad z_2 = x_2 + i y_2.$$

But if this point of view is adopted then the functions Φ_1 and Φ_2 must not be determined in the same region S which is occupied by the plate in reality, but, respectively, in some regions S_1 and S_2 obtained from S by affine transformation given by Formulas (8.9a) and (8.9b). Figure 9 8llustrates how regions S_1 and S_2 are obtained from S.

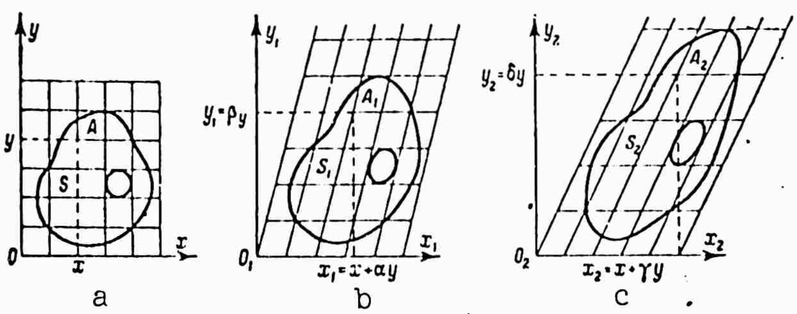

Fig. 9.

Thus, the plane problem for an anisotropic body may be regarded as the problem of determining the functions $\Phi_1(z_1)$ and $\Phi_2(z_2)$ satisfying

the boundary conditions (8.7) or (8.8) in the regions S_1 and S_2 (at points of the contours of regions corresponding to one another in an affine manner). In the general case this problem is rather complex, but it is possible to indicate a number of special cases of regions for which an exact solution can easily be obtained.

An investigation shows that the functions Φ_1 and Φ_2 must satisfy the following conditions within their regions*:

1) if the region of the plate S is finite and simply connected (the plate has no holes) then the functions Φ_1 and Φ_2 are holomorphic and single-valued in their regions S_1 and S_2;

2) if the region S is bounded by several contours or is an infinite plane with a cut (the plate is weakened by holes), but the equivalent vector (the resultant) of the forces applied to each of the contours is equal to zero then the functions Φ_1 and Φ_2 are holomorphic and single-valued in their regions S_1 and S_2;

3) if the region S is bounded by several contours or is an infinite plane with a cut (the plate has holes) and even if the equivalent vector (the resultant) on one contour is not equal to zero then the functions Φ_1 and Φ_2 will be multivalued. If, e.g., there is one hole in the plate and at its boundary act forces whose resultant has the components P_x and P_y then the functions Φ_1 and Φ_2 will increase by the increments Δ_1 and Δ_2 to be found from the following equations** if we pass around along any closed contour entirely lying in the region of the plate and encircling this hole:

$$\left.\begin{aligned}
&\Delta_1+\Delta_2+\bar{\Delta}_1+\bar{\Delta}_2=\frac{P_y}{h},\\
&\mu_1\Delta_1+\mu_2\Delta_2+\bar{\mu}_1\bar{\Delta}_1+\mu_2\bar{\Delta}_2=-\frac{P_x}{h},\\
&\mu_1^2\Delta_1+\mu_2^2\Delta_2+\bar{\mu}_1^2\bar{\Delta}_1+\bar{\mu}_2^2\Delta_2=-\frac{a_{16}}{a_{11}}\cdot\frac{P_x}{h}-\frac{a_{12}}{a_{11}}\cdot\frac{P_y}{h},\\
&\frac{1}{\mu_1}\Delta_1+\frac{1}{\mu_2}\Delta_2+\frac{1}{\bar{\mu}_1}\bar{\Delta}_1+\frac{1}{\bar{\mu}_2}\Delta_2=\frac{a_{12}}{a_{22}}\frac{P_x}{h}+\frac{a_{26}}{a_{22}}\cdot\frac{P_y}{h}.
\end{aligned}\right\} \qquad (8.10)$$

If we use the representation of the function F in terms of functions of the complex variables z_1' and z_2' according to Formula (7.21) we obtain instead of (8.2) and (8.3):

$$\left.\begin{aligned}\sigma_x &= 2\,\mathrm{Re}\,[\mu_1^2(1+\lambda_1)\varphi_1'(z_1')+\mu_2^2(1+\lambda_2)\varphi_2'(z_2')],\\ \sigma_y &= 2\,\mathrm{Re}\,[(1+\lambda_1)\varphi_1'(z_1')+(1+\lambda_2)\varphi_2'(z_2')],\\ \tau_{xy} &= -2\,\mathrm{Re}\,[\mu_1(1+\lambda_1)\varphi_1'(z_1')+\mu_2(1+\lambda_2)\varphi_2'(z_2')];\end{aligned}\right\} \quad (8.11)$$

$$\left.\begin{aligned}u &= 2\,\mathrm{Re}\,[p_1\varphi_1(z_1')+p_2\varphi_2(z_2')]-\omega y+u_0,\\ v &= 2\,\mathrm{Re}\,[q_1\varphi_1(z_1')+q_2\varphi_2(z_2')]+\omega x+v_0.\end{aligned}\right\} \quad (8.12)$$

We have here introduced the denotations

$$\left.\begin{aligned}&\varphi_1(z_1')=(1+\lambda_1)\frac{d\theta_1}{dz_1'},\quad \varphi_2(z_2')=(1+\lambda_2)\frac{d\theta_2}{dz_2'},\\ &\varphi_1'(z_1')=\frac{d\varphi_1}{dz_1'},\quad \varphi_2'(z_2')=\frac{d\varphi_2}{dz_2'}.\end{aligned}\right\} \quad (8.13)$$

The coefficients p_1, p_2, q_1, q_2 are determined from the preceding formulas (8.4). The stresses σ_n and τ_n on an arbitrary plane are found from the formulas which will be obtained from (8.6) by replacing $\Phi_1(z_1)$ and $\Phi_2'(z_2)$ by, respectively, the quantities $(1+\lambda_1)\varphi_1'(z_1')$ and $(1+\lambda_2)\varphi_2'(z_2')$. The boundary conditions for the functions $\varphi_1(z_1')$ and $\varphi_2'(z_2')$ coincide exactly with the conditions (8.7) and (8.8) for $\Phi_1(z_1)$ and $\Phi_2(z_2)$.

For an isotropic plate the well-known formulas of G.V. Kolosov and N.I. Muskhelishvili* are obtained on the basis of the general expression (7.18):

$$\left.\begin{aligned}\sigma_y-\sigma_x+2i\tau_{xy} &= 2\,[\bar{z}\varphi''(z)+\psi'(z)],\\ \sigma_x+\sigma_y &= 4\,\mathrm{Re}\,[\varphi'(z)];\end{aligned}\right\} \quad (8.14)$$

$$2\mu(u+iv)=\varkappa\varphi(z)-z\bar{\varphi}'(\bar{z})-\bar{\psi}(\bar{z}). \quad (8.15)$$

Here $\psi(z)=\chi'(z)$; $\bar{\varphi}'$, $\bar{\psi}$ are the functions conjugate to φ' and ψ; $\mu = G$ is the shear modulus; $\varkappa=\frac{3-\nu}{1+\nu}$; ν is the Poisson coefficient.

For given external forces X_n, Y_n the boundary conditions for the functions φ and ψ assume the form:

$$\varphi(z)+z\bar{\varphi}'(\bar{z})+\bar{\psi}(\bar{z})=\pm\int_0^s(iX_n-Y_n)\,ds+c \quad (8.16)$$

(the upper sign must be adopted if the outer contour is considered, the lower one if the hole contour is considered; c is the integration constant).

If the displacements are given on the contour of an isotropic plate then the boundary conditions may be written in the following form:

$$\varkappa\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)}=2\mu(u^{*}+iv^{*}). \qquad (8.17)$$

We shall not specially choose the case of complex parameters equal in pairs; by replacing the variable and changing the contour of the region S for which the problem must be solved it boils down to the case of an isotropic body.

The plane problem of the theory of elasticity of an anisotropic body may be reduced to integral equations of well investigated types, in which case various methods can be applied. The reduction of the plane problem to integral equations made it possible to study the problems of the existence and uniqueness of its solution with exhaustive completeness and to work out general methods of obtaining the solution in the general case. These problems were treated in a number of papers. S.G. Mikhlin considered the plane problem for a finite simply connected region for given external forces and reduced it to a system of integral equations with two unknown functions.* G.N. Savin investigated the case of an infinite region with a cut (a plate with a hole).** D.I. Sherman considered the case of a multiply connected region.*** In subsequent works D.I. Sherman reduced the plane problem for a multiply connected region for given external forces to one integral equation with one unknown function.**** The same problem for the case of given displacements was investigated considerably later by T.B. Ayzenberg; not only did he obtain an integral equation, but he also solved it for the special case of an anisotro[illegible]ate having the shape of a round disc.*****

The integral equation of the plane problem with one unknown function was obtained in a somewhat different way also in a work by I. Vekua.* We must also mention the work by V.D. Kupradze and M.O. Basheleyshvili in which it is shown that the plane problem may be reduced to the determination of the potentials of a simple or double layer; the densities of these potentials satisfy well-known integral equations.** Those solutions for simple regions which we shall present below were obtained by comparatively simple methods not connected with integral equations (we know, however, only one work in which the solution of a concrete special problem was found with the help of integral equations — the above-mentioned work by T.B. Ayzenberg).

Finally, we must mention the optical method of investigating the stresses in plates which are under the conditions of a generalized plane stressed state. The optical method is very efficient if the stresses in isotropic bodies are to be studied (particularly in those cases where it is cumbersome to seek the theoretical solution of a plane problem); it is set forth in detail, e.g., in the well-known book by Coker and Filon.*** The problem of applying the optical method for studying the plane stressed state of anisotropic bodies proves to be considerably more complex and only a little work has been done in this field, as yet. The most important results in this field, theoretical and experimental ones, were obtained by V.M. Krasnovyy and A.V. Stepanovyy.****

§9. DETERMINATION OF ELASTIC CONSTANTS FOR A NEW COORDINATE SYSTEM

When studying the plane stressed state of an anisotropic plate one may often encounter on the following problem: the elastic constants are known for some coordinate system x, y and the elastic constants for a new system x', y' must be found where the new system is rotated with

respect to the first one by an angle φ (Fig. 10). For an orthotropic plate usually the principal elastic constants are given; it may, however, prove that the use of the principal coordinate system is inconvenient, for some reason, such that the conversion of the elastic constants and complex parameters is necessary.

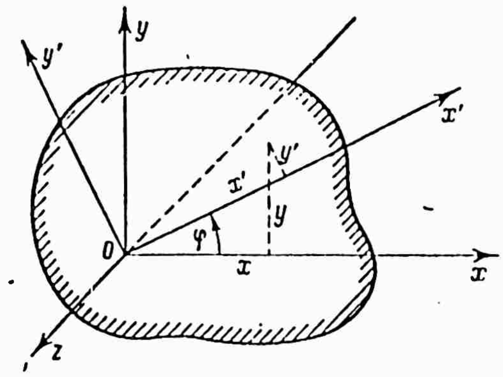

Fig. 10.

The formulas for the conversion of the elastic constants may be obtained in the following way.*

Let us consider the generalized plane stressed state of an anisotropic plate whose mid plane is chosen to be the xy plane. Let a_{ij} be the elastic constants for the x, y coordinate system, and a'_{ij} the elastic constants for the new axes x', y', rotated by an angle φ about the origin O with respect to x, y. Assuming that in the xy plane there are no principal directions of elasticity we have the equations of the generalized Hooke's law (for stress and strain components whose mean values have been taken over the thickness) and the expression for the elastic potential:

$$\left.\begin{aligned} \varepsilon_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{16}\tau_{xy}, \\ \varepsilon_y &= a_{12}\sigma_x + a_{22}\sigma_y + a_{26}\tau_{xy}, \\ \gamma_{xy} &= a_{16}\sigma_x + a_{26}\sigma_y + a_{66}\tau_{xy}; \end{aligned}\right\} \tag{9.1}$$

$$\bar{V} = \frac{1}{2}a_{11}\sigma_x^2 + a_{12}\sigma_x\sigma_y + a_{16}\sigma_x\tau_{xy} + \frac{1}{2}a_{22}\sigma_y^2 + a_{26}\sigma_y\tau_{xy} + \frac{1}{2}a_{66}\tau_{xy}^2. \tag{9.2}$$

For the new x', y' system we have:

$$\left.\begin{aligned} \varepsilon'_x &= a'_{11}\sigma'_x + a'_{12}\sigma'_y + a'_{16}\tau'_{xy}, \\ \varepsilon'_y &= a'_{12}\sigma'_x + a'_{22}\sigma'_y + a'_{26}\tau'_{xy}, \\ \gamma'_{xy} &= a'_{16}\sigma'_x + a'_{26}\sigma'_y + a'_{66}\tau'_{xy}; \end{aligned}\right\} \tag{9.3}$$

$$\begin{aligned} \bar{V} = \frac{1}{2}a'_{11}\sigma'^2_x &+ a'_{12}\sigma'_x\sigma'_y + a'_{16}\sigma'_x\tau'_{xy} + \\ &+ \frac{1}{2}a'_{22}\sigma'^2_y + a'_{26}\sigma'_y\tau'_{xy} + \frac{1}{2}a'_{66}\tau'^2_{xy}. \end{aligned} \tag{9.4}$$

We shall express σ_x, σ_y, τ_{xy} in terms of σ'_x, σ'_y, τ'_{xy} for which purpose

we make use of the formulas (8.5) which in our case assume the form:

$$\left.\begin{aligned}
\sigma_x &= \sigma_x' \cos^2\varphi + \sigma_y' \sin^2\varphi - 2\tau_{xy}' \sin\varphi\cos\varphi,\\
\sigma_y &= \sigma_x' \sin^2\varphi + \sigma_y' \cos^2\varphi + 2\tau_{xy}' \sin\varphi\cos\varphi,\\
\tau_{xy} &= (\sigma_x' - \sigma_y') \sin\varphi\cos\varphi + \tau_{xy}' (\cos^2\varphi - \sin^2\varphi).
\end{aligned}\right\} \qquad (9.5)$$

Furthermore, we substitute these values into the formula for the elastic potential (9.2) and compare the expression obtained with (9.4). Hence we find the sought formulas of the elastic-constant transformation in passing over to new axes:

$$\left.\begin{aligned}
a_{11}' &= a_{11}\cos^4\varphi + (2a_{12} + a_{66})\sin^2\varphi\cos^2\varphi + a_{22}\sin^4\varphi +\\
&\qquad + (a_{16}\cos^2\varphi + a_{26}\sin^2\varphi)\sin 2\varphi,\\
a_{22}' &= a_{11}\sin^4\varphi + (2a_{12} + a_{66})\sin^2\varphi\cos^2\varphi + a_{22}\cos^4\varphi -\\
&\qquad - (a_{16}\sin^2\varphi + a_{26}\cos^2\varphi)\sin 2\varphi,\\
a_{12}' &= a_{12} + (a_{11} + a_{22} - 2a_{12} - a_{66})\sin^2\varphi\cos^2\varphi +\\
&\qquad + \frac{1}{2}(a_{26} - a_{16})\sin 2\varphi\cos 2\varphi,\\
a_{66}' &= a_{66} + 4(a_{11} + a_{22} - 2a_{12} - a_{66})\sin^2\varphi\cos^2\varphi +\\
&\qquad + 2(a_{26} - a_{16})\sin 2\varphi\cos 2\varphi,\\
a_{16}' &= \left[a_{22}\sin^2\varphi - a_{11}\cos^2\varphi + \frac{1}{2}(2a_{12} + a_{66})\cos 2\varphi\right]\sin 2\varphi +\\
&\quad + a_{16}\cos^2\varphi(\cos^2\varphi - 3\sin^2\varphi) + a_{26}\sin^2\varphi(3\cos^2\varphi - \sin^2\varphi),\\
a_{26}' &= \left[a_{22}\cos^2\varphi - a_{11}\sin^2\varphi - \frac{1}{2}(2a_{12} + a_{66})\cos 2\varphi\right]\sin 2\varphi +\\
&\quad + a_{16}\sin^2\varphi(3\cos^2\varphi - \sin^2\varphi) + a_{26}\cos^2\varphi(\cos^2\varphi - 3\sin^2\varphi).
\end{aligned}\right\} \qquad (9.6)$$

We notice two invariants, i.e., two expressions which remain numerically equal on rotation by an arbitrary angle φ:

$$\left.\begin{aligned}
a_{11}' + a_{22}' + 2a_{12}' &= a_{11} + a_{22} + 2a_{12},\\
a_{66}' - 4a_{12}' &= a_{66} - 4a_{12}.
\end{aligned}\right\} \qquad (9.7)$$

If, in particular, the plate is orthotropic and the directions of the x and y axes coincide with the principal directions of elasticity the equations of the general Hooke's law (9.1) have the form:

$$\left.\begin{aligned}
\varepsilon_x &= \frac{1}{E_1}\sigma_x - \frac{\nu_2}{E_2}\sigma_y,\\
\varepsilon_y &= -\frac{\nu_1}{E_1}\sigma_x + \frac{1}{E_2}\sigma_y,\\
\gamma_{xy} &= \frac{1}{G}\tau_{xy}.
\end{aligned}\right\} \qquad (9.8)$$

Passing over to new axes x', y' we obtain the equations of the generalized Hooke's law (9.3), and introducing "technical constants" we

rewrite them in the following way:

$$\left.\begin{aligned}
\varepsilon'_x &= \frac{1}{E'_1}\sigma'_x - \frac{\nu'_2}{E'_2}\sigma'_y + \frac{\eta'_1}{E'_1}\tau'_{xy},\\
\varepsilon'_y &= -\frac{\nu'_1}{E'_1}\sigma'_x + \frac{1}{E'_2}\sigma'_y + \frac{\eta'_2}{E'_2}\tau'_{xy},\\
\gamma'_{xy} &= -\frac{\eta'_1}{E'_1}\sigma'_x + \frac{\eta'_2}{E'_2}\sigma'_y + \frac{1}{G'}\tau'_{xy}.
\end{aligned}\right\} \tag{9.9}$$

E'_1, E'_2 are Young's moduli; ν'_1, ν'_2 are the Poisson coefficients; G' is the shear modulus for the new directions; η'_1, η'_2 are the secondary coefficients which vanish in the fundamental system.* The moduli and coefficients for the new axes are determined by formulas resulting from (9.6):

$$\left.\begin{aligned}
\frac{1}{E'_1} &= \frac{\cos^4\varphi}{E_1} + \left(\frac{1}{G} - \frac{2\nu_1}{E_1}\right)\sin^2\varphi\cos^2\varphi + \frac{\sin^4\varphi}{E_2},\\
\frac{1}{E'_2} &= \frac{\sin^4\varphi}{E_1} + \left(\frac{1}{G} - \frac{2\nu_1}{E_1}\right)\sin^2\varphi\cos^2\varphi + \frac{\cos^4\varphi}{E_2},\\
\frac{1}{G'} &= \frac{1}{G} + \left(\frac{1+\nu_1}{E_1} + \frac{1+\nu_2}{E_2} - \frac{1}{G}\right)\sin^2 2\varphi,\\
\nu'_1 &= E'_1\left[\frac{\nu_1}{E_1} - \frac{1}{4}\left(\frac{1+\nu_1}{E_1} + \frac{1+\nu_2}{E_2} - \frac{1}{G}\right)\sin^2 2\varphi\right],\\
\nu'_2 &= \nu'_1\frac{E'_2}{E'_1},\\
\eta'_1 &= E'_1\left[\frac{\sin^2\varphi}{E_2} - \frac{\cos^2\varphi}{E_1} + \frac{1}{2}\left(\frac{1}{G} - \frac{2\nu_1}{E_1}\right)\cos 2\varphi\right]\sin 2\varphi,\\
\eta'_2 &= E'_2\left[\frac{\cos^2\varphi}{E_2} - \frac{\sin^2\varphi}{E_1} - \frac{1}{2}\left(\frac{1}{G} - \frac{2\nu_1}{E_1}\right)\cos 2\varphi\right]\sin 2\varphi.
\end{aligned}\right\} \tag{9.10}$$

The expressions

$$\left.\begin{aligned}
\frac{1}{E'_1} + \frac{1}{E'_2} - \frac{2\nu'_1}{E'_1} &= \frac{1}{E_1} + \frac{1}{E_2} - \frac{2\nu_1}{E_1},\\
\frac{1}{G'} + \frac{4\nu'_1}{E'_1} &= \frac{1}{G} + \frac{4\nu_1}{E_1}.
\end{aligned}\right\} \tag{9.11}$$

will be invariant.

In practice, also the following problem may arise: in an orthotropic plate the elastic constants a_{ij} referred to an arbitrary coordinate system x,y are known, and the principal elastic constants must be determined.

The problem is solved with the help of the two last formulas of

(9.6).

Let us designate the principal axes by x', y' and principal elastic constants by a'_{ij} in the given case. The unknown angle φ which is formed by the x axis with one of the principal directions is determined as the minimum angle satisfying simultaneously the two equations:

$$a'_{16} = 0, \quad a'_{26} = 0, \tag{9.12}$$

which will be reduced to the following equations after some simple transformations:

$$\left.\begin{aligned} \operatorname{tg} 2\varphi &= \frac{a_{16} + a_{26}}{a_{11} - a_{22}}, \\ \operatorname{tg} 4\varphi &= 2\,\frac{a_{16} - a_{26}}{a_{11} + a_{22} - 2a_{12} - a_{66}}. \end{aligned}\right\} \tag{9.13}$$

The condition for the existence of identical solutions to these two equations has the form:

$$\begin{aligned}(a_{16} - a_{26})(a_{11} - a_{22} + a_{16} + a_{26})(a_{11} - a_{22} - a_{16} - a_{26}) = \\ = (a_{16} + a_{26})(a_{11} - a_{22})(a_{11} + a_{22} - 2a_{12} - a_{66}).\end{aligned} \tag{9.14}$$

If this condition is not fulfilled there are no principal directions in the xy plane, i.e., the plate is not orthotropic.

The formulas for the recalculation of the given constants of plane deformation β_{ij} in passing over to new axes are identical with Formulas (9.6).

Example. Let us assume that we know the elastic constants of an anisotropic plate referred to an xy coordinate system, i.e.:

$$a_{16} = a_{26} \neq 0, \quad a_{11} = a_{22}, \quad a_{11} + a_{22} - 2a_{12} - a_{66} > 0.$$

The condition of the existence of principal directions (9.14) is, obviously, fulfilled. Equations (9.13) assume the form:

$$\operatorname{tg} 2\varphi = \infty, \quad \operatorname{tg} 4\varphi = 0. \tag{9.15}$$

From the first equation (9.15) we find: $\varphi = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots$, and from the second one: $\varphi = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \ldots$; the solutions of the first equation are also solutions of the second one. Consequently, we may put: $\varphi = \frac{\pi}{4}$. There are principal directions of elasticity; they are the directions

of the bisectrices of the angles between the x and y axes. We shall determine the principal elastic constants a'_{ij} from Formulas (9.6) by substituting in them the numerical values of the a_{ij} and $\varphi=\frac{\pi}{4}$.

§10. THE CONVERSION OF THE COMPLEX PARAMETERS IN PASSING OVER TO NEW AXES

If the complex parameters of the first kind μ_1 and μ_2 calculated for the x and y axes are given it is not necessary in passing over to the new axes x', y' to set up anew and to solve the fourth-degree equation (7.4). It is not difficult to derive formulas from which the complex parameters for any other coordinate system rotated with respect to the first one by the angle φ can be calculated (see Fig. 10) if the parameters for the first system are given.

We shall write the equation for the stress function F in a symbolic manner. In the old x, y system we have:

$$D_1 D_2 \overline{D}_1 \overline{D}_2 F = 0, \tag{10.1}$$

where

$$D_k = \frac{\partial}{\partial y} - \mu_k \frac{\partial}{\partial x}, \quad \overline{D}_k = \frac{\partial}{\partial y} - \overline{\mu}_k \frac{\partial}{\partial x} \quad (k = 1,\ 2); \tag{10.2}$$

μ_k, $\overline{\mu}_k$ are the roots of the equation

$$a_{11}\mu^4 - 2a_{16}\mu^3 + (2a_{12} + a_{66})\mu^2 - 2a_{26}\mu + a_{22} = 0. \tag{10.3}$$

We shall pass over to new axes x', y'; the transformation formulas have the form (see Fig. 10):

$$\left.\begin{aligned} x' &= x\cos\varphi + y\sin\varphi, \\ y' &= -x\sin\varphi + y\cos\varphi. \end{aligned}\right\} \tag{10.4}$$

Expressing the derivatives with respect to x and y in terms of the derivatives with respect to x' and y' we obtain:

$$\left.\begin{aligned} \frac{\partial}{\partial y} &= \cos\varphi\frac{\partial}{\partial y'} + \sin\varphi\frac{\partial}{\partial x'}, \\ \frac{\partial}{\partial x} &= -\sin\varphi\frac{\partial}{\partial y'} + \cos\varphi\frac{\partial}{\partial x'}. \end{aligned}\right\} \tag{10.5}$$

Reducing by a constant factor we shall rewrite Eq. (10.1) in the

form

$$D'_1 D'_2 \bar{D}'_1 \bar{D}'_2 F = 0, \qquad (10.6)$$

where

$$\left.\begin{aligned} D'_k &= \frac{\partial}{\partial y'} - \frac{\mu_k \cos\varphi - \sin\varphi}{\cos\varphi + \mu_k \sin\varphi} \cdot \frac{\partial}{\partial x'} = \frac{\partial}{\partial y'} - \mu'_k \frac{\partial}{\partial x'}, \\ \bar{D}'_k &= \frac{\partial}{\partial y'} - \frac{\bar{\mu}_k \cos\varphi - \sin\varphi}{\cos\varphi + \bar{\mu}_k \sin\varphi} \cdot \frac{\partial}{\partial x'} = \frac{\partial}{\partial y'} - \bar{\mu}'_k \frac{\partial}{\partial x'}. \end{aligned}\right\} \qquad (10.7)$$

Hence we also obtain formulas from which the complex parameters for the new axes will be determined*:

$$\mu'_1 = \frac{\mu_1 \cos\varphi - \sin\varphi}{\cos\varphi + \mu_1 \sin\varphi}, \quad \mu'_2 = \frac{\mu_2 \cos\varphi - \sin\varphi}{\cos\varphi + \mu_2 \sin\varphi}. \qquad (10.8)$$

Let us mention some important properties of the complex parameters which are found from an analysis of formulas (10.8):

1) If the parameters μ_1, μ_2 are complex numbers for some coordinate system x, y then also the parameters μ'_1, μ'_2 for any coordinate system x', y' rotated with respect to the first one by an angle φ will be complex, or, in particular, purely imaginary numbers. Conversely, if for some coordinate system the numbers μ_1, μ_2 proved to be real then also the corresponding numbers μ'_1, μ'_2 in an arbitrary coordinate system would be real numbers (which case is, however, excluded for an elastic plate if limiting cases are not taken into account).

2) If the parameters μ_1 and μ_2 for some coordinate system x, y were obtained unequal then also the corresponding μ'_1 and μ'_2 for any system x', y' rotated with respect to the first one by an angle φ will be unequal. Conversely, if for some coordinate system it has turned out that $\mu_1 = \mu_2$ then $\mu'_1 = \mu'_2$ for any other system.

3) If for some coordinate system one of the parameters proved to be equal to $i = \sqrt{-1}$ then for any other system rotated with respect to the first one the corresponding parameter will be equal to i, i.e., it will not change in the transition from one coordinate system to another.

With an isotropic plate both parameters are equal to i for any coordinate system as was already mentioned.

It is not difficult to obtain also formulas for the transformation of complex parameters of the second kind in the transition to new axes. Let λ_1, λ_2 be the complex parameters of the second kind for the x, y coordinate system to be determined from Formulas (7.20), λ_1', λ_2' the same quantities for the new system x', y' which is rotated with respect to the first one by the angle φ:

$$\lambda_1' = \frac{1 + i\mu_1'}{1 - i\mu_1'}, \quad \lambda_2' = \frac{1 + i\mu_2'}{1 - i\mu_2'}. \tag{10.9}$$

Substituting here the expressions from (10.8) we obtain the very simple formulas

$$\lambda_1' = \lambda_1 e^{-2\varphi i}, \quad \lambda_2' = \lambda_2 e^{-2\varphi i}. \tag{10.10}$$

Since the absolute value of the complex number $e^{-2\varphi i}$ is equal to unity it follows that

$$|\lambda_1'| = |\lambda_1|, \quad |\lambda_2'| = |\lambda_2|, \tag{10.11}$$

or, in other words, the absolute values of the complex parameters of the second kind retain constant values in any rotated coordinate system, i.e., are invariants.

The formulas (10.10) may be given a simple geometrical interpretation. Let for the coordinate system x, y $\lambda_k = \xi_k + i\eta_k$ and for the system x', y' $\lambda_k' = \xi_k' + i\eta_k'$. From (10.10) it follows that

$$\left.\begin{aligned} \xi_k' &= \xi_k \cos 2\varphi + \eta_k \sin 2\varphi, \\ \eta_k' &= -\xi_k \sin 2\varphi + \eta_k \cos 2\varphi \\ &(k = 1,\ 2). \end{aligned}\right\} \tag{10.12}$$

If a complex plane $\xi\eta$ is introduced the complex numbers λ_1 and λ_2 will be represented on it by vectors with lengths $|\lambda_1|$ and $|\lambda_2|$ which begin at the origin of coordinates and, generally, have arbitrary directions; the projections of these vectors on the ξ and η axes are, respectively, equal to ξ_1, η_1, and ξ_2, η_2. The formulas (10.12) show the

transition to a new coordinate system x', y' rotated with respect to x, y by an angle φ is equivalent to the transition to a new system ξ', η', rotated by an angle 2φ (Fig. 11) with respect to ξ, η. The real and imaginary parts of λ_1' and λ_2' are determined as the projections of the same vectors on the new axes ξ', η'.

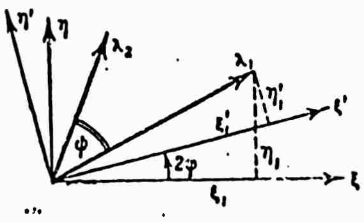

Fig. 11.

To each form of anisotropy of a body in the generalized plane stressed state or showing plane deformation corresponds a couple of completely determined vectors in the $\xi\eta$ plane. The lengths of these vectors are equal to $|\lambda_1|$ and $|\lambda_2|$, and the angle between them ψ has a determined value if both moduli are different from zero, and become indetermined if one of the moduli $|\lambda_k|$ or both are equal to zero. Thus, the anisotropy of a body in the case of a plane problem may entirely be characterized by the numerical values of three real quantities independent of the choice of the coordinate system, $|\lambda_1|$, $|\lambda_2|$ and ψ.

The formulas expressing these quantities in terms of the real and imaginary parts of the parameters of the first kind found for an arbitrarily chosen system of coordinates x, y have the form:

$$\left.\begin{aligned} |\lambda_1'| = |\lambda_1| = \sqrt{\frac{(1-\beta)^2+\alpha^2}{(1+\beta)^2+\alpha^2}}, \\ |\lambda_2'| = |\lambda_2| = \sqrt{\frac{(1-\delta)^2+\gamma^2}{(1+\delta)^2+\gamma^2}}; \end{aligned}\right\} \tag{10.13}$$

$$\left.\begin{aligned} \cos\psi &= \frac{(1-\alpha^2-\beta^2)(1-\gamma^2-\delta^2)+4\alpha\gamma}{\sqrt{[(1+\beta)^2+\alpha^2][(1-\beta)^2+\alpha^2][(1+\delta)^2+\gamma^2][(1-\delta)^2+\gamma^2]}}, \\ \sin\psi &= \frac{2[(1-\alpha^2-\beta^2)\gamma-(1-\gamma^2-\delta^2)\alpha]}{\sqrt{[(1+\beta)^2+\alpha^2][(1-\beta^2)+\alpha^2][(1+\delta)^2+\gamma^2][(1-\delta)^2+\gamma^2]}}. \end{aligned}\right\} \tag{10.14}$$

In the case of the isotropic body $|\lambda_1| = |\lambda_2| = 0$, and ψ has an undetermined value. In the limiting case where the anisotropy is expressed in the sharpest manner μ_1 and μ_2 are equal to zero or infinity, $|\lambda_1| = |\lambda_2| = 1$, $\psi = 0$ ($|\lambda_k|$ cannot be greater than one). If for some coordinate system it has proved that μ_1, μ_2 are purely imaginary num-

bers then the vectors expressing λ_1 and λ_2 will have the same or opposite directions, and, therefore, $\psi = 0$ or $\psi = \pi$.

§11. THE ELASTIC CONSTANTS FOR SOME ANISOTROPIC PLATES

Many research workers, among whom W. Voigt occupies an outstanding position, were concerned with problems of the experimental determination of the elastic constants of various crystalline substances (minerals). The numerical values of the elastic constants for many minerals are given, e.g., in the course on crystal physics by V. Voigt* and in the work by Auerbach,** references on this problem are also to be found in an article by Geckeler.*** Without presenting here those data which refer to crystals we shall give the numerical values of the elastic constants for three anisotropic materials (plates) of noncrystalline origin: for pine wood, delta wood, and plywood.

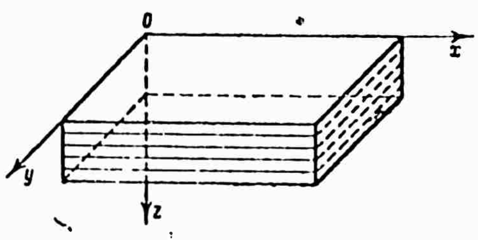

Fig. 12.

1. <u>Natural wood (pine wood)</u>. Let us consider a rectangular plate cut out of natural wood with regular annual layers, as shown in Fig. 12. If the inhomogeneity and the curvature of the layers is neglected three planes of structure symmetry can be distinguished in it, which, at the same time, are also the planes of elastic symmetry; one of them, yz, is normal to the wood fibers, the second (tangential) one, xy, is parallel to the planes of the annual layers, and the third (radial) one, xz, is orthogonal to the first two planes. All planes parallel to those mentioned are also planes of elastic symmetry, and the wood may in first approximation be regarded as a homogeneous orthotropic material. The equations of the generalized Hooke's law will be written in the forms (2.7) and (2.8); nine different elastic constants enter into them.****

Let a plate whose plane faces are parallel to the annual layers (the plate is not necessarily rectangular) be in a generalized plane stressed state. The equations of the generalized Hooke's law which connect the values of the stress and strain components whose mean values have been taken over the thickness are then written as follows:

$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{E_1}\sigma_x - \frac{\nu_2}{E_2}\sigma_y, \\ \varepsilon_y &= -\frac{\nu_1}{E_1}\sigma_x + \frac{1}{E_2}\sigma_y, \\ \gamma_{xy} &= \frac{1}{G}\tau_{xy}. \end{aligned}\right\} \tag{11.1}$$

The x axis is here directed along the fibers; E_1 is Young's modulus for the stretching (compression) along the fibers; E_2 is Young's modulus for the stretching (compression) along the directions lying in the plane of the annual layer and normal to the fibers; ν_1, ν_2 are the corresponding Poisson coefficients ($E_1\nu_2 = E_2\nu_1$); G is the shear modulus for the planes of the annual layers; the dashes by which the mean values were designated are discarded.

We shall give the numerical values of the elastic constants from Eqs. (11.1) for pine wood, taken from a work by A.L. Rabinovich*:

$$\left.\begin{aligned} &E_1 = 1\cdot 10^5\,\text{kg/cm}^2 \quad E_2 = 0{,}042\cdot 10^5\,\text{kg/cm}^2 \\ &\nu_2 = 0{,}01, \qquad\qquad\quad G = 0{,}075\cdot 10^5\,\text{kg/cm}^2 \end{aligned}\right\} \tag{11.2}$$

On the basis of these data we obtain the following values of the complex parameters:

$$\left.\begin{aligned} &\mu_1 = 3{,}26i, \qquad \mu_2 = 1{,}50i, \\ &\lambda_1 = -0{,}530, \quad \lambda_2 = -0{,}198, \\ &|\lambda_1| = 0{,}530, \quad |\lambda_2| = 0{,}198, \ \psi = 0. \end{aligned}\right\} \tag{11.3}$$

If the x and y axes change places with one another we shall obtain:

$$\left.\begin{aligned} &\mu_1 = 0{,}307i, \quad \mu_2 = 0{,}668i, \\ &\lambda_1 = 0{,}530, \quad \lambda_2 = 0{,}198. \end{aligned}\right\} \tag{11.4}$$

If the elastic constants for the principal directions are known, i.e., the longitudinal and the tangential one, we find the constants from formulas (9.10) and for an arbitrary direction in the xy plane. Thus,

Young's modulus E_φ for an angle φ with the x direction will be determined from the formula

$$E_\varphi = \frac{E_1}{\cos^4\varphi + \left(\frac{E_1}{G} - 2\nu_1\right)\sin^2\varphi\cos^2\varphi + \frac{E_1}{E_2}\sin^4\varphi}. \qquad (11.5)$$

Figure 13 shows the diagram of the variation of E_φ with the variation of φ for pine wood taken from the mentioned work by A.L. Rabinovich (page 41).

2. Delta wood. Slablike delta wood is produced of a number of wood layers (plywood) which have been impregnated by pressing in resin; one layer whose fibers are perpendicular to those of the rest of the layers is placed over ten layers with identical fiber direction.

In first approximation, a plate of delta wood may be considered to be a homogeneous orthotropic plate one plane of elastic symmetry of which is normal to the fibers of the predominant direction, and the second one is parallel to the mid plane. For a plate of delta wood in a generalized plane stressed state the equations of the generalized Hooke's law hold (11.1) (the mid plane is chosen to be the xy plane, the direction of the predominant fibers is chosen to be the direction of the x axis).

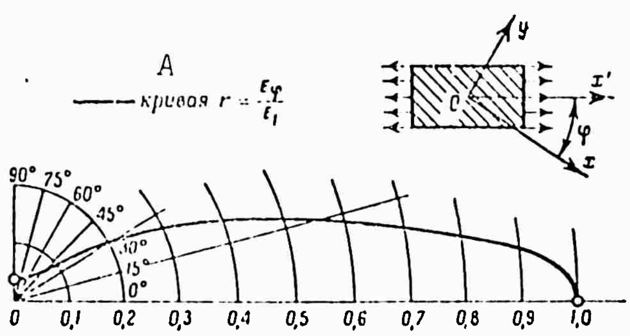

Fig. 13. A) Curve.

For the mean elastic constants we may choose:

$$\left.\begin{array}{ll} E_1 = 3{,}05 \cdot 10^5\ \mathrm{kg/cm^2} & E_2 = 0{,}467 \cdot 10^5\ \mathrm{kg/cm^2} \\ \nu_2 = 0{,}02, & G = 0{,}22 \cdot 10^5\ \mathrm{kg/cm^2} \end{array}\right\} \qquad (11.6)$$

The corresponding complex parameters are equal to:

$$\left.\begin{aligned} &\mu_1 = 3{,}62i, \quad \mu_2 = 0{,}700i, \\ &\lambda_1 = -0{,}567, \quad \lambda_2 = 0{,}172, \\ &|\lambda_1| = 0{,}567, \quad |\lambda_2| = 0{,}172, \quad \psi = \pi. \end{aligned}\right\} \qquad (11.7)$$

If the y axis is directed along the fibers rather than the x axis we obtain:

$$\left.\begin{aligned} &\mu_1 = 0{,}276i, \quad \mu_2 = 1{,}416i, \\ &\lambda_1 = 0{,}567, \quad \lambda_2 = -0{,}172. \end{aligned}\right\} \qquad (11.8)$$

Figure 14 shows the diagram of the variation of Young's modulus E_φ with the variation of the angle φ.*

3. Plywood. Also plywood may serve as an example of an anisotropic material. For the sake of definiteness, we shall focus our attention on birch plywood which is produced of an odd number of wood layers (plywood) glued to each other by a bakelite film and distributed symmetrically to the central layer; in this case the directions of the fibers of neighboring layers are mutually perpendicular (Fig. 15). The plywood plate is inhomogeneous, but if the plane stressed state is studied it may be regarded as homogeneous and, moreover, orthotropic, in the first approximation. One of the three planes of elastic symmetry coincides with the mid plane, the second one is perpendicular to the fibers of the outer layers (or, as they are called, to the casing fibers), and the third one is orthogonal to the first two planes.

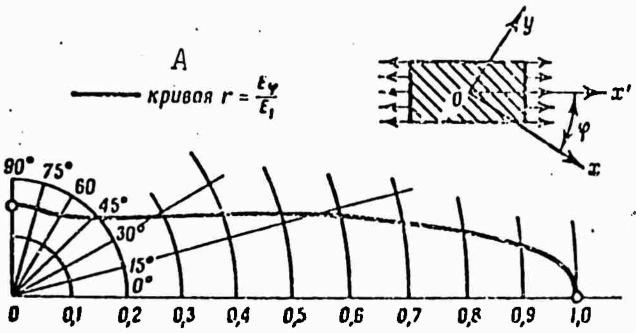

Fig. 14. A) Curve.

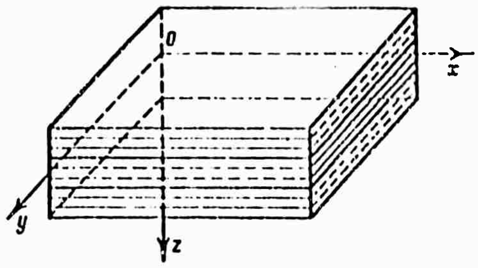

Fig. 15.

Considering the generalized plane stressed state of the above-mentioned plywood plate we choose the mid plate to be the xy plane, and the x axis parallel to the casing fibers. The equations of the generalized Hooke's law for such a plate regarded as an entirety will be written in the form (11.1) where E_1, E_2, ν_1, ν_2 must be understood to be the mean elastic constants for the plate as a whole; the latter depend on the elastic constants of the wood layers, their number and thickness.* We present the numerical values (the normalized ones) of the constants E_1, E_2, and G and Young's modulus E' for a stretching under an angle of 45° to the casing fibers for three-layer plywood with a thickness of 1; 1.5; 2; 2.5; 3; 4 and 5 taken from the "Handbook of the Airplane Constructor"**:

$$E_1 = 1{,}2 \cdot 10^5 \text{kg/cm}^2 \; E_2 = 0{,}6 \cdot 10^5 \text{kg/cm}^2$$
$$G = 0{,}07 \cdot 10^5 \text{kg/cm}^2 \; E' = 0{,}24 \cdot 10^5 \text{kg/cm}^2 \qquad (11.9)$$

In the "Handbook" the Poisson coefficients are not given, but they can be calculated from the first formula (9.10) by putting $\varphi = 45°$ in it and substituting the well-known values (11.9). Hence we obtain:

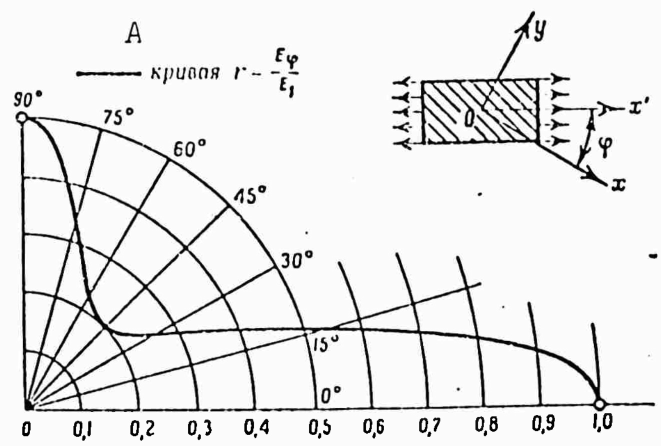

Fig. 16. A) Curve.

$$\nu_1 = 0{,}071, \quad \nu_2 = 0{,}036. \tag{11.10}$$

The complex parameters have the following values:

$$\left.\begin{array}{ll} \mu_1 = 4{,}11i, & \mu_2 = 0{,}343i, \\ \lambda_1 = -0{,}609, & \lambda_2 = 0{,}489, \\ |\lambda_1| = 0{,}609, & |\lambda_2| = 0{,}489, \quad \psi = \pi. \end{array}\right\} \tag{11.11}$$

If we direct the x axis across the casing fibers rather than along them we obtain:

$$\left.\begin{array}{ll} \mu_1 = 0{,}243i, & \mu_2 = 2{,}91i, \\ \lambda_1 = 0{,}609, & \lambda_2 = -0{,}489. \end{array}\right\} \tag{11.12}$$

Figure 16 shows a diagram of the variation of E_φ with the variation of the direction constructed on the basis of (11.5) and the numerical values (11.9)-(11.10).

§12. THE PLANE PROBLEM FOR A BODY WITH CYLINDRICAL ANISOTROPY

In §§ 5 and 6 the general equations of the plane problem for a homogeneous body were derived, in which parallel directions passing through different points are equivalent in the sense of the elastic properties. In a completely analogous way we may also obtain the general equations of the plane problem for a body with cylindrical anisotropy.*

Let us consider the elastic equilibrium of a plate of constant thickness with cylindrical anisotropy under the action of forces distributed along the boundary and of the volume forces. With respect to the elastic constants we shall make the following assumptions:

1) the axis of anisotropy g is normal to the mid plane of the plate (the point of intersection of the axis of anisotropy with the mid plane which will be called pole of anisotropy in the following may lie either inside the region of the plate or outside or at the boundary);

2) at each point there is a plane of elastic symmetry normal to the axis of anisotropy (and, consequently, parallel to the mid plane);

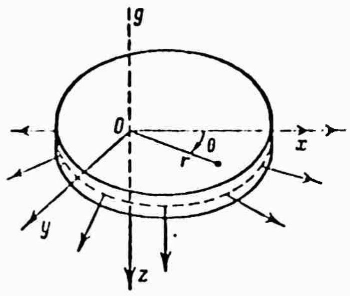

Fig. 17.

The surface and volume forces will be assumed parallel to the mid surface, distributed symmetrically with respect to this plane and slightly varying according to the thickness. The strains will be regarded as small.

The axis of anisotropy will be chosen as the z axis of the cylindrical coordinate system r, θ, z, directing the polar axis x arbitrarily in the mid plane (Fig. 17).

Let us designate by h the thickness of the plate, by R, Θ the projections of the volume forces (per unit volume) on the coordinate directions r, θ $(Z = 0)$ and consider the values of the stress components and displacement projections averaged over the thickness:

$$\left.\begin{aligned}
\bar{\sigma}_r &= \frac{1}{h}\int_{-h/2}^{h/2}\sigma_r\,dz, & \bar{\sigma}_\theta &= \frac{1}{h}\int_{-h/2}^{h/2}\sigma_\theta\,dz,\\
\bar{\sigma}_z &= \frac{1}{h}\int_{-h/2}^{h/2}\sigma_z\,dz, & \bar{\tau}_{r\theta} &= \frac{1}{h}\int_{-h/2}^{h/2}\tau_{r\theta}\,dz,\\
\bar{u}_r &= \frac{1}{h}\int_{-h/2}^{h/2}u_r\,dz, & \bar{u}_\theta &= \frac{1}{h}\int_{-h/2}^{h/2}u_\theta\,dz.
\end{aligned}\right\} \tag{12.1}$$

Moreover, we shall introduce the designations:

$$\bar{R} = \frac{1}{h}\int_{-h/2}^{h/2}R\,dz, \quad \bar{\Theta} = \frac{1}{h}\int_{-h/2}^{h/2}\Theta\,dz; \tag{12.2}$$

$$\bar{\varepsilon}_r = \frac{\partial\bar{u}_r}{\partial r}, \quad \bar{\varepsilon}_\theta = \frac{1}{r}\frac{\partial\bar{u}_\theta}{\partial\theta} + \frac{\bar{u}_r}{r}, \quad \bar{\gamma}_{r\theta} = \frac{1}{r}\frac{\partial\bar{u}_r}{\partial\theta} + \frac{\partial\bar{u}_\theta}{\partial r} - \frac{\bar{u}_\theta}{r} \tag{12.3}$$

and consider the case where the volume forces have a potential $\bar{U}(r, \theta)$, i.e., are determined by the formulas

$$\bar{R} = -\frac{\partial\bar{U}}{\partial r}, \quad \bar{\Theta} = -\frac{1}{r}\frac{\partial\bar{U}}{\partial\theta}. \tag{12.4}$$

Carrying out the operations of taking the mean values over the equilibrium equations in cylindrical coordinates (1.4) and the equa-

tions of the generalized Hooke's law corresponding to the given case of anisotropy and neglecting $\bar{\sigma}_z$ we obtain the system

$$\left.\begin{aligned} &\frac{\partial \bar{\sigma}_r}{\partial r} + \frac{1}{r}\cdot\frac{\partial \bar{\tau}_{r\theta}}{\partial \theta} + \frac{\bar{\sigma}_r - \bar{\sigma}_\theta}{r} + \bar{R} = 0, \\ &\frac{\partial \bar{\tau}_{r\theta}}{\partial r} + \frac{1}{r}\cdot\frac{\partial \bar{\sigma}_\theta}{\partial \theta} + \frac{2\bar{\tau}_{r\theta}}{r} + \bar{\Theta} = 0; \end{aligned}\right\} \tag{12.5}$$

$$\left.\begin{aligned} \bar{\varepsilon}_r &= a_{11}\bar{\sigma}_r + a_{12}\bar{\sigma}_\theta + a_{16}\bar{\tau}_{r\theta}, \\ \bar{\varepsilon}_\theta &= a_{12}\bar{\sigma}_r + a_{22}\bar{\sigma}_\theta + a_{26}\bar{\tau}_{r\theta}, \\ \bar{\gamma}_{r\theta} &= a_{16}\bar{\sigma}_r + a_{26}\bar{\sigma}_\theta + a_{66}\bar{\tau}_{r\theta}. \end{aligned}\right\} \tag{12.6}$$

Eliminating the displacements from the expressions (12.3) we obtain the compatibility equation

$$\frac{\partial^2 \bar{\varepsilon}_r}{\partial \theta^2} + r\,\frac{\partial^2 (r\bar{\varepsilon}_\theta)}{\partial r^2} - \frac{\partial^2 (r\bar{\gamma}_{r\theta})}{\partial r\,\partial \theta} - r\,\frac{\partial \bar{\varepsilon}_r}{\partial r} = 0. \tag{12.7}$$

We shall satisfy the equilibrium equations (12.5) by introducing the stress function $F(r, \theta)$ such that in the case of the isotropic body:

$$\left.\begin{aligned} \bar{\sigma}_r &= \frac{1}{r}\cdot\frac{\partial F}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 F}{\partial \theta^2} + \bar{U}, \\ \bar{\sigma}_\theta &= \frac{\partial^2 F}{\partial r^2} + \bar{U}, \\ \bar{\tau}_{r\theta} &= -\frac{\partial^2}{\partial r\,\partial \theta}\left(\frac{F}{r}\right). \end{aligned}\right\} \tag{12.8}$$

On the basis of the compatibility equation (12.7) and the relationships (12.6) and (12.8) we obtain a differential equation which is satisfied by the stress function:

$$\begin{aligned} &a_{22}\frac{\partial^4 F}{\partial r^4} - 2a_{26}\frac{1}{r}\cdot\frac{\partial^4 F}{\partial r^3\,\partial \theta} + (2a_{12} + a_{66})\frac{1}{r^2}\cdot\frac{\partial^4 F}{\partial r^2\,\partial \theta^2} - \\ &- 2a_{16}\frac{1}{r^3}\cdot\frac{\partial^4 F}{\partial r\,\partial \theta^3} + a_{11}\frac{1}{r^4}\cdot\frac{\partial^4 F}{\partial \theta^4} + 2a_{22}\frac{1}{r}\cdot\frac{\partial^3 F}{\partial r^3} - \\ &- (2a_{12} + a_{66})\frac{1}{r^3}\cdot\frac{\partial^3 F}{\partial r\,\partial \theta^2} + 2a_{16}\frac{1}{r^4}\cdot\frac{\partial^3 F}{\partial \theta^3} - a_{11}\frac{1}{r^2}\cdot\frac{\partial^2 F}{\partial r^2} - \\ &- 2(a_{16} + a_{26})\frac{1}{r^3}\cdot\frac{\partial^2 F}{\partial r\,\partial \theta} + (2a_{11} + 2a_{12} + a_{66})\frac{1}{r^4}\cdot\frac{\partial^2 F}{\partial \theta^2} + \\ &+ a_{11}\frac{1}{r^3}\cdot\frac{\partial F}{\partial r} + 2(a_{16} + a_{26})\frac{1}{r^4}\cdot\frac{\partial F}{\partial \theta} = \\ &= -(a_{12} + a_{22})\frac{\partial^2 \bar{U}}{\partial r^2} + (a_{16} + a_{26})\frac{1}{r}\cdot\frac{\partial^2 \bar{U}}{\partial r\,\partial \theta} - (a_{11} + a_{12})\frac{1}{r^2}\cdot\frac{\partial^2 \bar{U}}{\partial \theta^2} + \\ &+ (a_{11} - 2a_{22} - a_{12})\frac{1}{r}\cdot\frac{\partial \bar{U}}{\partial r} + (a_{16} + a_{26})\frac{1}{r^3}\cdot\frac{\partial \bar{U}}{\partial \theta}. \end{aligned} \tag{12.9}$$

This equation corresponds to Eq. (5.9) for a homogeneous plate. It is considerably more complex than Eq. (5.9) and contains arbitrary

functions F of different orders, from the first to the fourth one, and not only of the first order as was the case with the homogeneous equation. In view of the complexity of Eq. (12.9) it is not possible in this case to find a general expression for F in terms of arbitrary functions analogous to the expressions (7.16), (7.17) or (7.21).

If a plate with cylindrical anisotropy is, at the same time, also orthotropic, i.e., has three planes of elastic symmetry at every point, one of which is parallel to the mid plane, the other one passes through the axis of anisotropy, then Eqs. (12.6) will be written in the form:

$$\left.\begin{aligned} \bar{\varepsilon}_r &= \frac{1}{E_r}\bar{\sigma}_r - \frac{\nu_\theta}{E_\theta}\bar{\sigma}_\theta, \\ \bar{\varepsilon}_\theta &= -\frac{\nu_r}{E_r}\bar{\sigma}_r + \frac{1}{E_\theta}\bar{\sigma}_\theta, \\ \bar{\gamma}_{r\theta} &= \frac{1}{G_{r\theta}}\bar{\tau}_{r\theta}. \end{aligned}\right\} \tag{12.10}$$

E_r, E_θ are here Young's moduli for the stretching (compression) along the principal directions r and θ; ν_r, ν_θ are the Poisson coefficients and $G_{r\theta}$ is the shear modulus for the principal directions r, θ. For this case Eq. (12.9) simplifies and assumes the following form:

$$\begin{aligned} &\frac{1}{E_\theta}\cdot\frac{\partial^4 F}{\partial r^4} + \left(\frac{1}{G_{r\theta}} - \frac{2\nu_r}{E_r}\right)\frac{1}{r^2}\frac{\partial^4 F}{\partial r^2\,\partial\theta^2} + \frac{1}{E_r}\cdot\frac{1}{r^4}\cdot\frac{\partial^4 F}{\partial\theta^4} + \frac{2}{E_\theta}\cdot\frac{1}{r}\frac{\partial^3 F}{\partial r^3} - \\ &\quad - \left(\frac{1}{G_{r\theta}} - \frac{2\nu_r}{E_r}\right)\frac{1}{r^3}\cdot\frac{\partial^3 F}{\partial r\,\partial\theta^2} - \frac{1}{E_r}\cdot\frac{1}{r^2}\cdot\frac{\partial^2 F}{\partial r^2} + \\ &\quad + \left(2\,\frac{1+\nu_r}{E_r} + \frac{1}{G_{r\theta}}\right)\frac{1}{r^4}\cdot\frac{\partial^2 F}{\partial\theta^2} + \frac{1}{E_r}\cdot\frac{1}{r^3}\cdot\frac{\partial F}{\partial r} = \\ &= -\left[\frac{1-\nu_\theta}{E_\theta}\cdot\frac{\partial^2\bar{U}}{\partial r^2} + \frac{1-\nu_r}{E_r}\cdot\frac{1}{r^2}\cdot\frac{\partial^2\bar{U}}{\partial\theta^2} + \left(\frac{2}{E_\theta} - \frac{1+\nu_r}{E_r}\right)\frac{1}{r}\cdot\frac{\partial\bar{U}}{\partial r}\right]. \end{aligned} \tag{12.11}$$

The boundary conditions for the given forces at the plate boundary may be reduced to prescribing the first derivatives of the stress functions $\partial F/\partial r$ and $\partial F/\partial\theta$ at the contour of the region occupied by the plate.

The problem of plane deformation is completely analogous to the problem of the plane stressed state of the plate. If the body shown in Fig. 8 (§6) has the property of cylindrical anisotropy with the axis of anisotropy z directed parallel to the generatrix then Eqs. (12.5)-

(12.9) in which only the a_{ij} have to be replaced by the reduced strain coefficients

$$\beta_{ij} = a_{ij} - \frac{a_{i3}a_{j3}}{a_{33}} \qquad (12.12)$$
$$(i, j = 1, 2, 6).$$

hold for it.

Besides σ_r, σ_θ and $\tau_{r\theta}$ also the stress σ_z acting in the cross sections and equal to

$$\sigma_z = -\frac{1}{a_{33}}(a_{13}\sigma_r + a_{23}\sigma_\theta + a_{36}\tau_{r\theta}). \qquad (12.13)$$

is obtained in this case.

P.N. Zhitkov considered the generalized plane stressed state of an orthotropic body having cylindrical anisotropy in which the moduli of elasticity are functions of the coordinates r and θ. In this case a more complex equation of fourth order with variable coefficients* is obtained instead of Eq. (12.11).

Manuscript Page No. [Footnotes]

29* See our works: 1) K voprosu o vlyanii sosredotochnykh sil na paspredeleniye napryazheniy v anizotropnoy urpugoy srede [On the Problem of the Influence of Concentrated Forces on the Distribution of Stresses in an Anisotropic Elastic Medium], Prikladnaya matematika i mekhanika [Applied Mathematics and Mechanics], Vol. 3, No. 1, 1936; 2) Nekotoryye sluchai ploskoy zadachi teorii uprugosti anisotropnogo tela [Several Cases of the Plane Problem of the Theory of Elasticity of an Anisotropic Body], Sb. Eksperimental'nyye metody opredeleniya napryzheniy i deformatsiy v uprugoy i plasticheskoy zonakh [Experimental Methods of Determining the Stresses and Strains in the Elastic and Plastic Zones], ONTI [United Scientific and Technical Publishing Houses], 1935.

29** Muskhelishvili, N.I., Nekotoryye osnovnyye zadachi matematicheskoy teorii uprugosti [Several Basic Problems of the Mathematical Theory of Elasticity, Izd. AN SSSR [Publishing House of the Academy of Sciences of the USSR], Moscow, 1954, page 107.

30 N.I. Muskhelishvili chooses that direction to be the positive direction of passing along the contour for which the region remains on the left side, i.e., counterclockwise for the outer contour and clockwise for the contours of the openings (see his book mentioned, pages 113, 143).

32 See our works mentioned in §5.

35 See our work: "Ploskaya statisticheskaya zadacha teorii uprugosti anisotropnogo tela" [The Plane Statistical Problem of the Theory of Elasticity of an Anisotropic Body], Prikladnaya matematika i mekhanika [Applied Mathematics and Mechanics], Vol. 1, Edition 1, 1937.

37 See N.I. Muskhelishvili, "Nekotoryye osnovnyye zadachi matematicheskoy teorii uprugosti" [Some Basic Problems of the Mathematical Theory of Elasticity], Izd. AN SSSR [Publishing House of the Academy of Sciences of the USSR], Moscow, 1954, page 111.

39 See our work: "Ploskaya statisticheskaya zadacha teorii uprugosti anisotropnogo tela" [The Plane Statistical Problem of the Theory of Elasticity of an Anisotropic Body], Prikladnaya matematika i mekhanika [Applied Mathematics and Mechanics], Vol. 1, Edition 1, 1937, page 81.

41* See our mentioned work, pages 83-87.

41** The determinant of this system is equal to

$$d = \frac{4\beta\delta}{(\alpha^2 + \beta^2)(\gamma^2 + \delta^2)}[(\alpha - \gamma)^2 + (\beta - \delta)^2]\cdot[(\alpha - \gamma)^2 + (\beta + \delta)^2].$$

In the case of unequal complex parameters, obviously, always $d > 0$.

42 See the book by N.I. Muskhelishvili mentioned several times, pages 113, 114, 143.

43* Mikhlin, S.G., Ploskaya deformatsiya v anizotropnoy srede [The Plane Deformation in an Anisotropic Medium], Trudy Seysmologicheskogo instituta AN SSSR [Transactions of the Seismological Institute of the Academy of Sciences of the USSR], No. 76, 1936.

43** Savin, G.N., Osnovnaya ploskaya staticheskaya zadacha teorii urpugosti dlya anizotropnoy sredy [The Basic Plane Static Problem of the Theory of Elasticity for an Anisotropic Medium], Trudy instituta stroitel'noy mekhaniki Ukrainskoy Akademii nauk [Transactions of the Institute of Construction Mechanics of the Ukrainian Academy of Sciences], No. 32, 1938.

43*** Sherman, D.I., Ploskaya zadacha teorii uprugosti dlya anizotropy sredy [The Plane Problem of the Theory of Elasticity of an Anisotropic Medium], Trudy Seysmologicheskogo instituta AN SSSR, No. 86, 1938.

43**** Sherman, D.I.: 1) Novoye resheniye ploskoy zadachi teorii uprugosti dlya anizotropnoy sredy [A New Solution of the

Plane Problem of the Theory of Elasticity for an Anisotropic Medium], Doklady AN SSSR [Proceedings of the Academy of Sciences of the USSR], Vol. 32, No. 5, 1941; 2) K resheniyu ploskoy zadachi teorii uprugosti dlya anizotropnoy sredy [On the Solution of the Plane Problem of the Theory of Elasticity of an Anisotropic Medium], Prikladnaya matematika i mekhanika, Vol. 6, Edition 6, 1942.

43***** Ayzenberg, T.B., Ploskaya zadacha teorii uprugosti dlya anizotropnoy sredy pri zadannykh granichnykh smeshcheniyakh [The Plane Problem of the Theory of Elasticity for an Anisotropic Medium for Given Boundary Displacements] Sbornik statey Vsesoyuznogo zaochnogo Politekhnicheskogo instituta [Collection of Articles of the All-Union Correspondence-Course Polytechnical Institute], Edition 6, Moscow, 1954.

44* Il'ya Vekua, Prilozheniye metoda akademika N.I. Muskhelishvili k resheniyu granichnykh zadach ploskoy teorii uprugosti anizotropnoy sredy [Application of the Method of Academician N.I. Muskhelishvili to the Solution of the Boundary Problems of the Plane Theory of Elasticity of an Anisotropic Medium], Soobshcheniya Gruzinskogo filiala AN SSSR [Communications of the Georgian Branch of the Academy of Sciences of the USSR], Vol. 1, No. 10, 1940.

44** Kupradze, V.D. and Basheleyshvili, M.O., Novyye integral'nyye uravneniya anizotropnoy teorii uprugosti i ikh primeneniye dlya resheniya granichnykh zadach [New Integral Equations of the Anisotropic Theory of Elasticity and Their Application to the Solution of Boundary Problems], Soobshcheniya AN Gruzinskoy SSSR [Communication of the Academy of Sciences of the Georgian Socialist Soviet Republic], Vol. 15, No. 6 and 7, 1954.

44*** Coker, E. and Filon, L., Opticheskiy metod issledovaniya napryazheniy [Optical Method of Investigating Stresses], ONTI [United Scientific and Technical Publishing House], 1936.

44**** Krasnov, V.M., Ob opredelenii napryazheniy v kubicheskikh kristallakh opticheskim metodom [On the Determination of Stresses in Cubic Crystals by the Optical Method], Uch. zap. Leningradskogo gos. universiteta, seriya matem. nauk [Scientific Reports of the Leningrad State University, Series of Mathematical Sciences], No. 13, 1944, No. 87; 2) Krasnov, V.M. and Stepanov, A.V., Issledovaniye zarodyhsey razrusheniya opticheskim metodom [The Investigation of Nuclei of Destruction by the Optical Method], Zhurnal experimental'noy i teoreticheskoy fiziki [Journal of Experimental and Theoretical Physics], Vol. 23, No. 2 (8), 1952; 3) Krasnov, V.M. and Stepanov, A.V., Izucheniya opticheskim metodom napryazhennogo sostoyaniya anizotropnoy plastinki, nakhodyashcheysya pod deystviyem sosredotochnoy sily [The Investigation of the Stressed State of an Anisotropic Plate under the Action of a Concentrated Force by an Optical Method], loc. cit., Vol. 25, No. 1 (7), 1953.

45 This problem was worked out in the textbook by A. Lyav

(Lyav, A., Matematicheskaya teoriya uprugosti [Mathematical Theory of Elasticity], ONTI, Moscow-Leningrad, 1935, pages 163, 164), and in a particularly detailed manner in the works by P. Bekhterev (see the remark in §2); see also our book "Teoriya uprugosti anizotropnogo tela" [Theory of Elasticity of an Anisotropic Body], Gostekhizdat [State Publishing House of Theoretical and Technical Literature], Moscow-Leningrad, pages 33-45.

47 The constants η_1' and η_2' are called "coefficients of mutual influence of the first kind" by A.L. Rabinovich; they characterize the elongations due to the tangential stresses (see Rabinovich, A.L., Ob uprugikh postoyannykh i prochnosti anizotropnykh materialov [On the Elastic Constants and the Strength of Anisotropic Materials], Trudy TsaGI [Transactions of the Central Aero-Hydrodynamical Institute], No. 582, 1946).

50 These formulas were derived in our work "O kompleksnykh parametrakh vkhodyashchikh b obshchiye formuly nekotorykh zadach teorii uprugosti anizotropnogo tela [On the Complex Parameters Entering the General Formulas of Several Problems of the Theory of Elasticity of an Anisotropic Body], Uch. zap. Leningr. gos. un-ta, seriya fiz.-matem. nauk [Series of Physical and Mathematical Sciences], No. 13, 1944.

53* Voigt, W., Lehrbuch der Kristallphysik [Textbook of Crystal Physics], Leipzig-Berlin (Teubner), 1928.

53** Auerbach, F., Elastizitaet der Kristalle [Crystal Elasticity], Handbuch der Physikalischen und technischen Mechanik [Handbook and Physical and Technical Mechanics], Vol. 3, Leipzig, 1927.

53*** Geckeler, J.W., Elastizitaetstheorie anisotroper Koerper [Theory of Elasticity of Anisotropic Bodies], (Kristallelastizitaet) [Crystal Elasticity], Vol. 6, Berlin, 1928.

53**** Mitinskiy, A.N., Uprugiye postoyannyye drvesiny kak ortotropnogo materiala [Elastic Constants of Woods as an Orthotropic Material], Trudy Lesotekhnicheskoy akademii im. S.M. Kirova [Transactions of the S.M. Kirov Wood Engineering Academy], No. 63, 1948. In this work the literature on the problem of the mechanical properties of wood is cited.

54 Rabinovich, A.L., Ob uprugikh postoyannykh i prochnosti anizotropnykh materialov [On the Elastic Constants and the Strength of Anisotropic Materials], Trudy TsAGI, No. 582, page 40.

56 The numerical values of the moduli of delta wood and the diagram of Fig. 14 are taken from the above-mentioned work by A.L. Rabinovich (page 48).

57* See the work of A.L. Rabinovich, "O raschete ortotropnykh sloistykh paneley na rastyazheniye, sdvig i izgib" [On the

Calculation of Orthotropic Slab Panels With Respect to Stretching, Shear, and Bending], Ministerstvo aviatsionnoy promyshlennosti SSSR [Ministry of the Aviation Industry of the USSR], Trudy [Transactions], No. 675, 1948.

57** "Spravochnik aviakonstruktora" [Handbook of the Airplane Constructor"], Vol. 3, Prochnost' samoleta [Airplane Strength], Izd. TsAGI, 1939 (Table 63 on page 325).

58 The equations cited in this section were for the first time obtained in our work "Ploskaya zadacha teorii uprugosti dlya tela i tsilindricheskoy anizotropiyey" [The Plane Problem of the Theory of Elasticity for a Body with Cylindrical Anisotropy], Uch. zap. Saratovskogo gos. un-ta [Scientific Reports of the Sratov State University], Vol. 1 (14), No. 2, 1938.

62 Zhitkov, P.N., Ploskaya zadacha teorii uprugosti heodnorodnogo ortotropnogo tela v polyarnykh koordinatakh [The Plane Problem of the Theory of Elasticity of an Inhomogeneous Orthotropic Body in Polar Coordinates], Trudy Voronezhskogo gos. un-ta [Transactions of the Voronesh State University], Vol. 27, fiz.-matem. sbornik [Physical and Mathematical Collection], 1954.

Chapter 3

THE BENDING OF PLANE ANISOTROPIC BEAMS AND CURVED GIRDERS

§13. SIMPLEST CASES

In this chapter we consider several cases of stress distribution under the action of bending loads in a rectangular plane plate, a wedge-shaped console of rectangular cross section, and in a curved girder having the form of a part of a plane circular ring. In all cases we assume that at each point of the body there is a plane of elastic symmetry, parallel to its mid surface (which is taken to be the xy or $r\theta$ plane).

Let us start from the simplest cases of the equilibrium of a homogeneous anisotropic rectangular plate of constant thickness h which is in a generalized plane stressed state under the action of forces distributed along its boundary. In all cases where a homogeneous beam is considered we assume that the equations of the generalized Hooke's law connecting the values of the stress and strain components whose mean has been taken over the thickness have the form:

$$\left.\begin{aligned} \varepsilon_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{16}\tau_{xy}, \\ \varepsilon_y &= a_{12}\sigma_x + a_{22}\sigma_y + a_{26}\tau_{xy}, \\ \gamma_{xy} &= a_{16}\sigma_x + a_{26}\sigma_y + a_{66}\tau_{xy}. \end{aligned}\right\} \tag{13.1}$$

If the plate is orthotropic and the principal directions are chosen to be the directions of the x and y axes, then the coefficients a_{16}, a_{26} are equal to zero, and the rest is more conveniently expressed in terms of Young moduli, Poisson coefficients, and the shear modulus (the principal ones):

$$\left.\begin{array}{ll} a_{11}=\frac{1}{E_1}, & a_{22}=\frac{1}{E_2}, \\ a_{12}=-\frac{\nu_1}{E_1}=-\frac{\nu_2}{E_2}, & a_{66}=\frac{1}{G}. \end{array}\right\} \tag{13.2}$$

The solutions for the simplest cases are elementary and we present them without derivation.

1. Stretching. A rectangular plate is stretched by normal forces p which are uniformly distributed along its two sides (Fig. 18).

$$\sigma_x = p, \quad \sigma_y = \tau_{xy} = 0; \quad F = \frac{1}{2} py^2 \tag{13.3}$$

(p is the intensity of the forces or the force per unit area).

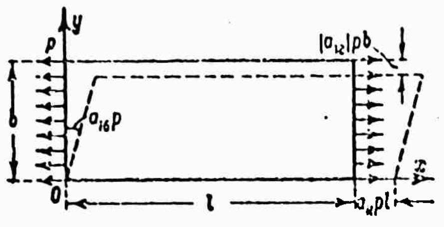

Fig. 18

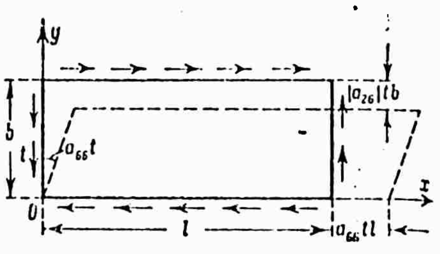

Fig. 19

The same stress distribution is obtained as in a stretched isotropic plane, and the deformations are determined from Eqs. (13.1):

$$\varepsilon_x = a_{11}p, \quad \varepsilon_y = a_{12}p, \quad \gamma_{xy} = a_{16}p. \tag{13.4}$$

A nonorthotropic plate is elongated under the action of stretching forces in the direction of the forces, and is contracted in the perpendicular direction (if only $a_{12} < 0$) and, besides, is distorted in the xy plane: the rectangular plate becomes oblique (see the dotted lines in Fig. 18). The distortion is determined by the constant a_{16}; an orthotropic plate remains rectangular.

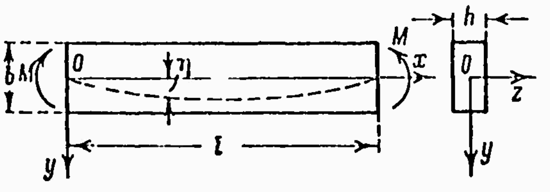

Fig. 20

2. Shear. Along the boundaries of a rectangular plate tangential forces of an intensity t are uniformly distributed (Fig. 19). We have:

$$\sigma_x = \sigma_y = 0, \quad \tau_{xy} = t; \qquad F = -txy; \tag{13.5}$$

$$\varepsilon_x = a_{16}t, \qquad \varepsilon_y = a_{26}t, \quad \gamma_{xy} = a_{66}t. \tag{13.6}$$

A nonorthotropic plate experiences elongations and lateral contractions according to the signs of a_{16} and a_{26}, besides the shear in the xy plane which is determined by the constant a_{66}. An orthotropic plate experiences pure shear without elongations.

3. Pure shear. Forces giving rise to moments M (in kg × cm) are distributed along the two sides of a beam-plate (Fig. 20).

We have:

$$\left.\begin{aligned} \sigma_x &= \frac{M}{J}y, \quad \sigma_y = \tau_{xy} = 0; \\ F &= \frac{M}{6J}y^3 \quad \left(J = \frac{hb^3}{12}\right). \end{aligned}\right\} \tag{13.7}$$

The same stress distribution is obtained as in an isotropic beam (if no account is taken of the local stresses in the neighborhood of the ends which, according to the Saint Venant principle, practically have no influence on the stresses in zones far from the loaded surfac es).

The displacements of the beam particles [found by integrating Eqs. (13.1)] are equal to:

$$\left.\begin{aligned} u &= \frac{M}{J}\left(a_{11}xy + \frac{1}{2}a_{16}y^2\right) - \omega y + u_0, \\ v &= \frac{M}{2J}(a_{12}y^2 - a_{11}x^2) + \omega x + v_0. \end{aligned}\right\} \tag{13.8}$$

Here ω, u_0, v_0 are constants expressing the "rigid" displacement of the beam in its mid-surface, i.e., which is not accompanied by deformation. The first of expressions (13.8) shows that the cross sections of a nonorthotropic beam do not remain plane; the distortion depends on the constant a_{16}. The cross sections of an orthotropic beam are not distorted in deformation.

The equation of the curved beam axis for fixed ends $x = 0$ and $x = l$ has the form:

$$\eta = \frac{Ma_{11}}{2J}(lx - x^2), \tag{13.9}$$

where η is the ordinate of the curved axis.

The curvature of the curved axis is equal to

$$\frac{1}{\rho} \approx -\eta'' = -\frac{Ma_{11}}{J} = -\frac{M}{E_1'J}. \tag{13.10}$$

The functional relationship between the curvature of the axis and the moment of flexure is the same as in the case of an isotropic beam, but instead of the modulus E (which is the same for all directions in an isotropic beam) the modulus E_1' for stretching (compression) along the axis occurs.

§14. BENDING OF A CONSOLE BY A TRANSVERSE FORCE

A beam with a cross section having the form of a narrow rectangle is fixed at one end and is bent by a transverse force P applied to the other end (Fig. 21). The solution is obtained with the help of a stress function in the form of a polynomial of the fourth degree*

$$F = \frac{P}{J}\left[-\frac{xy^3}{6} + \frac{b^2xy}{8} + \frac{a_{16}}{24a_{11}}(b^2y^2 - 2y^4)\right]. \tag{14.1}$$

The stress components are determined by the formulas:

$$\left.\begin{aligned} \sigma_x &= -\frac{P}{J}xy + \frac{P}{J}\cdot\frac{a_{16}}{a_{11}}\left(\frac{b^2}{12} - y^2\right), \\ \sigma_y &= 0, \\ \tau_{xy} &= -\frac{P}{2J}\left(\frac{b^2}{4} - y^2\right) \\ &\left(J = \frac{hb^3}{12}\right). \end{aligned}\right\} \tag{14.2}$$

These stresses satisfy exactly the conditions on the long sides $y = \pm \frac{b}{2}$ and in the cross sections they reduce to a force and a moment balancing the external force P.

In an orthotropic beam in which the axial direction x is the prin-

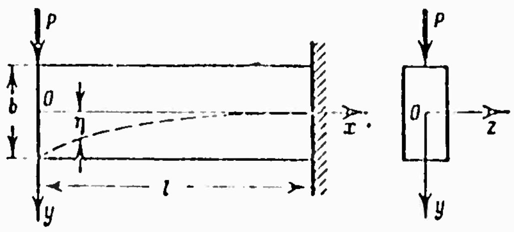

Fig. 21

cipal one $a_{16} = 0$, and the same stress distribution is obtained as in an isotropic beam*:

$$\sigma_x = -\frac{P}{J}xy, \quad \sigma_y = 0, \quad \tau_{xy} = -\frac{P}{2J}\left(\frac{b^2}{4} - y^2\right). \qquad (14.3)$$

For a beam with which the coefficient a_{16} is not equal to zero (the case of a nonorthotropic beam or an orthotropic beam with which the axial direction x is not the principal one) the stress σ_x is distributed according to a parabolic law rather than a linear one, along the cross section. The diagram of the distribution of σ_x over the cross section is shown (at an arbitrary scale and for $a_{16} > 0$) in Fig. 22; the dotted line shows the stress distribution in an isotropic beam. The greatest normal stress is obtained at the points $y = b/2$ or $y = -b/2$ of the fixed cross section; for $a_{16} > 0$ it is equal to

$$\sigma_{max} = \frac{6Pl}{hb^2}\left(1 + \frac{a_{16}}{a_{11}} \cdot \frac{b}{3l}\right) \qquad (14.4)$$

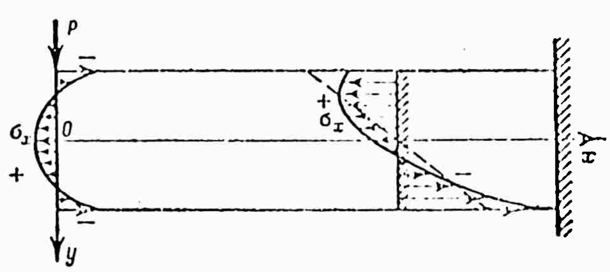

Fig. 22

(compressive stress) and for $a_{16} < 0$

$$\sigma_{max} = \frac{6Pl}{hb^2}\left(1 - \frac{a_{16}}{a_{11}} \frac{b}{3l}\right) \qquad (14.5)$$

(stretching stress).

We note that the formula for the curvature of the curved beam axis (both an orthotropic or nonorthotropic one) has the same form as in the case of pure bending, i.e.,

$$\frac{1}{\rho}=\frac{Ma_{11}}{J}=\frac{M}{E_1'J}, \tag{14.6}$$

but in this case the moment of flexure varies along the beam length:

$$M=-Px. \tag{14.7}$$

The parabolic distribution law of normal stresses in cross sections is not reflected in the equation of the curved axis which has the same form as in the case of an isotropic beam:

$$\eta=\frac{P}{6E_1'J}(x^3-3l^2x+2l^3). \tag{14.8}$$

E. Reissner particularly considered the limiting case of an orthotropic beam where Young's modulus E_2 for the y direction perpendicular to the x axis is negligibly small compared to the modulus E_1 for the axial direction. The solutions obtained were used by him to study the stresses and strains in a detail having the shape of a case.*

§15. BEAM BENDING BY A UNIFORMLY DISTRIBUTED LOAD

The stress distribution in a beam which is uniformly loaded along its whole length is obtained with the help of a stress function having the form of a fifth degree polynomial. The arbitrary constants entering this polynomial may always be chosen such that the stress on the long sides exactly satisfy the boundary conditions, and on the short sides reduce to forces and moments balancing the external load. We shall present the solutions for two cases of end fixing.**

1. Console. A beam with a cross section having the form of a narrow rectangle is fixed at one end and bent by a normal load q (per unit length), which is uniformly distributed along one of the long sides

(Fig. 23).

The following formulas are obtained for the stress components:

$$
\left.
\begin{aligned}
\sigma_x &= -\frac{qx^2y}{2J} + \frac{q}{h}\left[\frac{a_{16}}{a_{11}}\cdot\frac{x}{b}\left(1 - 12\frac{y^2}{b^2}\right) + \right.\\
&\quad \left. + 2\left(\frac{2a_{12}+a_{66}}{4a_{11}} - \frac{a_{16}^2}{a_{11}^2}\right)\left(\frac{4y^3}{b^3} - \frac{3y}{5b}\right)\right],\\
\sigma_y &= \frac{q}{2h}\left(-1 + 3\frac{y}{b} - 4\frac{y^3}{b^3}\right),\\
\tau_{xy} &= -\frac{qx}{2J}\left(\frac{b^2}{4} - y^2\right) - \frac{q}{h}\cdot\frac{a_{16}}{a_{11}}\left(\frac{y}{b} - \frac{4y^3}{b^3}\right)\\
&\qquad \left(J = \frac{hb^3}{12}\right);
\end{aligned}
\right\}
\qquad (15.1)
$$

The moment of flexure M and the crosscut force N in an arbitrary cross section x are equal to:

$$M = -\frac{qx^2}{2}, \quad N = -qx. \qquad (15.2)$$

The first terms of the expressions σ_x and τ_{xy} are the stresses determined by the elementary theory of bending, and the second terms which depend on the elastic constants are the additional stresses $\Delta\sigma_x$ and $\Delta\tau_{xy}$ which are not taken into account by the elementary theory. The formulas for the normal and tangential stresses in the cross section [the first and the second of (15.1)] may briefly be written in the following way:

$$
\left.
\begin{aligned}
\sigma_x &= \frac{M}{J}y + \Delta\sigma_x,\\
\tau_{xy} &= \frac{N}{2J}\left(\frac{b^2}{4} - y^2\right) + \Delta\tau_{xy}.
\end{aligned}
\right\}
\qquad (15.3)
$$

For an orthotropic beam with which the direction of the x axis coincides with one of the principal directions we obtain from (15.1):

$$
\left.
\begin{aligned}
\sigma_x &= -\frac{qx^2y}{2J} + \frac{q}{h}m\left(\frac{4y^3}{b^3} - \frac{3y}{5b}\right),\\
\sigma_y &= \frac{q}{2h}\left(-1 + 3\frac{y}{b} - 4\frac{y^3}{b^3}\right),\\
\tau_{xy} &= -\frac{qx}{2J}\left(\frac{b^2}{4} - y^2\right).
\end{aligned}
\right\}
\qquad (15.4)
$$

Here

$$m = \frac{2a_{12} + a_{66}}{2a_{11}} = \frac{1}{2}\left(\frac{E_1}{G} - 2\nu_1\right). \qquad (15.5)$$

Hence it is evident that the distribution of the stresses σ_y and τ_{xy} in an orthotropic beam does not differ from the distribution in an orthotropic beam. In the case of an isotropic material $m = 1.$*

Determining the displacements u and v by integrating Eqs. (13.1) we obtain the equation of the curved axis and the depth of curvature (the maximum flexure of the axis):

$$v = \frac{qa_{11}}{24J}(x^4 - 4l^3x + 3l^4) + \frac{qb^2}{80J}\left(3a_{12} + 4a_{66} - \frac{8}{3}\cdot\frac{a_{16}^2}{a_{11}}\right)(x-l)^2; \tag{15.6}$$

$$f = \frac{qa_{11}l^4}{8J} + \frac{qb^2l^2}{80J}\left(3a_{12} + 4a_{66} - \frac{8}{3}\cdot\frac{a_{16}^2}{a_{11}}\right). \tag{15.7}$$

The first term in the expression for f is the depth of curvature, determined by the elementary theory, and the second one is the correction obtained on the basis of a more rigorous theory of the plane stressed state.

The curvature of the curved axis is determined by the formula:

$$\frac{1}{\rho} = \frac{Ma_{11}}{J} + \frac{qb^2}{40J}\left(3a_{12} + 4a_{66} - \frac{8}{3}\cdot\frac{a_{16}^2}{a_{11}}\right). \tag{15.8}$$

In this case the law of proportionality between the curvature of the curved axis and the moment of flexure is no longer valid; the expression determined by the elementary theory must be supplemented by a constant correction term which depends on the elastic constants and the dimensions of the cross section.

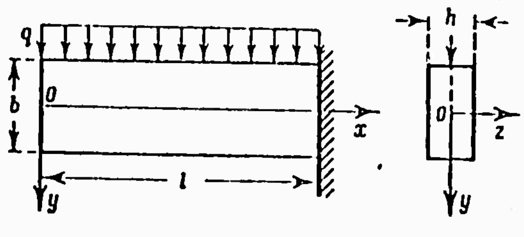

Fig. 23

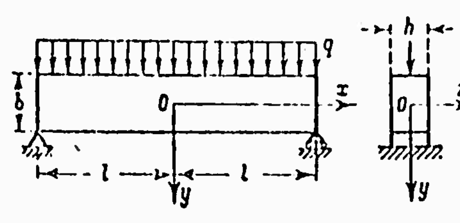

Fig. 24

2. <u>A beam on two supports</u>. For a beam hinged at the ends and bent by a uniform load (Fig. 24) the formulas for σ_y and τ_{xy} obtained are the same as for the console shown in Fig. 23, and the stress σ_x is de-

termined by the formula

$$\sigma_x = \frac{q}{2J}(l^2 - x^2)y + \frac{q}{h}\left[-\frac{a_{16}}{a_{11}}\cdot\frac{x}{b}\left(1 - 12\frac{y^2}{b^2}\right) + \right.$$
$$\left. + 2\left(\frac{2a_{12} + a_{66}}{4a_{11}} - \frac{a_{16}^2}{a_{11}^2}\right)\left(\frac{4y^3}{b^3} - \frac{3y}{5b}\right)\right]. \qquad (15.9)$$

The expressions for σ_x and τ_{xy} may be written in a short manner in the form (15.3), but in this case

$$M = \frac{q}{2}(l^2 - x^2), \quad N = -qx \qquad (15.10)$$

and the correction $\Delta\sigma_x$ has a somewhat different value (the constant a_{16} will enter with a minus sign).

We give here the equation of the curved axis and the formulas for the depth of curvature and the curvature:

$$\eta = \frac{qa_{11}}{24J}(x^4 - 6l^2x^2 + 5l^4) + \qquad (15.11)$$
$$+ \frac{qb^3}{80J}\left(3a_{12} + 4a_{66} + \frac{32}{3}\cdot\frac{a_{16}^2}{a_{11}}\right)(l^2 - x^2);$$
$$f = \frac{5qa_{11}l^4}{24J} + \frac{qb^2l^2}{80J}\left(3a_{12} + 4a_{66} + \frac{32}{3}\cdot\frac{a_{16}^2}{a_{11}}\right); \qquad (15.12)$$
$$\frac{1}{\rho} = \frac{Ma_{11}}{J} + \frac{qb^3}{40J}\left(3a_{12} + 4a_{66} + \frac{32}{3}\cdot\frac{a_{16}^2}{a_{11}}\right). \qquad (15.13)$$

Using the stress function in the form of an integral polynomial also the stress distribution in a homogeneous beam under the action of the proper weight may be obtained. Having solved this problem we come to the following conclusion. The distribution of the stresses σ_x and τ_{xy} due to the proper weight obtained in a beam fixed at one end or supported at the ends is exactly the same as in a beam which is loaded by uniformly distributed normal forces with an intensity of $q = \gamma bh$ (per unit length), where γ is the specific weight of the material. As to the normal stresses σ_y in the longitudinal sections expressing the action of the longitudinal beam layers one each other they are in both cases determined by the law

$$\sigma_y = -\frac{\gamma}{2}y\left(1 - \frac{4y}{b^3}\right). \qquad (15.14)$$

§16. BEAM BENDING UNDER A LOAD WHICH IS DISTRIBUTED ACCORDING TO A LINEAR LAW

With the help of a stress function having the form of a sixth-degree polynomial it is easy to obtain the solution for a beam loaded by normal forces distributed along the length according to a linear law. As in the cases considered earlier the solution will exactly satisfy the conditions on the long sides and approximately those at the short ones where the stresses, generally speaking, will be reduced to moments and forces.*

1. Console. The beam-plate is fixed at one end and bent by a normal load distributed according to a linear law. Placing the coordinate axes as in Fig. 25 we give the equation of a load referred to the unit length in the form:

$$q = q_0 \frac{x}{l}. \tag{16.1}$$

Here q_0 is the maximum value of the load (at the place of fixation).

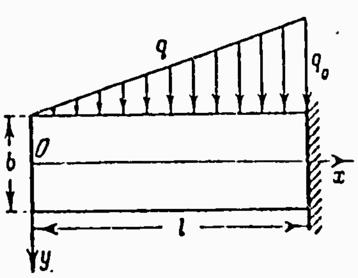

Fig. 25

The moment of flexure and the crosscut force are equal to:

$$M = -\frac{q_0 x^3}{6l}, \quad N = -\frac{q_0 x^2}{2l}. \tag{16.2}$$

The final formulas for the stress components read:

$$\left.\begin{aligned} \sigma_x &= \frac{M}{J} y + \frac{q_0 m}{5h} \cdot \frac{x}{l} \left(20 \frac{y^3}{b^3} - 3 \frac{y}{b}\right), \\ \sigma_y &= -\frac{q_0 x}{2hl} \left(-1 + 3 \frac{y}{b} - 4 \frac{y^3}{b^3}\right), \\ \tau_{xy} &= \frac{N}{2J} \left(\frac{b^2}{4} - y^2\right) - \frac{q_0 b}{80hl} m \left(1 - 24 \frac{y^2}{b^2} + 80 \frac{y^4}{b^4}\right) \\ &\left[m = \frac{1}{2}\left(\frac{E_1}{G} - 2\nu_1\right); \quad J = \frac{hb^3}{12}\right]. \end{aligned}\right\} \tag{16.3}$$

In the formulas (16.3) the first terms of the expressions for σ_x and τ_{xy} are the stresses obtained from the elementary theory, and the second terms are the corrections $\Delta\sigma_x$ and $\Delta\tau_{xy}$ given by the rigorous theory.

The equation of the curved axis is obtained by determining the displacements u and v; it has the form

$$\eta = \frac{q_0}{120E_1Jl}(x^5 - 5l^4x + 4l^5) - \frac{q_0b^3}{240E_1Jl}\times \\ \times\left(\frac{4E_1}{G} - 3\nu_1\right)(x^3 - 3l^2x + 2l^3). \quad (16.4)$$

The depth of curvature and the curvature are determined from the formulas:

$$f = \frac{q_0l^4}{30E_1J} - \frac{q_0b^2l^2}{120E_1J}\left(\frac{4E_1}{G} - 3\nu_1\right); \quad (16.5)$$

$$\frac{1}{\rho} = \frac{M}{E_1J} + \frac{q_0b^2}{40E_1Jl}\left(\frac{4E_1}{G} - 3\nu_1\right)x. \quad (16.6)$$

The correction term in the formula for the curvature is a linear function of x.

2. <u>A beam on two supports</u>. A beam-plate supported at the ends is bent by a load whose equation has the form

$$q = q_0\frac{l+x}{2l}, \quad (16.7)$$

where q_0 is the maximum value of the load (Fig. 26).

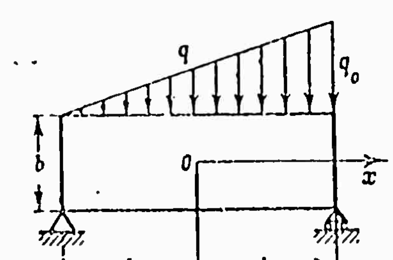

Fig. 26

We present the formulas for the moment of flexure, the crosscut force, and the stresses:

$$\left.\begin{aligned} M &= \frac{q_0}{12l}(l^2 - x^2)(3l + x), \\ N &= \frac{q_0}{12l}(l^2 - 6lx - 3x^2); \end{aligned}\right\} \quad (16.8)$$

$$\left.\begin{aligned} \sigma_x &= \frac{M}{J}y + \frac{q_0}{10hl}m(l + x)\left(20\frac{y^3}{b^3} - 3\frac{y}{b}\right), \\ \sigma_y &= \frac{q_0}{4hl}(l + x)\left(-1 + 3\frac{y}{b} - 4\frac{y^3}{b^3}\right), \\ \tau_{xy} &= \frac{N}{2J}\left(\frac{b^2}{4} - y^2\right) - \frac{q_0}{160hl}m\left(1 - 24\frac{y^2}{b^2} + 80\frac{y^4}{b^4}\right). \end{aligned}\right\} \quad (16.8)$$

The equation of the curved axis and the expression for its curvature will be in this case:

$$\eta = \frac{q_0}{720E_1Jl}(3x^5 + 15lx^4 - 10l^2x^3 - 90l^3x^2 + 7l^4x + 75l^5) - \quad (16.10)$$
$$- \frac{q_0b^2}{480E_1Jl}\left(\frac{4E_1}{G} - 3\nu_1\right)(x^3 + 3lx^2 - l^2x - 3l^3);$$

$$\frac{1}{\rho} = \frac{M}{E_1J} + \frac{q_0b^2}{80E_1Jl}\left(\frac{4E_1}{G} - 3\nu_1\right)(l + x). \quad (16.11)$$

The determination of the solution for a nonorthotropic beam does not, in principle, present any difficulties, but all calculations and final formulas and equations will be more cumbersome (owing to the fact that the constants a_{16} and a_{26} are different from zero).

§17. BEAM BENDING BY AN ARBITRARY LOAD

Using a stress function in the form of an integral polynomial the stresses in a homogeneous beam acted upon by a normal load which is distributed along the whole length according to the law

$$q = q_0 + \sum_{k=1}^{n} q_k \left(\frac{x}{l}\right)^k. \quad (17.1)$$

may be found. If the load, as a function of x, is given in the form of a polynomial of degree n then the corresponding stress function must be taken in the form of an integral polynomial of degree $n + 5$ with respect to x and y. It may be represented in the form of a sum of homogeneous polynomials

$$F = \sum_{k=2}^{n+5} P_k(x, y), \quad (17.2)$$

where

$$P_k(x, y) = A_{k0}x^k + A_{k1}x^{k-1}y + A_{k2}x^{k-2}y^2 + \ldots + A_{kk}y^k, \quad (17.3)$$

and the A_{ki} are constant coefficients. To the polynomials of zeroth and first degree, obviously, correspond stresses equal to zero, and they may be discarded. The polynomials of second and third degree satisfy the equation for the stress function (7.1) for arbitrary values of the coefficients. Each of the polynomials of higher degrees – fourth, fifth, etc. – contains four arbitrary coefficients, and the rest of the coefficients are expressed in terms of these four coefficients. This fact may easily be verified by requiring that the function P_k be a solution to Eq. (7.1) (i.e., substituting P_k into the left-hand side of

this equation and putting the result of the substitution equal to zero).

It is always possible to dispose of the arbitrary constants entering the expression (17.2) such that the stresses exactly satisfy the conditions on the long sides of the beam. On the short sides (i.e., at the ends) it is, in general, not possible to satisfy the conditions exactly by the function (17.2); it is only possible to require that the stresses at the ends, depending on the method of their fixation, be reduced to forces and moments balancing the external load.

If, e.g., we want to obtain the stress distribution in a beam-plate due to a load given in the form of a quadratic function of x (according to a parabolic law) the stress function must be chosen in the form of a seventh-degree polynomial, or, which is the same, in the form of a sum of homogeneous polynomials from the second to the seventh degree, inclusively. The solution is obtained in a rather cumbersome form which cannot be reduced. We only note that in all cases where the load is given by Eq. (17.1) the final formulas for the normal and tangential stresses in the cross section have the form:

$$\left.\begin{aligned} \sigma_x &= -\frac{M}{J}y + \Delta\sigma_x, \\ \tau_{xy} &= \frac{N}{2J}\left(\frac{b^2}{4} - y^2\right) + \Delta\tau_{xy}, \end{aligned}\right\} \qquad (17.4)$$

where M and N are the moment of flexure and the crosscut force in the given section. The first terms are the stresses obtained according to the elementary theory of bending, and the second ones are the correction terms not taken into account by the elementary theory. An analogous form has also the expression for the curvature of the curved axis

$$\frac{1}{\rho} = \frac{M}{E_1'J} + \Delta\left(\frac{1}{\rho}\right); \qquad (17.5)$$

the second term is not taken into account by the elementary theory.

The more general case of the elastic equilibrium of an anisotropic

rectangular plate (band) acted upon by normal and tangential forces on both long sides was studied by A.A. Kurdyumov.* This author seeks the stress function in the form of a sum

$$F = \sum_{k=0}^{n} f_k(y)\, x^k \tag{17.6}$$

and gives a method of determining the coefficients f_k (which are polynomials with respect to powers of y) by taking account of the boundary conditions and the equilibrium conditions. A somewhat different method of solving the same problem for an orthotropic band ("method of successive approximations") was proposed later on by L.N. Vorob'yev.**

If a load is distributed along a beam according to a more complex law particularly in those cases where only a part of the beam length is loaded the stress distribution may be obtained with the help of Fourier series. We shall consider the way in which this method is applied with the help of the example of an orthotropic beam supported at the ends and bent by a normal load distributed symmetrically with respect to the middle according to an arbitrary law. The other cases of a single-bay beam – a console and a beam on two supports loaded by an asymmetric load may be studied by the same method with unimportant modifications in the details.***

Let the planes of elastic symmetry be parallel to the beam faces, and, consequently, the axial direction be the principal one.

With the coordinate axes placed as shown in Fig. 27, we shall expand the load q, as a function of x, in a Fourier series in the interval $(-l, l)$ where this load is given. The series will contain only cosines and a constant term

$$q = q_0 + \sum_{m=1}^{\infty} q_m \cos\frac{m\pi x}{l}, \tag{17.7}$$

where

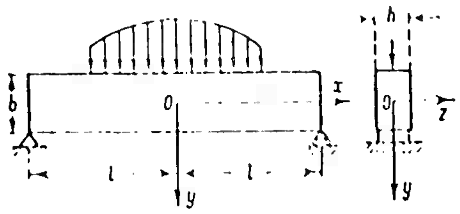

Fig. 27

$$q_0 = \frac{1}{l}\int_0^l q\,dx, \quad q_m = \frac{2}{l}\int_0^l q\cos\frac{m\pi x}{l}\,dx.$$

The boundary conditions on the long sides $y = \pm b/2$ have the form:

$$\left.\begin{aligned} &\text{for } y = \frac{b}{2} && \sigma_y = \tau_{xy} = 0; \\ &\text{for } y = -\frac{b}{2} && \sigma_y = -q_0 - \sum_{m=1}^{\infty} q_m \cos\frac{m\pi x}{l}, \quad \tau_{xy} = 0. \end{aligned}\right\} \qquad (17.8)$$

The equation for the stress function has the form:

$$\frac{1}{E_2}\cdot\frac{\partial^4 F}{\partial x^4} + \left(\frac{1}{G} - \frac{2\nu_1}{E_1}\right)\frac{\partial^4 F}{\partial x^2\partial y^2} + \frac{1}{E_1}\cdot\frac{\partial^4 F}{\partial y^4} = 0. \qquad (17.9)$$

The stress distribution due to the constant load q_0 is known to us [see Formula (15.9) and the second and third formulas (15.1)]. The function giving the stress distribution due to the load

$$q_m \cos\frac{m\pi x}{l},$$

will be sought in the form

$$F_m = f_m(y)\cos\frac{m\pi x}{l}. \qquad (17.10)$$

Substituting the function F_m into Eq. (17.9) we obtain the ordinary differential equation for the function f_m:

$$\frac{1}{E_1}f_m^{IV} - \left(\frac{1}{G} - \frac{2\nu_1}{E_1}\right)\left(\frac{m\pi}{l}\right)^2 f_m'' + \frac{1}{E_2}\left(\frac{m\pi}{l}\right)^4 f_m = 0. \qquad (17.11)$$

The form of the function f_m depends on the roots of the characteristic equation

$$s^4 - \left(\frac{E_1}{G} - 2\nu_1\right)s^2 + \frac{E_1}{E_2} = 0. \qquad (17.12)$$

If all moduli E_1, E_2, G are finite and not equal to zero three cases are possible.

Case 1. The roots are real, unequal; we shall designate them by:

$$\pm s_1, \quad \pm s_2 \quad (s_1 > 0, \quad s_2 > 0).$$

Case 2. The roots are real, equal; we shall designate them by:

$$\pm s \quad (s > 0).$$

Case 3. The roots are complex; we shall designate them by:

$$s \pm ti, \quad -s \pm ti \quad (s > 0, \ t > 0).$$

Purely imaginary roots cannot be obtained since the numbers s_1 and s_2 are connected with the complex parameters of the plane stressed state by simple relations $s_1 = -i\mu_1$, $s_2 = -i\mu_2$ [see Eq. (7.5)].

In case 1 the stress function F_m has the form:

$$F_m = \left(A_m \operatorname{ch}\frac{s_1 m\pi y}{l} + B_m \operatorname{sh}\frac{s_1 m\pi y}{l} + C_m \operatorname{ch}\frac{s_2 m\pi y}{l} + D_m \operatorname{sh}\frac{s_2 m\pi y}{l}\right)\cos\frac{m\pi x}{l} \qquad (17.13)$$

(A_m, B_m, C_m, D_m are arbitrary constants, ch and sh are the hyperbolic functions).

In case 2

$$F_m = \left[(A_m + B_m y)\operatorname{ch}\frac{sm\pi y}{l} + (C_m + D_m y)\operatorname{sh}\frac{sm\pi y}{l}\right]\cos\frac{m\pi x}{l}. \qquad (17.14)$$

In case 3 F_m can be represented as follows:

$$F_m = \left[\left(A_m \operatorname{ch}\frac{sm\pi y}{l} + B_m \operatorname{sh}\frac{sm\pi y}{l}\right)\cos\frac{tm\pi y}{l} + \left(C_m \operatorname{ch}\frac{sm\pi y}{l} + D_m \operatorname{sh}\frac{sm\pi y}{l}\right)\sin\frac{tm\pi y}{l}\right]\cos\frac{m\pi x}{l}. \qquad (17.15)$$

In order to construct a solution for the beam shown in Fig. 27 we assume a stress function in the form of a sum of expressions (17.13) [or, respectively, (17.14) and (17.15)]. Moreover, from formulas (5.7) (for $\overline{U} = 0$) we determine the corresponding stresses, add to them the stresses due to the constant load q_0, and require that the conditions (17.8) be satisfied. From the boundary conditions we obtain a system of

equations for the determination of the constants A_m, B_m, C_m, D_m (for each m a separate system is obtained); we find all constants by solving these equations. As a result, we obtain formulas for the stresses in the form of series of rather complex structure. The stresses σ_x at the beam ends may be reduced to moments with the help of which we shall get rid of the superposition of the solution for the case of pure bending.

Problems of this kind for an anisotropic beam, apparently, have not yet led to numerical results.

P.P. Kufarev and V.A. Sveklo have given the general solution of the problem of the elastic equilibrium of an infinitely long anisotropic band loaded along the boundaries by normal and tangential forces.* Directing the x axis parallel to the boundaries the authors assume that the external forces, as functions of x, are absolutely integrable in the interval $(-\infty, +\infty)$ and may be represented in the form of Fourier integrals. The functions $\Phi_1'(z_1)$ and $\Phi_2'(z_2)$ in terms of which the stresses are expressed are determined in the form of double integrals with infinite limits.

§18. THE BENDING OF A COMPOSITE (MULTILAYER) BEAM

With the help of stress functions in the form of integral polynomials the stress distribution for several cases of beam bending may be obtained where the beam consists of an arbitrary number of anisotropic bands of identical thickness. We shall here consider only the case of an orthotropic console bent by a force and a moment.**

Let be given a beam soldered or glued together of an arbitrary number of orthotropic bands of identical thickness, but with different elastic properties; one of its ends is rigidly fixed, and the other one is acted upon by a load resulting in a moment M and a transverse force P. The stresses in each layer as well as the equation of the bent axis

and the bending strength must be determined.

Identifying the origin of coordinates with the topmost point of the free end we place the x axis along the upper edge, as shown in Fig. 28. We shall number the layers consecutively, starting from the topmost

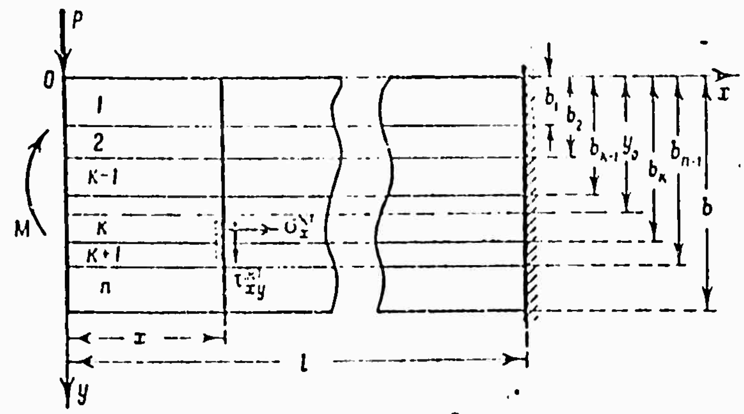

Fig. 28

layer, and introduce the following designations: n is the number of layers; l is the beam length; b is the height; h is the thickness (identical for all bands); $\sigma_x^{(k)}$, $\sigma_y^{(k)}$, $\tau_{xy}^{(k)}$, u_k, v_k are the stress components and displacements in the kth layer; $E_1^{(k)}$, $E_2^{(k)}$, $\nu_1^{(k)}$, $\nu_2^{(k)}$, $G^{(k)}$ are the elastic constants (the principal ones) for this layer; b_{k-1} and b_k are the distances from the upper edge to the upper and lower boundaries of the kth layer ($k = 1, 2, \ldots, n$; $b_0 = 0$, $b_n = b$).

The stress components (the mean values taken over the thickness will be expressed in terms of the stress function F_k (for each band separately)

$$\sigma_x^{(k)} = \frac{\partial^2 F_k}{\partial y^2}, \quad \sigma_y^{(k)} = \frac{\partial^2 F_k}{\partial x^2}, \quad \tau_{xy}^{(k)} = -\frac{\partial^2 F_k}{\partial x \, \partial y}. \qquad (18.1)$$

The function F_k satisfies the equation

$$\frac{1}{E_2^{(k)}} \cdot \frac{\partial^4 F_k}{\partial x^4} + \left(\frac{1}{G^{(k)}} - \frac{2\nu_1^{(k)}}{E_1^{(k)}}\right) \frac{\partial^4 F_k}{\partial x^2 \, \partial y^2} + \frac{1}{E_1^{(k)}} \cdot \frac{\partial^4 F_k}{\partial y^4} = 0, \qquad (18.2)$$

The displacements projections (also mean values over the thickness) will be determined from the equations

$$\left.\begin{aligned}
\frac{\partial u_k}{\partial x} &= \frac{1}{E_1^{(k)}}\sigma_x^{(k)} - \frac{\nu_2^{(k)}}{E_2^{(k)}}\sigma_y^{(k)},\\
\frac{\partial v_k}{\partial y} &= -\frac{\nu_1^{(k)}}{E_1^{(k)}}\sigma_x^{(k)} + \frac{1}{E_2^{(k)}}\sigma_y^{(k)},\\
\frac{\partial u_k}{\partial y} + \frac{\partial v_k}{\partial x} &= \frac{1}{G^{(k)}}\tau_{xy}^{(k)}.
\end{aligned}\right\} \qquad (18.3)$$

The boundary conditions on the upper and lower edges have the form:

$$\left.\begin{aligned}
&\text{for } y = 0 \qquad \sigma_y^{(1)} = \tau_{xy}^{(1)} = 0;\\
&\text{for } y = b \qquad \sigma_y^{(n)} = \tau_{xy}^{(n)} = 0.
\end{aligned}\right\} \qquad (18.4)$$

Since slipping of the bands is excluded we have the following conditions on the contact surfaces.

$$\left.\begin{aligned}
\text{for } y = b_{k-1} \qquad &\sigma_y^{(k-1)} = \sigma_y^{(k)}, \quad \tau_{xy}^{(k-1)} = \tau_{xy}^{(k)},\\
&u_{k-1} = u_k, \quad v_{k-1} = v_k.
\end{aligned}\right\} \qquad (18.5)$$

The stresses in each cross section must balance the external force and moment (this is also valid for the end sections $x = 0$, $x = l$); hence follow, in addition, the conditions

$$\left.\begin{aligned}
&\sum_{k=1}^{n}\int_{b_{k-1}}^{b_k}\sigma_x^{(k)}\,dy = 0, \qquad \sum_{k=1}^{n}\int_{b_{k-1}}^{b_k}\sigma_x^{(k)}y\,dy = \frac{M - Px}{h},\\
&\qquad\qquad \sum_{k=1}^{n}\int_{b_{k-1}}^{b_k}\tau_{xy}^{(k)}\,dy = -\frac{P}{h}.
\end{aligned}\right\} \qquad (18.6)$$

The solution is obtained with the help of stress functions of the form

$$F_k = A_k xy + B_k y^2 + C_k xy^2 + D_k y^3 + E_k xy^3, \qquad (18.7)$$

which satisfy Eq. (18.2) for arbitrary values of the coefficients.

The final results boil down to the following. The stress components are determined by the formulas:

$$\sigma_x^{(k)} = \frac{6E_1^{(k)}}{hS}(M - Px)(2S_1 y - S_2), \quad \sigma_y^{(k)} = 0 \qquad (18.8)$$

$$(k = 1, 2, \ldots, n);$$

$$\left.\begin{aligned}
\tau_{xy}^{(k)} &= \frac{6P}{hS}\left\{S_1\left[\sum_{i=1}^{k-1}(b_i^2 - b_{i-1}^2)E_1^{(i)} + (y^2 - b_{k-1}^2)E_1^{(k)}\right]\right. \\
&\quad \left. - S_2\left[\sum_{i=1}^{k-1}(b_i - b_{i-1})E_1^{(i)} + (y - b_{k-1})E_1^{(k)}\right]\right\} \\
&\qquad (k = 2,\ 3,\ \ldots,\ n-1); \\
\tau_{xy}^{(1)} &= -\frac{6PE_1^{(1)}}{hS}(S_1 y - S_2)\,y, \\
\tau_{xy}^{(n)} &= \frac{6PE_1^{(n)}}{hS}[S_2 - (b_n + y)S_1](b_n - y).
\end{aligned}\right\} \quad (18.9)$$

Here we have introduced the designations:

$$\left.\begin{aligned}
S_1 &= \sum_{k=1}^{n}(b_k - b_{k-1})E_1^{(k)}, \\
S_2 &= \sum_{k=1}^{n}(b_k^2 - b_{k-1}^2)E_1^{(k)}, \\
S_3 &= \sum_{k=1}^{n}(b_k^3 - b_{k-1}^3)E_1^{(k)}, \\
S &= 4S_1S_3 - 3S_2^2.
\end{aligned}\right\} \quad (18.10)$$

The neutral axis is at a distance of

$$y_0 = \frac{S_2}{2S_1}; \quad (18.11)$$

from the upper edge, it is impossible to predict which layer will contain this axis.

The equation of the bent axis (i.e., the line into which the neutral axis goes over in bending) has the same form as for a homogeneous console with some digidity D:

$$\eta = \frac{P}{6D}(x^3 - 3l^2x + 2l^3) - \frac{M}{2D}(x - l)^2. \quad (18.12)$$

This rigidity is determined by the formula:

$$D = \frac{hS}{12S_1}. \quad (18.13)$$

For the curvature of the bent axis an expression coinciding with that given by the elementary theory of bending is obtained:

$$\frac{1}{\rho} = \frac{M - Px}{D}. \quad (18.14)$$

We note that among all elastic constants only Young's moduli for

the axial directions, $E_1^{(k)}$, enter all above-mentioned formulas. All formulas simplify to a certain extent if the layers have the same height equal to b/n. In particular, the rigidity formula (18.13) assumes the form

$$D = \overline{E}_1 J, \tag{18.15}$$

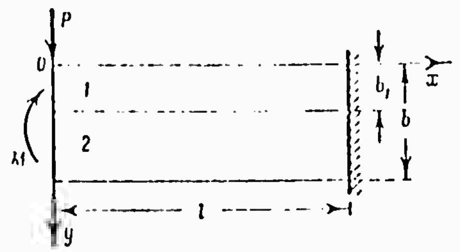

Fig. 29

where $J = hb^3/12$ is the moment of inertia of the section of the whole beam, and $\overline{E}_1$ is a constant connected with Young's moduli in the following way:

$$\overline{E}_1 = \frac{4\sum_{k=1}^{n} E_1^{(k)} \cdot \sum_{k=1}^{n} E_1^{(k)}(3k^2 - 3k + 1) - 3\left[\sum_{k=1}^{n} E_1^{(k)}(2k - 1)\right]^2}{n^3 \sum_{k=1}^{n} E_1^{(k)}}. \tag{18.16}$$

The distance of the neutral axis from the upper edge is in this case equal to

$$y_0 = \frac{b}{2n} \cdot \frac{\sum_{k=1}^{n} E_1^{(k)}(2k - 1)}{\sum_{k=1}^{n} E_1^{(k)}}. \tag{18.17}$$

For a beam soldered or glued together of two bands with heights b_1 and $b - b_1$ (Fig. 29) we obtain:

$$\left.\begin{aligned} \sigma_x^{(1)} &= \frac{6E_1^{(1)}}{hS}(M - Px)(2S_1 y - S_2), \\ \sigma_x^{(2)} &= \frac{E_1^{(2)}}{E_1^{(1)}}\sigma_x^{(1)}; \end{aligned}\right\} \tag{18.18}$$

$$\left.\begin{aligned} \tau_{xy}^{(1)} &= -\frac{6PE_1^{(1)}}{hS}(S_1 y^2 - S_2 y), \\ \tau_{xy}^{(2)} &= -\frac{6PE_1^{(2)}}{hS}(S_2 - S_1 b - S_1 y)(b - y). \end{aligned}\right\} \tag{18.19}$$

Here

$$\left.\begin{aligned} &S_1 = b_1 E_1^{(1)} + (b - b_1)E_1^{(2)}, \qquad S_2 = b_1^2 E_1^{(1)} + (b^2 - b_1^2)E_1^{(2)}, \\ &S = b_1^4 (E_1^{(1)})^2 + 2b_1(b - b_1)(2b^2 - bb_1 + b_1^2)E_1^{(2)}E_1^{(1)} + (b - b_1)^4 (E_1^{(2)})^2. \end{aligned}\right\} \tag{18.20}$$

<u>Example</u>. A beam is composed of two bands with a height ratio of

1:3, and a ratio of the moduli for the longitudinal directions of 1:9. We must distinguish two cases.

Case 1. $E_1^{(1)} = E_1$, $E_1^{(2)} = 9E_1$, i.e., the upper, narrower band has a lower Young's modulus.

Using formulas (18.11)-(18.20) we obtain the following results.

The neutral axis lies in the zone of the second band and its distance from the upper edge of the beam is $y_0 = 0.61b$. Introducing the abbreviations

$$m = \frac{M - Px}{hb^2}, \qquad p = \frac{P}{hb},$$

we obtain the following values of the stresses σ_x and τ_{xy} at the upper face, on the contact surface of the bands, on the neutral line, and near the lower face:

$$\left.\begin{aligned}
&(\sigma_x^{(1)})_{y=0} = -1,61m, \quad (\tau_{xy}^{(1)})_{y=0} = 0;\\
&(\sigma_x^{(1)})_{y=b/4} = -0,95m, \quad (\sigma_x^{(2)})_{y=b/4} = -8,51m;\\
&(\tau_{xy}^{(1)})_{y=b/4} = (\tau_{xy}^{(2)})_{y=b/4} = -0,32p;\\
&(\sigma_x^{(2)})_{y=y_0} = 0, \quad (\tau_{xy}^{(2)})_{y=y_0} = -1,84p;\\
&(\sigma_x^{(2)})_{y=b} = 9,35m, \quad (\tau_{xy}^{(2)})_{y=b} = 0.
\end{aligned}\right\} \qquad (18.21)$$

In a homogeneous console the maximum normal stress is equal to $6m$ and the maximum tangential stress is equal to $1.5p$.

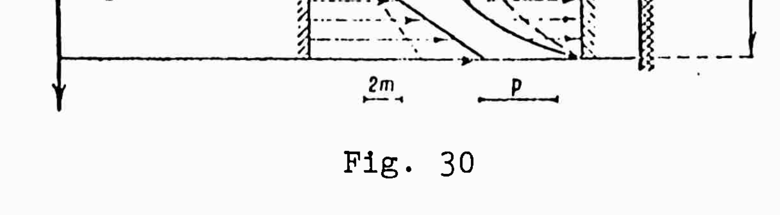

Fig. 30

The distribution of the stresses σ_x and τ_{xy} in the cross section is shown in Fig. 30. The dotted lines are the curves showing the stresses in a homogeneous console.

Case 2. $E_1^{(1)} = 9E_1$, $E_1^{(2)} = E_1$, i.e., the upper band has a higher Young's modulus.

The neutral axis coincides with the contact line of the bands $y_0 = 0.25b$. The stresses at the characteristic points have the following values:

$$\left.\begin{aligned} &(\sigma_x^{(1)})_{y=0} = -12m, && (\tau_{xy}^{(1)})_{y=0} = 0; \\ &(\sigma_x^{(1)})_{y=y_0} = (\sigma_x^{(2)})_{y=y_0} = 0, && (\tau_{xy}^{(1)})_{y=y_0} = (\tau_{xy}^{(2)})_{y=y_0} = -1,5p; \\ &(\sigma_x^{(2)})_{y=b} = 4m, && (\tau_{xy}^{(2)})_{y=b} = 0. \end{aligned}\right\} \quad (18.22)$$

The diagrams showing the distribution of the stresses σ_x and τ_{xy} in the cross section are shown in Fig. 31.

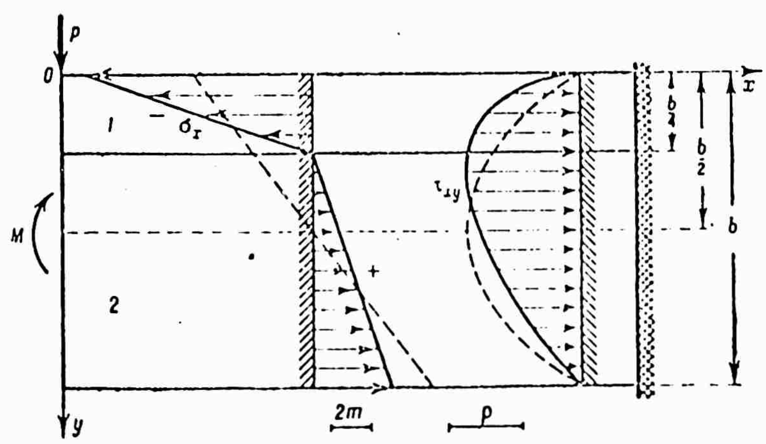

Fig. 31

Comparing the diagrams of Fig. 30 and 31 we may notice that the maximum stresses are obtained in a band with great Young's modulus. In both cases the maximum normal stress in a composite console exceeds (considerably enough) the maximum stress in the same homogeneous console.

§19. BENDING OF A BEAM WITH VARIABLE MODULI OF ELASTICITY

In the preceding section we considered a beam in which the modu of elasticity vary discontinuously along the height, i.e., when we p from one layer to the neighboring one. It is not difficult to obtain solution also for a beam with moduli of elasticity varying continuou along the height. We restrict ourselves to the case of an orthotropi console with which the principal directions are parallel to the sides, and which is bent by a force P and a moment M (Fig. 32).

We shall assume that the beam experiences small strains and is governed by the equations of the generalized Hooke's law:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{1}{E_1}\sigma_x - \frac{\nu_2}{E_2}\sigma_y, \\ \frac{\partial v}{\partial y} &= -\frac{\nu_1}{E_1}\sigma_x + \frac{1}{E_2}\sigma_y, \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= \frac{1}{G}\tau_{xy}, \end{aligned}\right\} \tag{19.1}$$

where E_1, E_2, ν_1, ν_2 and G are arbitrary continuous functions of y.

The stress components (the mean values over the thickness) satisfy the equilibrium conditions

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \tag{19.2}$$

and the boundary conditions

$$\text{при } y = 0,\ y = b \qquad \sigma_y = \tau_{xy} = 0. \tag{19.3}$$

Besides, in each cross section the stresses must balance the external load; hence follow other three conditions:

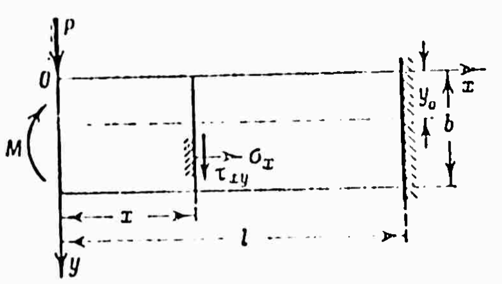

Fig. 32

$$\left.\begin{aligned} \int_0^b \sigma_x\,dy &= 0, \\ \int_0^b \sigma_x y\,dy &= \frac{M - Px}{h}, \\ \int_0^b \tau_{xy}\,dy &= -\frac{P}{h}. \end{aligned}\right\} \tag{19.4}$$

We shall assume that the general character of the stress distribution in the given beam is the same as in a homogeneous console, i.e.,

$$\sigma_x = -\frac{M - Px}{h} f'(y), \qquad \sigma_y = 0, \qquad \tau_{xy} = \frac{P}{h} f(y). \tag{19.5}$$

These expressions satisfy the equations of equilibrium (19.2); the unknown function *f(y)* will be determined from Eqs. (19.1) and the conditions (19.3) and (19.4). From the first two equations (19.1) we find the displacements; they will be expressed in terms of *f(y)*. Requiring that u and v satisfy also the third equation (19.1) we obtain the equation

$$\left[\frac{f'(y)}{E_1}\right]'' = 0. \tag{19.6}$$

Hence we find the expression for the function *f(y)*:

$$f(y) = c\int E_1 y\,dy + d\int E_1\,dy + e, \tag{19.7}$$

where c, d, e are arbitrary constants. All arbitrary constants will be determined from the conditions (19.3) and (18.4). As a result we obtain the following final expressions for the stress components:

$$\left.\begin{aligned} \sigma_x &= -\frac{6(M - Px)}{hS} E_1(y)(2S_1 y - S_2), \\ \sigma_y &= 0, \\ \tau_{xy} &= \frac{6P}{hS}\int_0^y E_1(y)(2S_1 y - S_2)\,dy. \end{aligned}\right\} \tag{19.8}$$

Here

$$\left.\begin{aligned} S_1 &= \int_0^b E_1\,dy, \qquad S_2 = 2\int_0^b E_1 y\,dy, \\ S &= 12\left[\int_0^b E_1\,dy \cdot \int_0^b E_1 y^2\,dy - \left(\int_0^b E_1 y\,dy\right)^2\right]. \end{aligned}\right\} \tag{19.9}$$

The neutral axis where $\sigma_x = 0$ is at a distance of

$$y_0 = \frac{S_2}{2S_1} \tag{19.10}$$

from the upper face.

The expressions for the displacements which we shall not present here show that a console with variable moduli is bent, as a homogeneous console with a rigidity D equal to

$$D = \frac{hS}{12S_1}, \tag{19.11}$$

and Eq. (18.12) and formula (18.14) prove to hold for it.

Only Young's modulus E_1 for the axial direction x enters the formulas for the stresses and the rigidity; the rest of the moduli of elasticity are not selected in the values of the stresses and the rigidity. The displacements u and v of any point off from the neutral axis will depend on G and ν_1, the modulus E_2, however, will not enter any formulas.*

Let, e.g., the modulus E_1 vary along the height symmetrically with respect to the geometrical axis of the beam and be expressed by a quadratic function of y:

$$E_1 = E_1^0 + \frac{E_1''}{4b^2}(2y - b)^2. \tag{19.12}$$

For such a beam the neutral line coincides with the geometrical axis $y = b/2$. The stress components will be determined by the formulas:

$$\left.\begin{aligned}
\sigma_x &= \frac{6(M - Px)}{hb^3(E_1^0 + 0{,}15E_1'')}\left[E_1^0 + \frac{E_1''}{4b^2}(2y - b)^2\right](2y - b),\\
\sigma_y &= 0,\\
\tau_{xy} &=\\
&= -\frac{6P}{hb^3(E_1^0 + 0{,}15E_1'')}\left[E_1^0 + \frac{E_1''}{4b^2}(b^2 - 2by + 2y^2)\right](by - y^2).
\end{aligned}\right\} \tag{19.13}$$

For the rigidity we obtain the formula

$$D = \bar{E}_1 J, \tag{19.14}$$

where J is the moment of inertia of the cross section and

$$\bar{E}_1 = E_1^0 + 0{,}15E_1''. \tag{19.15}$$

<u>Example</u>. The Young's modulus of a beam for the axial direction is given by formula (19.12) where the ratio of its maximum value to its

minimum value is equal to 5. Two cases are here possible.

1) The maximum value E_1 is at the outer edges;

$$(E_1)_{y=0} = 5(E_1)_{y=b/2}, \qquad E_1'' = 16E_1^0.$$

The normal stress in each section achieves the maximum values at the beam edges, and the tangential stress on the neutral axis. Putting

$$m = \frac{M - Px}{hb^3}, \qquad p = \frac{P}{hb},$$

we obtain on the basis of formulas (19.13)

$$\sigma_{max} = 8{,}82\,m, \qquad \tau_{max} = 1{,}32p. \qquad (19.16)$$

The diagram of the distribution of the stresses σ_x and τ_{xy} in the cross sections is shown in Fig. 33 (the dotted lines show the variation of the stresses in a homogeneous beam, for which $\sigma_{max} = 6m$, $\tau_{max} =$ $= 1.5p$).

2) The maximum value of E_1 is on the natural axis;

$$(E_1)_{y=0} = 0{,}2(E_1)_{y=b/2}, \qquad E_1'' = -3{,}2E_1^0.$$

The maximum value of the tangential stress is obtained on the neutral axis, and the normal stress attains the maximum values at the points of the section which are at a distance of 0.18b from the outer edges:

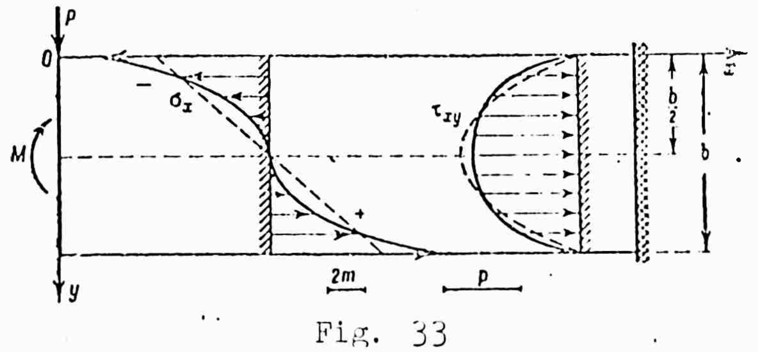

Fig. 33

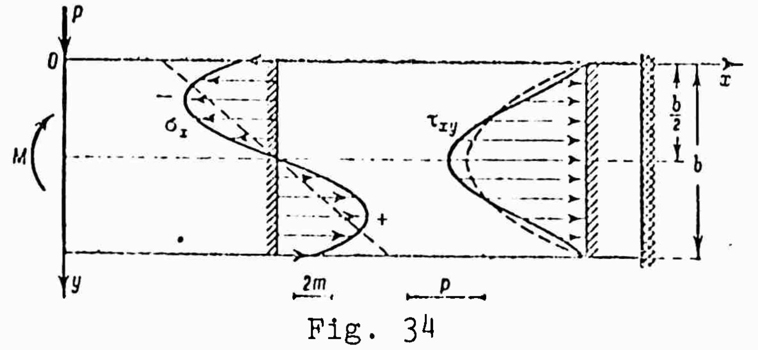

Fig. 34

$$\sigma_{max} = 4{,}96m, \qquad \tau_{max} = 1{,}73p. \tag{19.17}$$

At the points $y = 0$ and $y = b$

$$\sigma_x = \pm 2{,}31m, \qquad \tau_{xy} = 0. \tag{19.18}$$

The distribution of the stresses, in particular the normal ones, differs remarkably from the distribution in the preceding case; it is gi en in Fig. 34.

§20. THE DEFORMATION OF A WEDGE-SHAPED CONSOLE BY A FORCE APPLIED TO ITS TOP

Let us consider the elastic equilibrium of a wedge-shaped console of rectangular cross section in which the broad end is rigidly fixed, under the action of a force P applied to its top. It is assumed that the console material is homogeneous and anisotropic, but generally not orthotropic (the existence of planes of elastic symmetry parallel to the mid surface is always assumed).

We shall regard the console as an infinite wedge, i.e., as a body whose region is limited by two infinite straight lines starting from the top O. We shall choose the wedge top to be the origin of coordinates and place the x axis arbitrarily in the mid plane; we shall also use polar coordinates, counting the polar angles θ from the x axis. We shall designate the angles of inclination of the faces to the x axis by ψ_1 and ψ_2 (obviously, the angle at the top is equal to $\psi_1 + \psi_2$) and the angle of inclination of the force to the x axis by ω (Fig. 35).

The problem consists in choosing such a solution of Eq. (5.10) which permits the equilibrium conditions and the conditions on the faces to be satisfied:

$$\text{for } \theta = -\psi_1 \text{ and } \theta = \psi_2$$
$$\sigma_\theta = \tau_{r\theta} = 0, \tag{20.1}$$

and will determine the stress components tending to zero with increasing distance from the top.

As is shown by an investigation* the stress function giving the solution of the problem has the form

$$F = r\Phi_1(\theta), \tag{20.2}$$

where Φ_1 is a function of the polar angle only, which will be found from Eq. (5.10). To determine this function Eq. (5.10) should be written in the symbolic form (7.2):

$$D_1 D_2 D_3 D_4 F = 0. \tag{20.3}$$

The operators D_k in polar coordinates assume the form

$$D_k = (\sin\theta - \mu_k\cos\theta)\frac{\partial}{\partial r} + (\cos\theta + \mu_k\sin\theta)\frac{1}{r}\frac{\partial}{\partial\theta} \tag{20.4}$$
$$(k = 1,\ 2,\ 3,\ 4).$$

The function Φ_1 will be determined in four stages, by successive integration of four first-order equations after which the stress components in polar coordinates will be found from the formulas:

$$\sigma_r = \frac{1}{r}\cdot\frac{\partial F}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 F}{\partial\theta^2}, \qquad \sigma_\theta = \frac{\partial^2 F}{\partial r^2}, \qquad \tau_{r\theta} = -\frac{\partial^2}{\partial r\,\partial\theta}\left(\frac{F}{r}\right). \tag{20.5}$$

Omitting all intermediate calculations we shall present the final formulas for these stresses:

$$\sigma_r = \frac{1}{r}\cdot\frac{A\cos\theta + B\sin\theta}{L(\theta)}, \qquad \sigma_\theta = \tau_{r\theta} = 0, \tag{20.6}$$

where

$$L(\theta) = a_{11}\cos^4\theta - 2a_{16}\sin\theta\cos^3\theta + (2a_{12} + a_{66})\sin^2\theta\cos^2\theta - 2a_{26}\sin^3\theta\cos\theta + a_{22}\sin^4\theta. \tag{20.7}$$

Obviously, the boundary conditions (20.1) are fulfilled, and the stress tends to zero with increasing distance from the point where the force is applied.

The arbitrary constants A and B are determined from the equilibrium conditions of the wedge part cut out by a circular section of arbitrary radius r described from the top as the center (the dotted line in Fig. 35). We obtain the following equations for these constants:

$$\left.\begin{aligned} A\int_{-\psi_1}^{\psi_1}\frac{\cos^2\theta}{L(\theta)}\,d\theta+B\int_{-\psi_1}^{\psi_1}\frac{\sin\theta\cos\theta}{L(\theta)}\,d\theta&=-\frac{P}{h}\cos\omega,\\ A\int_{-\psi_1}^{\psi_1}\frac{\sin\theta\cos\theta}{L(\theta)}\,d\theta+B\int_{-\psi_1}^{\psi_1}\frac{\sin^2\theta}{L(\theta)}\,d\theta&=-\frac{P}{h}\sin\omega \end{aligned}\right\} \qquad (20.8)$$

(h is the wedge thickness).

The calculations simplify if the console is orthotropic and the direction of its axis coincides with the principal one. Then

$$L(\theta)=\frac{\cos^4\theta}{E_1}+\left(\frac{1}{G}-\frac{2\nu_1}{E_1}\right)\sin^2\theta\cos^2\theta+\frac{\sin^4\theta}{E_2}. \qquad (20.9)$$

In the given case the function $L(\theta)$ is the reciprocal value of Young's modulus E_r for the radial direction $L = 1/E_r$ [see the first formula (9.10) where we must put $\varphi = \theta$].

The calculation of the integrals entering Eqs. (20.8) for an orthotropic wedge is not very laborious. We present here the expressions of the corresponding indefinite integrals for the case where the complex parameters are purely imaginary and unequal:

$$\mu_1=\beta i,\quad \mu_2=\delta i,$$

$$\left.\begin{aligned} I_1&=\int E_r\cos^2\theta\,d\theta=\frac{1}{a_{11}(\beta^2-\delta^2)}\left[\beta\,\mathrm{arctg}(\beta\,\mathrm{tg}\,\theta)-\delta\,\mathrm{arctg}(\delta\,\mathrm{tg}\,\theta)\right],\\ I_2&=\int E_r\sin\theta\cos\theta\,d\theta=\frac{1}{2a_{11}(\beta^2-\delta^2)}\ln\frac{\cos^2\theta+\beta^2\sin^2\theta}{\cos^2\theta+\delta^2\sin^2\theta},\\ I_3&=\int E_r\sin^2\theta\,d\theta=-\frac{1}{a_{11}(\beta^2-\delta^2)}\left[-\frac{1}{\beta}\,\mathrm{arctg}(\beta\,\mathrm{tg}\,\theta)+\frac{1}{\delta}\,\mathrm{arctg}(\delta\,\mathrm{tg}\,\theta)\right]. \end{aligned}\right\} \qquad (20.10)$$

The formulas for the stresses in the case of equal complex parameters ($\mu_1=\mu_2=\beta i$) and in case 3 ($\mu_1=\alpha+\beta i,\ \mu_2=-\alpha+\beta i$) may always be obtained from the formulas for the case of purely imaginary unequal parameters by passing to the limit, δ tending to β, or, respectively, by replacing β and δ by the quantities $\beta + \alpha i$ and $\beta - \alpha i$.

We must make the following remarks with respect to the general character of the stress distribution. The stress distribution in a wedge which is deformed by a force is "radial" or "ray-like"; the radial stress σ_r at an arbitrary point is, at the same time, the principal

stress, whereas the other principal stress σ_θ acting on the radial surface is equal to zero. According to the relationship between the elastic constants and the angle of inclination ω three cases are possible:

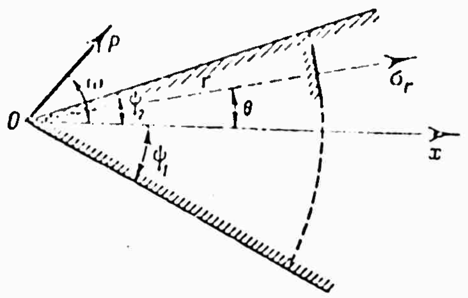

Fig. 35

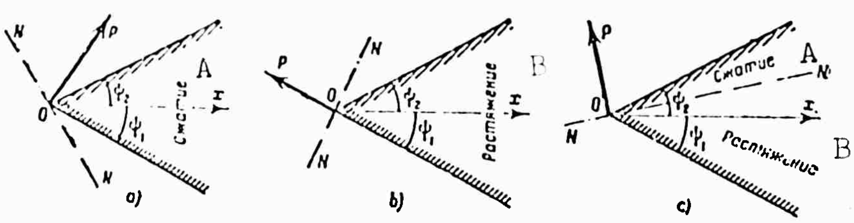

Fig. 36. A) Compression; B) stretching.

1) all parts of the wedge are compressed, 2) all parts of the wedge are stretched, and 3) there is a neutral line on which all stresses vanish in the region of the wedge; on one side of the neutral line the material is compressed and on the other one it is stretched (Fig. 36abc).

The position of the neutral line with respect to the x axis is determined by the angle θ for which

$$\mathrm{tg}\,\theta_0 = -\frac{A}{B}. \tag{20.11}$$

The points where the principal stress σ_r has the same value σ_0, positive or negative, lie on a fourth-order curve; on one side of the neutral line there are the curves corresponding to the compressive stresses (negative σ_0), and on the other side the curves corresponding to the stretching stresses (positive σ_0). The equation of the family of curves of equal stresses in Cartesian coordinates have the form

$$a_{11}x^4 - 2a_{16}x^3y + (2a_{12} + a_{66})x^2y^2 - 2a_{26}xy^3 + a_{22}y^4 - \frac{1}{\sigma_0}(x^2 + y^2)(Ax + By) = 0. \tag{20.12}$$

These curves are always closed although only some arcs lie in the region of the wedge; they all pass through the top and touch the neutral line at this point. We shall encounter on other curves of this type in the following chapter.

For a console of an isotropic material the stress components wi be found from the formulas (the x axis is placed along the axis of sy metry, the angle at the top is equal to 2ψ):

$$\sigma_r = -\frac{2P}{hr}\left(\frac{\cos\omega\cos\theta}{2\psi+\sin 2\psi} + \frac{\sin\omega\sin\theta}{2\psi-\sin 2\psi}\right), \quad \sigma_\theta = \tau_{r\theta} = 0. \qquad (20.13)$$

The curves of equal stresses become circles passing through the top and touching the neutral axis.

We note that in a wedge with cylindrical anisotropy (with a pole of anisotropy at the top) the stress distribution due to a force does not depend on the elastic constants and exactly coincides with the di tribution in an isotropic wedge of the same form [formulas (20.13)].

§21. THE BENDING OF A WEDGE-SHAPED CONSOLE BY A MOMENT

Let a wedge-shaped console as considered in §20 be bent by a moment M applied to the top.

We shall place the polar axis x along the symmetry axis of the wedge. The angle at the top will be designated by 2ψ (Fig. 37). As before, we consider the console as an infinite wedge. On the faces the following conditions must be fulfilled: for $\theta = \pm\psi$

$$\sigma_\theta = 0, \tau_{r\theta} = 0; \qquad (21.1)$$

besides, the stress components must tend to zero with increasing distance from the top. The stress function which makes it possible to satisfy these conditions and the equilibrium conditions has the form*

$$F = \Phi_0(\theta) \qquad (21.2)$$

(it does not depend on r). With the help of the symbolic representation (20.3) of the equation for the stress function the function Φ_0 will be determined in four stages by integrating four first-order

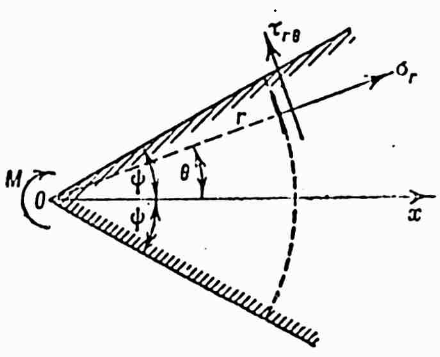

Fig. 37

equations.

As a result we obtain the following formulas for the stress components satisfying the conditions (21.1):

$$
\left.
\begin{aligned}
\sigma_r &= -\frac{A}{r^2}\left[\frac{2\sin 2\theta}{L(\theta)} + (\cos 2\theta - \cos 2\psi)\frac{L'(\theta)}{L^2(\theta)}\right],\\
\sigma_\theta &= 0,\\
\tau_{r\theta} &= \frac{A}{r^2}\cdot\frac{\cos 2\theta - \cos 2\psi}{L(\theta)}.
\end{aligned}
\right\}
\qquad (21.3)
$$

Here

$$
\left.
\begin{aligned}
L(\theta) &= a_{11}\cos^4\theta - 2a_{16}\sin\theta\cos^3\theta + \\
&+ (2a_{12} + a_{66})\sin^2\theta\cos^2\theta - 2a_{26}\sin^3\theta\cos\theta + a_{22}\sin^4\theta,\\
L'(\theta) &= \frac{dL}{d\theta}.
\end{aligned}
\right\}
\qquad (21.4)
$$

The constant A will be determined from the equilibrium conditions of the wedge part cut out by a circular section (the dotted line in Fig. 37) and proves to be equal to

$$
A = \frac{M}{h}\cdot\frac{1}{\displaystyle\int_{-\psi}^{\psi}\frac{\cos 2\theta - \cos 2\psi}{L(\theta)}\,d\theta}. \qquad (21.5)
$$

In the case of an orthotropic console, in which the principal direction coincides with the direction of the geometrical axis x and the complex parameters are purely imaginary and unequal we have:

$$
\int \frac{\cos 2\theta - \cos 2\psi}{L(\theta)}\,d\theta = 2(I_1\sin^2\psi - I_3\cos^2\psi), \qquad (21.6)
$$

where I_1 and I_3 are integrals to be calculated from the formulas (20.10).

If the principal direction in an orthotropic console does not coincide with the direction of the geometrical axis it is more convenient to choose it as the direction of the x axis; instead of (21.3) we obtain somewhat more complex formulas.

With respect to the general character of the stress distribution we must note that when the bending is carried out by a moment this distribution is no longer "radial" (the tangential stress $\tau_{r\theta}$ is not equal

to zero, and, therefore, σ_r is not a principal stress). If in the case of console deformation by a force the stresses varied inversely proportionally to the distance from the point where the force is applied then the decrease of the stresses in the case of deformation under the action of a moment of flexure will be more rapid: they vary inversely proportionally to the square of the distance from the point where the moment is applied.

In an isotropic console the stress distribution is determined by the formulas:

$$\left.\begin{aligned} \sigma_r &= -\frac{2M}{h(\sin 2\varphi - 2\varphi\cos 2\varphi)}\cdot\frac{\sin 2\theta}{r^2},\\ \sigma_\theta &= 0,\\ \tau_{r\theta} &= \frac{M}{h(\sin 2\varphi - 2\varphi\cos 2\varphi)}\cdot\frac{\cos 2\theta - \cos 2\varphi}{r^2}. \end{aligned}\right\} \qquad (21.7)$$

If the wedge has cylindrical anisotropy and the pole of anisotropy coincides with the top the equations of the generalized Hooke's law will be written for it in the form (12.6). The stress function for such a wedge satisfies Eq. (12.9) where $\overline{U} = 0$, and in the case of bending by a moment (Fig. 37) has the form

$$F = \Phi_0(\theta) = (A\cos n\theta + B\sin n\theta)e^{m\theta} + Ce^{s\theta} + D, \qquad (21.8)$$

where s and $m + ni$ are the roots of the equation

$$a_{11}s^3 + 2a_{16}s^2 + (2a_{11} + 2a_{12} + a_{66})s + 2(a_{16} + a_{26}) = 0. \qquad (21.9)$$

In particular, for an orthotropic wedge with cylindrical anisotropy

$$F = A\cos n\theta + B\sin n\theta + C\theta + D, \qquad (21.10)$$

where

$$n = \sqrt{2 + \frac{2a_{12} + a_{66}}{a_{11}}}. \qquad (21.11)$$

The constants A, B, C are determined as in the case of a homogeneous wedge, from the conditions on the faces and the equilibrium conditions.

§22. THE BENDING OF A WEDGE-SHAPED CONSOLE BY A DISTRIBUTED LOAD

Combining stress functions of the form

$$F = r^k \Phi_k(\theta), \qquad (22.1)$$

where $k = 2, 3, \ldots$, the stresses in a wedge-shaped console along whose faces a load given in the form of an algebraic function of the distance r is distributed may be obtained. If, e.g., a load given by the formula

$$q = q_0 + \sum_{k=1}^{n} q_k r^k, \qquad (22.2)$$

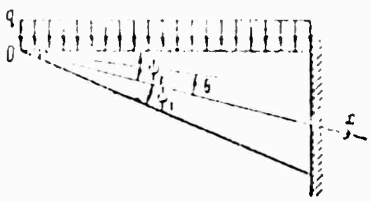

Fig. 38

is distributed along the whole length of the face the corresponding stress function must be assumed in the form of the sum

$$F = \sum_{k=2}^{n+2} r^k \Phi_k(\theta). \qquad (22.3)$$

In particular, for a console bent by a uniformly distributed load q (per unit length) the solution is found with the help of a stress function of the form*

$$F = r^2 \Phi_2(\theta). \qquad (22.4)$$

If the x axis is placed arbitrarily (Fig. 38) the conditions on the faces will be written in the following way:

$$\left.\begin{array}{ll} \text{for } \theta = \varphi_1 & \sigma_\theta = \tau_{r\theta} = 0; \\ \text{for } \theta = \varphi_2 & \sigma_\theta = -\frac{q}{h}, \ \tau_{r\theta} = 0. \end{array}\right\} \qquad (22.5)$$

For the function Φ_2 we obtain the expression

$$\Phi_2(\theta) = A\cos 2\theta + B\sin 2\theta + C\Psi(\theta) + D. \qquad (22.6)$$

A, B, C, D are here arbitrary constants, and $\Psi(\theta)$ is a function of the angle θ of rather complex structure. In the general case, if the complex parameters have the form:

$$\mu_1 = \alpha + \beta i, \quad \mu_2 = \gamma + \delta i,$$

and we introduce the designations

$$a = \frac{\alpha}{\alpha^2 + \beta^2}, \quad b = \frac{\beta}{\alpha^2 + \beta^2}, \quad c = \frac{\gamma}{\gamma^2 + \delta^2}, \quad d = \frac{\delta}{\gamma^2 + \delta^2}, \qquad (22.7)$$

we obtain the following expression for Φ:

$$
\begin{aligned}
\varphi(\theta) = {} & bd\{(a-c)\sin^2\theta + (a^2 - c^2 + b^2 - d^2)\sin\theta\cos\theta + \\
& + [c(a^2+b^2) - a(c^2+d^2)]\cos^2\theta\} \times \\
& \times \ln\frac{\sin^2\theta + 2a\sin\theta\cos\theta + (a^2+b^2)\cos^2\theta}{\sin^2\theta + 2c\sin\theta\cos\theta + (c^2+d^2)\cos^2\theta} - \\
& - d\{4(a-c)b^2(\sin\theta + a\cos\theta)\cos\theta + \\
& + [(a-c)^2 + d^2 - b^2][\sin^2\theta + 2a\sin\theta\cos\theta + (a^2-b^2)\cos^2\theta]\} \times \\
& \times \operatorname{arctg}\frac{b\cos\theta}{\sin\theta + a\cos\theta} + b\{4(a-c)d^2(\sin\theta + c\cos\theta)\cos\theta - \\
& - [(a-c)^2 + b^2 - d^2][\sin^2\theta + 2c\sin\theta\cos\theta + (c^2-d^2)\cos^2\theta]\} \times \\
& \times \operatorname{arctg}\frac{d\cos\theta}{\sin\theta + c\cos\theta}.
\end{aligned}
\tag{22.8}
$$

The general formulas for the stress components have the form:

$$
\left.
\begin{aligned}
\sigma_r &= -2A\cos 2\theta - 2B\sin 2\theta + C(2\varphi + \varphi'') + D, \\
\sigma_\theta &= 2(A\cos 2\theta + B\sin 2\theta + C\varphi + D), \\
\tau_{r\theta} &= 2A\sin 2\theta - 2B\cos 2\theta - C\varphi'.
\end{aligned}
\right\}
\tag{22.9}
$$

The constants *A*, *B*, *C*, *D* will be determined from the boundary conditions (22.5); the final expressions are complex, and we cannot give them here. The stresses in a console bent by a uniformly distributed load do not depend on the distance *r*.

For a console of an orthotropic material the expression for the function Φ is simpler, but in the case of an isotropic material $a = c = 0$, $b = d = 1$, and we obtain the quite simple function $= \theta$.

The stress function for an isotropic console has the form:

$$F = r^2(A\cos 2\theta + B\sin 2\theta + C\theta + D). \tag{22.10}$$

Among other cases of bending of a console by a continuously distributed load we mention the bending by normal forces varying according to a linear law $q = q_1 x$ (Fig. 39), taking account of the proper weight. In this case the stress distribution in an anisotropic console exactly coincides with the distribution of the same isotropic console. This is the case because

$$
\begin{aligned}
F = r^3\Phi_3(\theta) = Ax^3 + Bx^2y + \\
+ Cxy^2 + Dy^3
\end{aligned}
\tag{22.11}
$$

for arbitrary values of the constants is a solution both of Eq. (5.10)

and the biharmonic equation. If the loaded side is horizontal then i any homogeneous console, both anisotropic and isotropic, shown in Fi 39, the stresses due to the external load with account taken of the proper weight will be determined from the formulas:

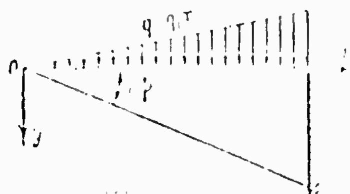

Fig. 39

$$\left.\begin{aligned} \sigma_x &= -\left(\frac{q_1}{h}\operatorname{ctg}^2\beta + \gamma\right)\operatorname{ctg}^2\beta\,(x - 2y\operatorname{ctg}^2\beta), \\ \sigma_y &= -\frac{q_1}{h}x - \gamma y, \\ \tau_{xy} &= -\left(\frac{q_1}{h}\operatorname{ctg}^2\beta + \gamma\right)\operatorname{ctg}^2\beta\; y \end{aligned}\right\} \quad (22.12)$$

(h is the console height, γ is the specific weight of the material). The anisotropy of the material will only influence the strains and d placements whose expressions will depend on the elastic constants.

The solution for the general case of bending of an anisotropic console under the action of a load distributed along the faces accord ing to an arbitrary law or given in the form of concentrated forces w s obtained by V.M. Abramov.* The console was regarded as an infinite wedge; the solution was found with the halp of a Mellin integral and the stress components are represented in the form of integrals.

Another method of solving this problem was proposed by P.P. Kufarev**; the expressions for the stresses are also represented in the form of integrals with infinite limits. The general solutions mentioned were not led to numerical results, as yet.

All things set forth, with appropriate variations in details, may be referred also to a wedge with cylindrical anisotropy with which the pole of anisotropy coincides with the top. In the case of bending by a uniform load, e.g., (Fig. 38) the stress function has the form:

$$F = r^2\Phi_2(\theta) = r^2[(A\cos n\theta + B\sin n\theta)e^{m\theta} + Ce^{s\theta} + D], \quad (22.13)$$

where s, m and n have the same values as for the wedge bent by a moment [see Eq. (21.9)]. In particular, for an orthotropic wedge with cylindrical asymmetry we obtain:

$$F = r^2(A\cos 2\theta + B\sin 2\theta + C\theta + D). \quad (22.14)$$

§23. PURE BENDING OF A CURVED GIRDER WITH CYLINDRICAL ANISOTROPY

Let us consider the elastic equilibrium of a curved girder in the form of a part of a plane circular ring under the action of forces applied to the ends and reduced to moments. We assume that the girder has the property of cylindrical anisotropy in which case the pole of anisotropy is at the common center of the circles whose arcs form the outer and the inner edges of the girder. Besides the planes of elastic symmetry parallel to the mid surface we do not assume any elements of elastic symmetry. Fig. 40 shows a section of the girder through the mid plane. The pole of anisotropy is assumed to be the origin of coordinates, the polar axis x is directed along the axis of symmetry. We shall designate by a and b the inside and the outside radii, by c the ratio a/b, by h the thickness of the girder, and by M the value of the moments. We shall assume the angle between the end radii to have an arbitrary value smaller than 2π.

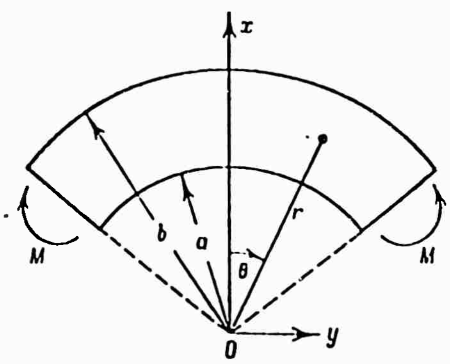

Fig. 40

The equations of the generalized Hooke's law will be written in the form:

$$\left.\begin{aligned} \varepsilon_r &= a_{11}\sigma_r + a_{12}\sigma_\theta + a_{16}\tau_{r\theta}, \\ \varepsilon_\theta &= a_{12}\sigma_r + a_{22}\sigma_\theta + a_{26}\tau_{r\theta}, \\ \gamma_{r\theta} &= a_{16}\sigma_r + a_{26}\sigma_\theta + a_{66}\tau_{r\theta}. \end{aligned}\right\} \quad (23.1)$$

The stress components (mean values over the thickness) are expressed in terms of the stress function $F(r, \theta)$ according to the formulas (12.8) where $\overline{U} = 0$, and the function F satisfies Eq. (12.9) (where we also must put $\overline{U} = 0$). The conditions on line constituting the girder outline are obvious: on the curves $r = a$ and $r = b$ the stresses are equal to zero, and at the ends the stresses are reduced to the moments M.

The solution is obtained with the help of the function F which does not depend on the polar angle θ.* The function looks as follows:

$$F = f_0(r) = A + Br^2 + Cr^{1+k} + Dr^{1-k}, \tag{23.2}$$

where

$$k = \sqrt{\frac{a_{11}}{a_{22}}} = \sqrt{\frac{E_\theta}{E_r}}, \tag{23.3}$$

and E_r, E_θ are Young's moduli for the stretching (compression) in the radial and tangential directions r and θ which are, in general, no principal directions of elasticity.

Having determined the constants from the boundary conditions we obtain the following final expressions for the stress components:

$$\left.\begin{aligned}
\sigma_r &= -\frac{M}{b^2hg}\left[1 - \frac{1-c^{k+1}}{1-c^{2k}}\left(\frac{r}{b}\right)^{k-1} - \frac{1-c^{k-1}}{1-c^{2k}}\,c^{k+1}\left(\frac{b}{r}\right)^{k+1}\right],\\
\sigma_\theta &= -\frac{M}{b^2hg}\left[1 - \frac{1-c^{k+1}}{1-c^{2k}}\,k\left(\frac{r}{b}\right)^{k-1} + \frac{1-c^{k-1}}{1-c^{2k}}\,kc^{k+1}\left(\frac{b}{r}\right)^{k+1}\right],\\
\tau_{r\theta} &= 0.
\end{aligned}\right\} \tag{23.4}$$

Here

$$g = \frac{1-c^2}{2} - \frac{k}{k+1}\cdot\frac{(1-c^{k+1})^2}{1-c^{2k}} + \frac{kc^2}{k-1}\cdot\frac{(1-c^{k-1})^2}{1-c^{2k}}. \tag{23.5}$$

The stress distribution obtained is the same for all cross (radial) sections and depends only on the ratio of the constants a_{11}/a_{22}. The normal stress σ_θ in the cross section is neither governed by a linear, nor a hyperbolic law.

The normal stresses on the outer and inner contours are equal to:

$$(\sigma_\theta)_b = -\frac{M}{b^2hg}\cdot\frac{1-k+2kc^{k+1}-(1+k)c^{2k}}{1-c^{2k}}; \tag{23.6}$$

$$(\sigma_\theta)_a = \frac{M}{b^2hg}\cdot\frac{(1-k)c^{2k}+2kc^{k-1}-(1+k)}{1-c^{2k}}. \tag{23.7}$$

One of these values will be the maximum for the whole girder, but which cannot be indicated beforehand if the numerical value of k is unknown. The displacements u_r, u_θ in the radial and tangential directions may easily be determined from Eqs. (23.1).

We note that in the case where $a_{11} = a_{22}$, i.e., $E_r = E_\theta$, $k = 1$ formulas for the stresses coincide exactly with the formulas for the isotropic girder*:

$$\left.\begin{aligned}
\sigma_r &= -\frac{M}{b^2hg}\left\{c^2\ln c\left[1-\left(\frac{b}{r}\right)^2\right]-(1-c^2)\ln\frac{b}{r}\right\},\\
\sigma_\theta &= -\frac{M}{b^2hg}\left\{1-c^2+c^2\ln c\left[1+\left(\frac{b}{r}\right)^2\right]-(1-c^2)\ln\frac{b}{r}\right\},\\
\tau_{r\theta} &= 0,
\end{aligned}\right\} \qquad (23.8)$$

where

$$g = \left(\frac{1-c^2}{2}\right)^2 - c^2(\ln c)^2. \qquad (23.9)$$

§24. THE BENDING OF A CURVED CURVILINEAR-ANISOTROPIC GIRDER BY A FORCE APPLIED TO ITS END

Let a curved girder having the shape of a part of a plane circular ring be fixed at one end and be deformed by forces distributed at the other end and resulting in a force P applied to the center of the section. It is assumed that the girder has the property of cylindrical anisotropy with the pole in the center of the circles whose arcs form the girder outline. We shall choose the pole to be the origin of coordinates, place the polar axis x along the radius corresponding to the loaded end, and designate by ω the angle of inclination of the force to the x axis (Fig. 41). The value of the angle between the end sections will be assumed arbitrary, but not more than 2π.

To start with, we shall concentrate our attention on the case of an orthotropic girder with cylindrical anisotropy which, besides the plane of elastic symmetry parallel to the mid plane has two other ones at each point: the radial and the tangential one. In this case $a_{16} = a_{26} = 0$ and the equations of the generalized Hooke's law may be written in the form (12.10):

$$
\left.
\begin{aligned}
\varepsilon_r &= \frac{1}{E_r}\sigma_r - \frac{\nu_\theta}{E_\theta}\sigma_\theta,\\
\varepsilon_\theta &= -\frac{\nu_r}{E_r}\sigma_r + \frac{1}{E_\theta}\sigma_\theta,\\
\gamma_{r\theta} &= \frac{1}{G_{r\theta}}\tau_{r\theta}
\end{aligned}
\right\} \qquad (24.1)
$$

(the dashes over the denotations for the stresses and strains are omitted).

To determine the stress function F we use Eq. (12.11) in which $\bar{U} = 0$ must be put. The solution is obtained with the help of a stress function of the form*

$$
F = f_1(r)\cos\theta + f_1^*(r)\sin\theta = (Ar^{1+\beta} + Br^{1-\beta} + Cr + Dr\ln r)\cos\theta + (A'r^{1+\beta} + B'r^{1-\beta} + C'r + D'r\ln r)\sin\theta. \qquad (24.2)
$$

A, B, ..., D' are here arbitrary constants and

$$
\beta = \sqrt{1 + \frac{a_{11} + 2a_{12} + a_{66}}{a_{22}}} = \sqrt{1 + \frac{E_\theta}{E_r}(1 - 2\nu_r) + \frac{E_\theta}{G_{r\theta}}}. \qquad (24.3)
$$

Having determined all constants from the conditions at the boundaries $r = a$ and $r = b$, where $\sigma_r = \tau_{r\theta} = 0$, and requiring that the stresses at the free end result in a force P we obtain:

$$
\left.
\begin{aligned}
\sigma_r &= \frac{P}{bhg_1}\cdot\frac{b}{r}\left[\left(\frac{r}{b}\right)^{\beta} + c^{\beta}\left(\frac{b}{r}\right)^{\beta} - 1 - c^{\beta}\right]\sin(\theta + \omega),\\
\sigma_\theta &= \frac{P}{bhg_1}\cdot\frac{b}{r}\left[(1+\beta)\left(\frac{r}{b}\right)^{\beta} + (1-\beta)\left(\frac{b}{r}\right)^{\beta}c^{\beta} - 1 - c^{\beta}\right]\sin(\theta + \omega),\\
\tau_{r\theta} &= -\frac{P}{bhg_1}\cdot\frac{b}{r}\left[\left(\frac{r}{b}\right)^{\beta} + c^{\beta}\left(\frac{b}{r}\right)^{\beta} - 1 - c^{\beta}\right]\cos(\theta + \omega).
\end{aligned}
\right\} \qquad (24.4)
$$

We have used here the denotations:

$$
c = \frac{a}{b}, \quad g_1 = \frac{2}{\beta}(1 - c^{\beta}) + (1 + c^{\beta})\ln c. \qquad (24.5)
$$

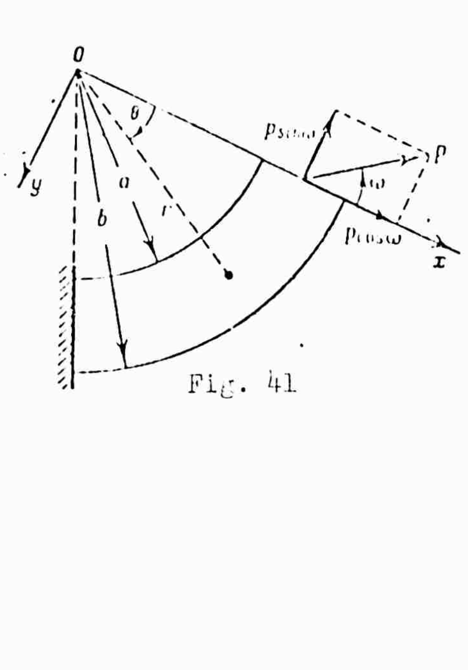

Fig. 41

The normal stresses obtained are maximum in the girder section perpendicular to the line of action of the force; in these sections the tangential stresses vanish.**

The tangential stresses attain maximum absolute values in the sections on the line of action of the force; the normal stresses are there equal to zero. In any given section $\theta = \theta_0$ the normal stress σ_θ will be obtained maximum near the inner boundary $r = a$; it is equal

$$(\sigma_\theta)_a = \frac{P\sin(\theta_0+\omega)}{bhg_1}\,\frac{\beta(1-c^{\beta})}{c}. \tag{24.6}$$

If the elastic constants of an orthotropic girder satisfy the condition

$$\frac{E_\theta}{E_r}(1-2\nu_r)+\frac{E_\theta}{G_{r\theta}}=3, \tag{24.7}$$

then β = 2 and the stress distribution will be exactly the same as is obtained in an isotropic girder*:

$$\left.\begin{aligned}
\sigma_r &= \frac{P}{bhg_1}\left[\frac{r}{b}+c^2\left(\frac{b}{r}\right)^3-(1+c^2)\frac{b}{r}\right]\sin(\theta+\omega),\\
\sigma_\theta &= \frac{P}{bhg_1}\left[3\frac{r}{b}-c^2\left(\frac{b}{r}\right)^3-(1+c^2)\frac{b}{r}\right]\sin(\theta+\omega),\\
\tau_{r\theta} &= -\frac{P}{bhg_1}\left[\frac{r}{b}+c^2\left(\frac{b}{r}\right)^3-(1+c^2)\frac{b}{r}\right]\cos(\theta+\omega);
\end{aligned}\right\} \tag{24.8}$$

$$g_1 = 1-c^2+(1+c^2)\ln c. \tag{24.9}$$

A.S. Kosmodamianskiy studied the more general case where the girder shown in Fig. 41 is not orthotropic [the coefficients $_{16}$ and $_{26}$ of Eqs. (23.1) are not equal to zero, and the stress function satisfies Eq. (12.9) where $\overline{U}$ = 0]. The solution for this case is found with the help of the stress function**

$$\begin{aligned}F = {}&(Ar^{1+\beta+\alpha i}+Br^{1-\beta+\alpha i}+Cr+Dr\ln r)e^{\theta i}+\\
&+(\overline{A}r^{1+\beta-\alpha i}+\overline{B}r^{1-\beta-\alpha i}+\overline{C}r+\overline{D}r\ln r)e^{-\theta i}.\end{aligned} \tag{24.10}$$

A, B, C, D are here arbitrary constants, generally complex, to be determined from the boundary conditions and the conditions at the free end; $\overline{A}$, $\overline{B}$, $\overline{C}$, $\overline{D}$ are the conjugate quantities;

$$\alpha=\frac{a_{26}}{a_{22}},\quad \beta=\sqrt{1+\frac{a_{11}+2a_{12}+a_{66}}{a_{22}}-\left(\frac{a_{26}}{a_{22}}\right)^2}. \tag{24.11}$$

A.S. Kosmodamianskiy considered also numerical examples and constructed diagrams of the stress distribution in the sections of a nonorthotropic girder with given elastic constants for several values of the ratio *c*. The analysis of the results obtained enabled him to make a number of conclusions the most important of which boil down to the following:

1) the maximum value of the stress σ_θ by far exceeds the maximum

values of σ_r and $\tau_{r\theta}$ and is obtained, as in the case of the orthotropic girder, at the inner boundary of the section (for $r = a$);

2) the constant a_{16} does not affect the value of the stresses;

3) with decreasing ratio b/a the distribution law of the normal stresses across the section approaches a linear one, and the distribution law of the tangential stresses tends to a parabolic one.

§25. THE BENDING OF A CURVED CURVILINEAR-ANISOTROPIC GIRDER UNDER A DISTRIBUTED LOAD

The solution of the problem of the bending of a curved girder under a load uniformly distributed along the curvilinear edge may be obtained with the help of a stress function in the form of a sum

$$F = f_0(r) + f_1(r)\cos\theta + f_1^{\bullet}(r)\sin\theta. \tag{25.1}$$

The first term has the form (23.2), and the second and the third ones are determined by Formula (24.2) for an orthotropic girder, and for a nonorthotropic one by Formula (24.10). All constants will be found from the boundary conditions which can always be satisfied exactly at the curvilinear boundaries, while at the ends the filfillment is approximate.

Let us consider as an example an orthotropic girder with cylindrical anisotropy, supported at the ends and bent by a normal load which is uniformly distributed along the outer boundary (Fig. 42). The common center of the circles bounding the girder which is at the same time also the pole of anisotropy will be chosen to be the origin of coordinates, and the axis of symmetry will be identified with the polar axis x. We shall assume that both supports are hinged and designed such that the support reactions form the same angles ψ with the axis of symmetry. We shall designate by q the load per unit length and by 2φ the angle between the end sections of the girder which is, in general, not equal

to 2ψ $\left(\varphi < \frac{\pi}{2}\right)$.

The boundary conditions on the curvilinear sides have the form:

$$\left.\begin{aligned} &\text{for } r=a \quad \sigma_r=0, \quad \tau_{r\theta}=0; \\ &\text{for } r=b \quad \sigma_r=-\frac{q}{h}, \quad \tau_{r\theta}=0. \end{aligned}\right\} \qquad (25.2)$$

At the ends the stresses will result in a radial force (reactio R and in a tangential force R_θ; the following conditions must be fulfilled there:

for $\theta=\pm\varphi$

$$\left.\begin{aligned} &\int_a^b \sigma_\theta\, dr = -\frac{R_\theta}{h} = -\frac{qb}{h}\cdot\frac{\sin\varphi\sin(\varphi-\psi)}{\cos\psi}, \\ &\int_a^b \sigma_\theta r\, dr = 0, \\ &\int_a^b \tau_{r\theta}\, dr = \pm\frac{R_r}{h} = \pm\frac{qb}{h}\cdot\frac{\sin\varphi\cos(\varphi-\psi)}{\cos\psi}. \end{aligned}\right\} (25.3)$$

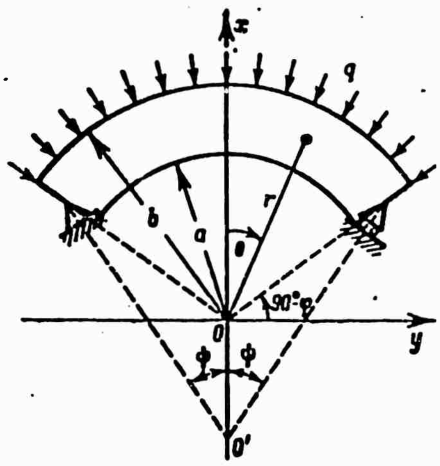

Fig. 42

Putting $f_1^* = 0$ we determine the constants entering expression (25.1) from the conditions (25.2) and (25.3) and obtain the following final formulas for the stresses:

$$\left.\begin{aligned} \sigma_r &= \frac{q}{h}\left[P+Q\left(\frac{r}{b}\right)^{k-1}+R\left(\frac{b}{r}\right)^{k+1}\right]+\frac{q}{bhg_1}\cdot\frac{b}{r}\left[\left(\frac{r}{b}\right)^{\beta}+\right. \\ &\left.+c^{\beta}\left(\frac{b}{r}\right)^{\beta}-(1+c^{\beta})\right]\frac{\cos(\varphi-\psi)}{\cos\psi}\cos\theta, \\ \sigma_\theta &= \frac{q}{h}\left[P+Qk\left(\frac{r}{b}\right)^{k-1}-Rk\left(\frac{b}{r}\right)^{k+1}\right]+\frac{q}{bhg_1}\times \\ &\times\frac{b}{r}\left[(1+\beta)\left(\frac{r}{b}\right)^{\beta}+(1-\beta)c^{\beta}\left(\frac{b}{r}\right)^{\beta}-(1+c^{\beta})\right]\frac{\cos(\varphi-\psi)}{\cos\psi}\cos\theta, \\ \tau_{r\theta} &= \frac{q}{bhg_1}\cdot\frac{b}{r}\left[\left(\frac{r}{b}\right)^{\beta}+c^{\beta}\left(\frac{b}{r}\right)^{\beta}-(1+c^{\beta})\right]\frac{\cos(\varphi-\psi)}{\cos\psi}\sin\theta. \end{aligned}\right\} \quad (25.4)$$

Here we have used the designations:

$$
\left.\begin{aligned}
P &= \frac{1}{2(k^2-1)(1-c^{2k})g}[2k(k-1)(1-c^{k+1})+\\
&\quad +2k(k+1)c^{k+1}(1-c^{k-1})-(k^2-1)(1+c)(1-c^{2k})m],\\
Q &= \frac{1}{2(k-1)(1-c^{2k})g}[-(k-1)(1-c^2)-2kc^2(1-c^{k-1})+\\
&\quad +(k-1)(1+c)(1-c^{k+1})m],\\
R &= \frac{1}{2(k+1)(1-c^{2k})g}[(k+1)c^{2k}(1-c^2)-2kc^{k+1}(1-c^{k+1})-\\
&\quad -(k+1)(1+c)c^{2k}(1-c^{1-k})m],\\
c &= \frac{a}{b}, \quad m=\frac{\sin\varphi\sin(\varphi-\psi)}{\cos\psi};
\end{aligned}\right\} \qquad (25.5)
$$

k, g, β and g_1 are, respectively, determined from the formulas (23.3), (23.5), (24.3), and (24.5).

The normal stresses σ_θ in an arbitrary section $\theta = \theta_0$ attain the maximum values either at the inner boundary $r = a$ or at the outer one $r = b$ according to relationships between the parameters k, β and other parameters entering formulas (25.4)-(25.5). At these points we have:

$$
(\sigma_\theta)_a = \frac{q}{h}(P+Qkc^{k-1}-Rkc^{-k-1})-\frac{q\beta}{hg_1}(1-c^\beta)\frac{\cos(\varphi-\psi)}{\cos\psi}\cos\theta_0; \qquad (25.6)
$$

$$
(\sigma_\theta)_b = \frac{q}{h}(P+Qk-Rk)+\frac{q\beta}{hg_1}(1-c^\beta)\frac{\cos(\varphi-\psi)}{\cos\psi}\cos\theta_0. \qquad (25.7)
$$

For an isotropic girder we obtain the following formulas instead of the formulas (25.4) and (25.5):

$$
\left.\begin{aligned}
\sigma_r &= \frac{q}{h}\left[P+Q+2Q\ln\frac{r}{b}+R\left(\frac{b}{r}\right)^2\right]+\\
&\quad +\frac{q}{hg_1}\left[\frac{r}{b}+c^2\left(\frac{b}{r}\right)^3-(1+c^2)\frac{b}{r}\right]\frac{\cos(\varphi-\psi)}{\cos\psi}\cos\theta,\\
\sigma_\theta &= \frac{q}{h}\left[P+3Q+2Q\ln\frac{r}{b}-R\left(\frac{b}{r}\right)^2\right]+\\
&\quad +\frac{q}{hg_1}\left[3\frac{r}{b}-c^2\left(\frac{b}{r}\right)^3-(1+c^2)\frac{b}{r}\right]\frac{\cos(\varphi-\psi)}{\cos\psi}\cos\theta,\\
\tau_{r\theta} &= \frac{q}{hg_1}\left[\frac{r}{b}+c^2\left(\frac{b}{r}\right)^3-(1+c^2)\frac{b}{r}\right]\frac{\cos(\varphi-\psi)}{\cos\psi}\sin\theta;
\end{aligned}\right\} \qquad (25.8)
$$

$$
\left.\begin{aligned}
P &= -\frac{1}{4g}[2(1-c^2)-4c^2\ln c-4c^2(\ln c)^2+\\
&\quad +2c^2(1+c)m\ln c-(1-c^2)(1+c)m],\\
Q &= -\frac{1}{4g}[-(1-c^2)+2c^2\ln c+(1-c^2)(1+c)m],\\
R &= -\frac{c^2}{4g}[-(1-c^2)+2\ln c-2(1+c)m\ln c]
\end{aligned}\right\} \qquad (25.9)
$$

[the expressions for g and g_1 have the forms (23.9) and (24.9)].

At the end points of the section $\theta = \theta_0$ of an isotropic girder the

normal stresses are equal to:

$$(\sigma_\theta)_a = \frac{q}{2hg}[2(1-c^2)\ln c - (1+c)(1-c^2+2\ln c)m] - \\ -\frac{2q}{hg_1 c}(1-c^2)\frac{\cos(\varphi-\psi)}{\cos\psi}\cos\theta_0; \tag{25.10}$$

$$(\sigma_\theta)_b = \frac{q}{4hg}[(1-c^2)^2 + 4c^2(\ln c)^2 - 2(1+c)(1-c^2+2c^2\ln c)m] + \\ +\frac{2q}{hg_1}(1-c^2)\frac{\cos(\varphi-\psi)}{\cos\psi}\cos\theta_0. \tag{25.11}$$

If the device of the supports is such that the reactions to them have parallel directions we obtain $\varphi = 0$, $m = 1$; the formulas for the stresses simplify a little.

For a semicircle arc supported at the ends as shown in Fig. 43 and loaded by a uniform normal pressure we have:

$$\varphi = \frac{\pi}{2}, \quad \psi = 0, \quad m = 1,$$

$$\cos(\varphi - \psi) = 0.$$

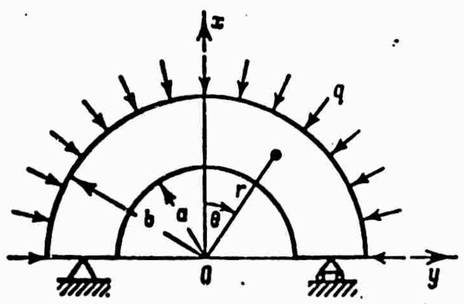

Fig. 43

The stress components will not depend on the angle θ and will be found from the formulas:

$$\left.\begin{aligned} \sigma_r &= \frac{q}{h}\left[P + Q\left(\frac{r}{b}\right)^{k-1} + R\left(\frac{b}{r}\right)^{k+1}\right], \\ \sigma_\theta &= \frac{q}{h}\left[P + Qk\left(\frac{r}{b}\right)^{k-1} - Rk\left(\frac{b}{r}\right)^{k+1}\right], \\ \tau_{r\theta} &= 0. \end{aligned}\right\} \tag{25.12}$$

The coefficients *P*, *Q*, *R* are determined from the formulas (25.5) in which we must put $m = 1$.

It is relatively easy to obtain a solution also for the general case where the curved girder having the form of a part of a plane ring is deformed by normal and tangential forces distributed along the curvilinear sides in an arbitrary manner. Each of the given forces must be espanded in a Fourier series, i.e., represented in the form

$$q_0 + \sum_{n=1}^{\infty}(q_n\cos n\theta + q_n'\sin n\theta). \tag{25.13}$$

The stress function must be sought in the form

$$F = f_0(r) + C\theta + Ar\theta \sin\theta + Br\theta \cos\theta +$$
$$+ \sum_{n=1}^{\infty} [f_n(r) \cos n\theta + f_n^*(r) \sin n\theta]. \qquad (25.14)$$

Each of the functions $f_n(r)$ and $f_n^*(r)$ will be determined from ordinary differential equations which are obtained on the basis of Eq. (12.9) [or (12.11) in the case of an orthotropic girder]. All constants entering expression (25.14) will be found from the conditions on the curvilinear sides and at the ends; the first conditions can always be exactly satisfied, the second one only in an approximate manner.

§26. THE STRESS DISTRIBUTION IN A RING-SHAPED PLATE WITH CYLINDRICAL ANISOTROPY

Let us consider the elastic equilibrium of a plate having the shape of a whole circular concentric ring with cylindrical anisotropy and compressed along the outside and inside outlines by uniformly distributed normal forces. We shall assume that the pole of anisotropy coincides with the ring center and that there are no elements of elastic symmetry except for the planes parallel to the mid plane. Having solved this problem we shall in the same way obtain the solution of an analogous problem concerning the stress distribution in a tube of a material with cylindrical anisotropy, under the action of internal and external pressures. The latter problem for a tube with cylindrical anisotropy of a special form was solved by Saint Venant and Voigt.*

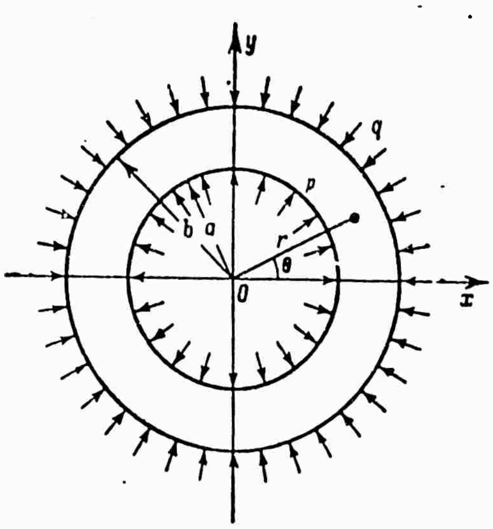

Fig. 44

As was already indicated we consider the more general case of a nonorthotropic ring to which corresponds a nonorthotropic tube.

Choosing the pole of anisotropy (the ring center) to be the orig of coordinates we shall place the polar axis x arbitrarily (Fig. 44). We shall designate by p and q the values of the internal and external pressure per unit of surface and by a and b the inside and outside radii of the ring.

The stresses will be found with the help of the stress function (23.2) independent of θ and represented by the formulas*

$$\left.\begin{aligned} \sigma_r &= \frac{pc^{k+1}-q}{1-c^{2k}}\left(\frac{r}{b}\right)^{k-1} - \frac{p-qc^{k-1}}{1-c^{2k}}c^{k+1}\left(\frac{b}{r}\right)^{k+1}, \\ \sigma_\theta &= \frac{pc^{k+1}-q}{1-c^{2k}}k\left(\frac{r}{b}\right)^{k-1} + \frac{p-qc^{k-1}}{1-c^{2k}}kc^{k+1}\left(\frac{b}{r}\right)^{k+1}, \\ \tau_{r\theta} &= 0. \end{aligned}\right\} \qquad (26.1)$$

Here

$$c = \frac{a}{b}, \quad k = \sqrt{\frac{a_{11}}{a_{22}}} = \sqrt{\frac{E_\theta}{E_r}}. \qquad (26.2)$$

The distribution of the stresses is the same for all radial sections and depends only on the ratio of the Young's moduli for stretching (compression) in the tangential and radial directions. The stress distribution obtained is the same both in an orthotropic and a nonorthotropic ring for which a_{16} and a_{26} are not equal to zero. The influence of the radial planes of elastic symmetry will show up only after deformation: if there are such planes the radial sections remain plane; if there are no such planes the radial sections are distorted.

The displacements of the plate points in radial and tangential directions u_r and u_θ will be found from Eqs. (23.1) expressing the generalized Hooke's law. We present the formulas for the displacements in an orthotropic plate ("rigid" displacements not accompanied by deformation are eliminated):

$$\left.\begin{aligned} u_r &= \frac{b}{E_\theta(1-c^{2k})}\Big[(pc^{k+1}-q)(k-\nu_\theta)\left(\frac{r}{b}\right)^k + \\ &\quad + (p-qc^{k-1})c^{k+1}(k+\nu_\theta)\left(\frac{b}{r}\right)^k\Big], \\ u_\theta &= 0. \end{aligned}\right\} \qquad (26.3)$$

E_θ, ν_θ are here elastic constants for the directions r, θ (the princ pal directions) from the equations of the generalized Hooke's law (24.1).

If the anisotropic material has Young's moduli equal for the ra al and tangential directions the stress distribution obtained is the same as in an isotropic ring. Putting $k = 1$ we obtain the well-known Lame solution from formulas (26.1)*:

$$\left.\begin{aligned} \sigma_r &= \frac{pc^2 - q}{1 - c^2} - \frac{p - q}{1 - c^2} c^2 \left(\frac{b}{r}\right)^2, \\ \sigma_\theta &= \frac{pc^2 - q}{1 - c^2} + \frac{p - q}{1 - c^2} c^2 \left(\frac{b}{r}\right)^2, \\ \tau_{r\theta} &= 0. \end{aligned}\right\} \tag{26.4}$$

One special case is worth mentioning. If we put $c = 0$ and $p = 0$ in formulas (26.1) we obtain the stress distribution in a solid disk with cylindrical anisotropy compressed on the edge by normal forces q:

$$\sigma_r = -q\left(\frac{r}{b}\right)^{k-1}, \quad \sigma_\theta = -qk\left(\frac{r}{b}\right)^{k-1}, \quad \tau_{r\theta} = 0. \tag{26.5}$$

In an isotropic disk the material will be compressed uniformly, but in a curvilinear-anisotropic disk the stresses will vary along the diameter. For disks of materials for which $E_\theta > E_r$, $k > 1$ and the stresses decrease the nearer we get to the center, and in the center become zero. If, however, $E_\theta < E_r$ then $k < 1$ and, as is seen from formulas (26.5), the stresses will tend to infinity with decreasing distance from the center, and, close to the center, i.e., the center of anisotropy, a stress concentration takes place. The curves giving the stres distribution of σ_r and σ_θ along the disk diameter are shown in Fig. 4 for $k = 1$, $k > 1$ and $k < 1$.

The formulas for the stresses in a tube with cylindrical anisotropy under the action of internal and external pressures p and q have the form (26.1), but for a tube

$$k = \sqrt{\frac{\beta_{11}}{\beta_{22}}} = \sqrt{\frac{a_{11}a_{33} - a_{13}^2}{a_{22}a_{33} - a_{23}^2}}. \tag{26.6}$$

Besides, in the cross sections of a tube with fixed ends acts a normal pressure

$$\sigma_z = -\frac{1}{a_{33}}(a_{13}\sigma_r + a_{23}\sigma_\theta). \tag{26.7}$$

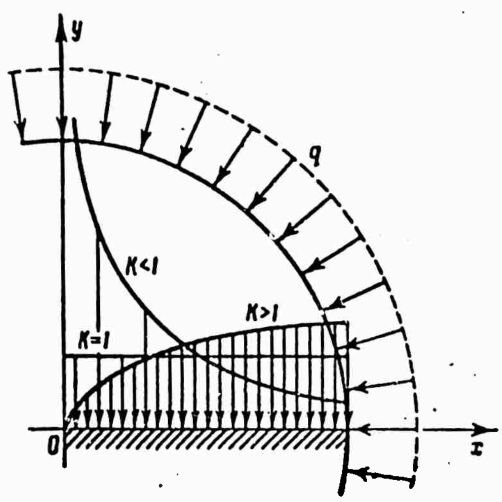

Fig. 45.

With the help of a stress function of the form (25.14) the solution for the general case of elastic equilibrium of a ring may be obtained when arbitrary normal and tangential forces (fulfilling the equilibrium conditions) are distributed along its inside and outside boundaries $r = a$ and $r = b$. To start with, it is necessary to represent the functions expressing the distribution of the normal and tangential forces along the boundaries in terms of Fourier series (25.13). We shall restrict ourselves to these general remarks, all the more since apparently none of the problems of this kind for a curvilinear-anisotropic ring, except for the above-mentioned one, has been finished.

§27. THE STRESS DISTRIBUTION IN A COMPOSITE CURVILINEAR-ANISOTROPIC RING

Let us consider the following problem. Let be given a round plate with a round opening at the center, composed of an arbitrary number of layers having the shape of concentric rings of the same thickness h, and the property of cylindrical anisotropy. We assume that each layer is orthotropic where the poles of anisotropy of all layers are at the center, and the layers are connected with each other in a rigid manner, i.e., they are soldered or glued together at the contact surfaces. Along the boundary of the opening and the outside boundary the normal

forces are uniformly distributed. The stresses in each layer must be determined.

An analog of this problem is the problem of the stress distribution in a multilayer curvilinear-anisotropic tube which is acted upon by internal and external pressure.

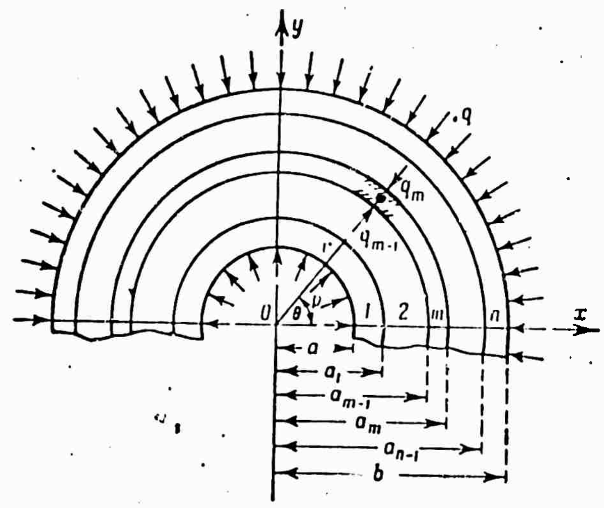

Fig. 46

We shall choose the plate center (the pole of anisotropy) to be the origin of coordinates and place the polar axis x along an arbitrary radius (Fig. 46). We introduce the designations: n is the number of layers; a, b are the inside and outside radii of the whole composite ring; p, q are the internal and external pressures on the unit area; a_{m-1}, a_m are the inside and outside radii of the layer No. m; $\sigma_r^{(m)}$, $\sigma_\theta^{(m)}$, $\tau_{r\theta}^{(m)}$, $u_r^{(m)}$, $u_\theta^{(m)}$ are the stress components and displacement projections and $E_r^{(m)}$, $E_\theta^{(m)}$, $\nu_\theta^{(m)}$ — are the elastic constants (for the principal directions r, θ) from the equations of the generalized Hooke's law of the type (24.1):

$$c_m = \frac{a_{m-1}}{a_m}, \quad k_m = \sqrt{\frac{E_\theta^{(m)}}{E_r^{(m)}}} \qquad (27.1)$$

(the subscript m indicates the number of the layer; m = 1, 2, ..., n;

$a_0 = a,\ a_n = b$).

The stress components satisfy the boundary conditions:

$$\left.\begin{array}{ll}\text{for } r=a & \sigma_r^{(1)}=-p, \quad \tau_{r\theta}^{(1)}=0;\\ \text{for } r=b & \sigma_r^{(n)}=-q, \quad \tau_{r\theta}^{(n)}=0.\end{array}\right\} \tag{27.2}$$

On the contact surfaces of adjacent layers we have the conditions:

$$\left.\begin{array}{ll}\text{for } r=a_{m-1} & \sigma_r^{(m-1)}=\sigma_r^{(m)}, \quad \tau_{r\theta}^{(m-1)}=\tau_{r\theta}^{(m)},\\ & u_r^{(m-1)}=u_r^{(m)}, \quad u_\theta^{(m-1)}=u_\theta^{(m)}.\end{array}\right\} \tag{27.3}$$

It is obvious that the stresses and displacements in each layer will only depend on the distance r where $u_\theta^{(m)} = 0$. Designating by q_{m-1} and q_m the normal forces acting on the inside and outside surfaces of the mth layer we obtain with the help of formulas (26.1)-(26.3) of the preceding section:

$$\left.\begin{aligned}\sigma_r^{(m)} &= \frac{q_{m-1}c_m^{k_m+1}}{1-c_m^{2k_m}}\left[\left(\frac{r}{a_m}\right)^{k_m-1}-\left(\frac{a_m}{r}\right)^{k_m+1}\right]+\\ &\quad+\frac{q_m}{1-c_m^{2k_m}}\left[-\left(\frac{r}{a_m}\right)^{k_m-1}+c_m^{2k_m}\left(\frac{a_m}{r}\right)^{k_m+1}\right],\\ \sigma_\theta^{(m)} &= \frac{q_{m-1}c_m^{k_m+1}k_m}{1-c_m^{2k_m}}\left[\left(\frac{r}{a_m}\right)^{k_m-1}+\left(\frac{a_m}{r}\right)^{k_m+1}\right]-\\ &\quad-\frac{q_mk_m}{1-c_m^{2k_m}}\left[\left(\frac{r}{a_m}\right)^{k_m-1}+c_m^{2k_m}\left(\frac{a_m}{r}\right)^{k_m+1}\right],\\ \tau_{r\theta}^{(m)} &= 0;\end{aligned}\right\} \tag{27.4}$$

$$\begin{aligned}u_r^{(m)} &= \frac{q_{m-1}a_mc_m^{k_m+1}}{E_\theta^{(m)}(1-c_m^{2k_m})}\left[(k_m-\nu_\theta^{(m)})\left(\frac{r}{a_m}\right)^{k_m}+(k_m+\nu_\theta^{(m)})\left(\frac{a_m}{r}\right)^{k_m}\right]-\\ &\quad-\frac{q_ma_m}{E_\theta^{(m)}(1-c_m^{2k_m})}\left[(k_m-\nu_\theta^{(m)})\left(\frac{r}{a_m}\right)^{k_m}+(k_m+\nu_\theta^{(m)})c_m^{2k_m}\left(\frac{a_m}{r}\right)^{k_m}\right]\end{aligned} \tag{27.5}$$

$$(m=1,\ 2,\ \ldots,\ n;\quad q_0=p,\ q_n=q).$$

These expressions satisfy the boundary conditions (27.2) and the first, the second, and the fourth condition of (27.3). Requiring that the radial displacements of adjoining points of neighboring layers be equal we obtain the equations for the determination of the unknown forces q_m:

$$q_{m+1}a_{m+1}\alpha_{m+1}+q_ma_m\beta_m+q_{m-1}a_{m-1}\alpha_m=0 \quad (m=1,\ 2,\ \ldots,\ n-1). \tag{27.6}$$

Here we have used the designations:

$$
\left.\begin{aligned}
\alpha_m &= \frac{2k_m}{E_\theta^m}\,\frac{c_m^{k_m}}{1-c_m^{2k_m}},\\
\beta_m &= \frac{1}{E_\theta^{(m)}}\left(\nu_\theta^{(m)} - k_m\frac{1+c_m^{2k_m}}{1-c_m^{2k_m}}\right) - \frac{1}{E_\theta^{(m+1)}}\times\\
&\times\left(\nu_\theta^{(m+1)} + k_{m+1}\frac{1+c_{m+1}^{2k_{m+1}}}{1-c_{m+1}^{2k_{m+1}}}\right).
\end{aligned}\right\} \qquad (27.7)
$$

Eq. (27.6) is in its idea analogous to the three-moment equation in the theory of solid beams, and it may be called "three-force equation."

Assigning to m successively the values of the integers from 1 to $n-1$ and paying attention to the fact that q_0 and q_n are given and equal to the given pressures p and q, we shall gradually determine all forces q_m, and, at the same time, also the stresses in each layer.

The normal stresses in the radial sections near the inside and outside surfaces of the layer No. m will be found from the formulas

$$
\left(\sigma_\theta^{(m)}\right)_{a_{m-1}} = \frac{q_{m-1}\left(1+c_m^{2k_m}\right) - 2q_m c_m^{k_m-1}}{1-c_m^{2k_m}}\,k_m; \qquad (27.8)
$$

$$
\left(\sigma_\theta^{(m)}\right)_{a_m} = \frac{2q_{m-1}c_m^{k_m+1} - q_m\left(1+c_m^{2k_m}\right)}{1-c_m^{2k_m}}\,k_m. \qquad (27.9)
$$

We shall mention the main results for a ring glued together from two curvilinear-anisotropic rings and loaded by an external pressure only (Fig. 47).

In this case $n = 2$, $q_0 = p$, $q_2 = 0$ and we obtain from Eq. (27.6) only

$$
q_1 = 2pc_1^{k_1+1}k_1\lambda, \qquad (27.10)
$$

where

$$
\left.\begin{aligned}
\lambda &= \frac{1}{\left(1-c_1^{2k_1}\right)\left[k_1\dfrac{1+c_1^{2k_1}}{1-c_1^{2k_1}} - \nu_\theta^{(1)} + \dfrac{E_\theta^{(1)}}{E_\theta^{(2)}}\left(k_2\dfrac{1+c_2^{2k_2}}{1-c_2^{2k_2}} + \nu_\theta^{(2)}\right)\right]},\\
c_1 &= \frac{a}{a_1}, \quad c_2 = \frac{a_1}{b}, \quad k_1 = \sqrt{\frac{E_\theta^{(1)}}{E_r^{(1)}}}, \quad k_2 = \sqrt{\frac{E_\theta^{(2)}}{E_r^{(2)}}}.
\end{aligned}\right\} \qquad (27.11)
$$

From formulas (27.8) and (27.9) we obtain the following values of the normal pressure σ_θ near the inside surface $r = a$, the contact surface $r = a_1$ and the outside surface $r = b$:

$$\left(\sigma_\theta^{(1)}\right)_a = pk_1 \frac{1 + c_1^{2k_1} - 4k_1\lambda c_1^{2k_1}}{1 - c_1^{2k_1}}, \tag{27.12}$$

$$\left(\sigma_\theta^{(1)}\right)_{a_1} = 2pk_1c_1^{k_1+1} \frac{1 - k_1\lambda\left(1 + c_1^{2k_1}\right)}{1 - c_1^{2k_1}}, \tag{27.13}$$

$$\left(\sigma_\theta^{(2)}\right)_{a_1} = 2pk_1k_2c_1^{k_1+1}\lambda \frac{1 + c_2^{2k_2}}{1 - c_2^{2k_2}}, \tag{27.14}$$

$$\left(\sigma_\theta^{(2)}\right)_b = 4pk_1k_2\lambda \frac{c_1^{k_1+1}c_2^{k_2+1}}{1 - c_2^{2k_2}}. \tag{27.15}$$

The maximum stress for the whole composite (double) ring will be found from one of formulas (27.12)-(27.15), but from which cannot be said beforehand. Finally, the problem of the maximum stress can be solved only by giving the numerical values of the ratios of the moduli of elasticity and the radii.

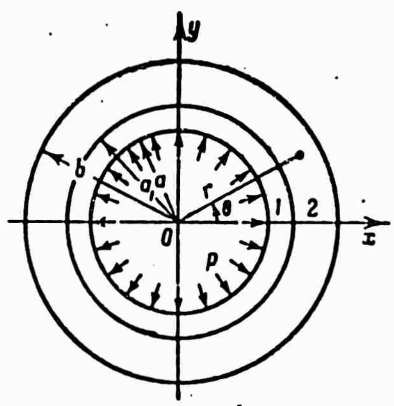

Fig. 47

If the layers from which the ring is composed are not orthotropic the stresses will be determined by the same formulas (27.4), (27.8), (27.9) or (27.12)-(27.15), but in this case the quantities $\frac{1}{E_r}$, $\frac{1}{E_\theta}$, $-\frac{\nu_\theta}{E_\theta}$ must be replaced by the constants a_{11}, a_{22}, and a_{12}, respectively, from the equations of the generalized Hooke's law (23.1). In a nonorthotropic ring the displacements u_θ are not equal to zero, and the radial sections will be distorted for this reason.

The formulas for the stresses in a composite multilayer tube (case of plane deformation) are obtained from the formulas for the stresses in a ring by replacing the constants a_{ij} in the latter, respectively, by the quantities

$$\beta_{ij}=a_{ij}-\frac{a_{i3}a_{j3}}{a_{33}} \quad (i,\ j=1,\ 2,\ 6).$$

§28. THE STRESS DISTRIBUTION IN A RING WITH VARIABLE MODULI OF ELASTICITY

Let us assume that with the curvilinear-anisotropic ring shown in Fig. 44 which is deformed by uniformly distributed pressures p and q the moduli of elasticity are not constant, but arbitrary functions of the distance r. We consider the behavior of the solution of the problem if an orthotropic ring experiencing small deformations.

If the radial planes are planes of elastic symmetry it is obvious that the stress distribution will only depend on r in which case all points will be displaced along the radii in the deformation. Consequently, $u_\theta=0,\ u_r=u_r(r),\ \gamma_{r\theta}=0,$ and from the third equation of the generalized Hooke's law $\tau_{r\theta} = 0$. For the three unknown functions σ_r, σ_θ and u_r we obtain three equations (in which the primes denote the derivatives with respect to the sole variable r):

$$\sigma_r'+\frac{\sigma_r-\sigma_\theta}{r}=0; \tag{28.1}$$

$$\left.\begin{aligned} u_r'&=\frac{1}{E_r}\sigma_r-\frac{\nu_\theta}{E_\theta}\sigma_\theta,\\ \frac{u_r}{r}&=-\frac{\nu_r}{E_r}\sigma_r+\frac{1}{E_\theta}\sigma_\theta. \end{aligned}\right\} \tag{28.2}$$

$E_r,\ E_\theta,\ \nu_r,\ \nu_\theta$ — are, respectively, Young's moduli and the Poisson coefficients for the principal directions of elasticity (of radial r and tangential θ) of the function of the variable r. Eliminating the displacement u_r from Eqs. (28.2) we obtain:

$$\frac{\sigma_r}{E_r}-\frac{\nu_\theta\sigma_\theta}{E_\theta}+\left(\frac{\nu_r r}{E_r}\sigma_r\right)'-\left(\frac{r}{E_\theta}\sigma_\theta\right)'=0. \tag{28.3}$$

The stress function does not depend on θ: $F = f_0(r)$; consequently,

$$\sigma_r=\frac{f_0'}{r},\qquad \sigma_\theta=f_0''. \tag{28.4}$$

Substituting these expressions into (28.3) we obtain the equation for

the function f_0:

$$f_0''' + \left(\frac{1}{r} - \frac{E_\theta'}{E_\theta}\right) f_0'' + \left(\frac{\nu_0 E_\theta'}{r E_0} - \frac{\nu_0'}{r} - \frac{E_\theta}{r^2 E_r}\right) f_0' = 0. \qquad (28.5)$$

The general integral of this equation with variable coefficients has the form

$$f_0' = A\varphi_1(r) + B\varphi_2(r), \qquad (28.6)$$

where φ_1, φ_2 are linearly independent special solutions to Eq. (28.5), and A and B are arbitrary constants; hence

$$\left.\begin{aligned} \sigma_r &= A\frac{\varphi_1}{r} + B\frac{\varphi_2}{r}, \\ \sigma_\theta &= A\varphi_1' + B\varphi_2'. \end{aligned}\right\} \qquad (28.7)$$

The constants A and B will be determined from the boundary conditions:

$$\left.\begin{aligned} &\text{for } r = a \quad \sigma_r = -p; \\ &\text{for } r = b \quad \sigma_r = -q. \end{aligned}\right\} \qquad (28.8)$$

To determine the special solutions φ_1 and φ_2 we must know how E_r, E_θ and ν_0 depend on r. Particularly simple will the determination of these special solutions be in the case where the Poisson coefficients are constant, and Young's moduli vary along the radius according to a power law:

$$\left.\begin{aligned} &E_r = E_{rm} r^m, \quad E_\theta = E_{\theta m} r^m, \\ &\nu_\theta = \text{const}, \quad \nu_r = \nu_0 \frac{E_{rm}}{E_{\theta m}}, \end{aligned}\right\} \qquad (28.9)$$

where m is an arbitrary real number, positive or negative, integer or fraction. In this case, Eq. (28.5) is integrated with elementary functions, and we obtain:

$$\varphi_1 = r^{n_1}, \quad \varphi_2 = r^{-n_2}, \qquad (28.10)$$

where

$$\left.\begin{aligned} n_1 &= \frac{1}{2}\left[\sqrt{m^2 + 4(k^2 - m\nu_0)} + m\right], \\ n_2 &= \frac{1}{2}\left[\sqrt{m^2 + 4(k^2 - m\nu_0)} - m\right], \\ k &= \sqrt{\frac{E_\theta}{E_r}} = \sqrt{\frac{E_{\theta m}}{E_{rm}}}. \end{aligned}\right\} \qquad (28.11)$$

Designating by c the ratio of the radii: $c = a/b$ we obtain the

following stress distribution:

$$
\left.\begin{aligned}
\sigma_r &= \frac{pc^{n_2+1}-q}{1-c^{n_1+n_2}}\left(\frac{r}{b}\right)^{n_1-1} - \frac{p-qc^{n_1-1}}{1-c^{n_1+n_2}}\,c^{n_1+1}\left(\frac{b}{r}\right)^{n_2+1},\\
\sigma_\theta &= \frac{pc^{n_2+1}-q}{1-c^{n_1+n_2}}\,n_1\left(\frac{r}{b}\right)^{n_1-1} + \frac{p-qc^{n_1-1}}{1-c^{n_1+n_2}}\,c^{n_2+1}n_2\left(\frac{b}{r}\right)^{n_2+1},\\
\tau_{r\theta} &= 0.
\end{aligned}\right\} \tag{28.12}
$$

For constant moduli (m = 0) we have $n_1 = n_2 = k$, and we obtain the formulas (26.1) derived above from (28.12).

With somewhat different denotations Eq. (28.5) was obtained in a work by P.N. Zhitkov.* This author considered also the special cases where Young's moduli and the Poisson coefficients are linear functions of the distance r and they are represented by exponential functions (in both cases the ratio E_θ/E_r is assumed constant). P.N. Zhitkov shows that the moduli of elasticity depend on the distance r in this way for pressed wood (page 2(of his work). In both cases Eq. (28.5) assumes a rather complex form and its solutions are expressed in terms of hypergeometrical series or degenerate hypergeometrical functions.

Manu-script Page No.	[Footnotes]
70	See the work: Lekhnitskiy, S.G., Nekotoryye sluchai ploskoy zadachi teorii uprugosti anizotropnogo tela [Several Cases of the Plane Problem of the Theory of Elasticity of an Anisotropic Body], Sbornik "Eksperimental'nyye metody opredeleniya napryazheniy i deformatsiy v uprugoy i plasticheskoy zonakh" [Collection "Experimental Methods of Determining the Stresses and Strains in the Elastic and Plastic Zones"], ONTI [United Scientific and Technical Publishing Houses], 1935, pages 158-161. In this work a coordinate system with different directions of the axes is used in studying the bending of a console.
71	See, e.g., Timoshenko, S.P., Teoriya uprugosti [Theory of Elasticity], ONTI, 1937, page 45.
72	E. Reissner, A contribution to the theory of elasticity of nonisotropic material (with application to problems of bend-

ing and torsion), Philosophical Magazine, ser. 7, Vol. 30, No. 202, 1940.

72** See our work mentioned in the preceding section (pages 161-164).

74 See, e.g., Timoshenko, S.P., Teoriya uprugosti, ONTI, 1937, page 50.

76 The solutions set forth in this section we obtained by Ye.V. Orlova in the diploma work "Izgib anizotropnykh balok s secheniyem v forme uzkogo nryamoygol'nika popoechnoy nagruzkoy, raspredelennoy po lineynomu i parabolicheskomu zakonu" [The Bending of Anisotropic Beams with a Cross Section Having the Form of a Narrow Rectangle Under the Action of a Transverse Load Distributed According to Linear and Parabolic Laws], (Saratovskiy gos. un-t) [Saratov State University] 1948.

80* Kurdyumov, A.A., O reshenii v polinomakh ploskoy zadachi teorii uprugosti dlya pryamougol'noy anizotropnoy polosy [On the Solution of the Plane Problem of the Theory of Elasticity in Polynomials for a Rectangular Anisotropic Band], Prikladnaya matematika i mekhanika [Applied Mathematics and Mechanics], Vol. 9, No. 4, 1945.

80** Vorob'yev, L.N., Ob odnom reshenii ploskoy zadachi v polinomakh dlya pryamougol'noy plastinki [On One Solution to the Plane Problem in Polynomials for a Rectangular Orthotropic Plate DAN USSR [Proceedings of the Academy of Sciences of the Ukrainian Socialist Soviet Republic], 1954, No. 5.

80*** Problems on the bending of an isotropic beam with the help of Fourier series were solved by Ribiere, Filon, Bleich, etc. (Literature on this problem is given in the textbook by S.P. Timoshenko, Teoriya urpugosti, ONTI, 1937, pages 56-63.

83* Kufarev, P.P. and Sveklo, V.A., Opredeleniye napryazheniy v anizotropnoy polose [The Determination of Stresses in an Anisotropic Band], DAN SSSR [Proceedings of the Academy of Sciences of the USSR], Vol. 32, No. 9, 1941.

83** See our work "On the Calculation of the Strength of Composite Beams," Vestnik inzhenerov i tekhnikov [Herald for Engineers and Technicians], 1935, No. 9.

92 We note that all formulas for a beam with moduli of elasticity continuously varying with the height may also be obtained in another way: by passing to the limit of the formulas for a beam composed of bands with the same height, making the number of bands infinitely great.

95 See the work: Lekhnitskiy, S.G., Nekotoryye sluchai ploskoy zadachi teorii uprugosti anizotropnogo tela [Several Cases of the Plane Problem of the Theory of Elasticity of an Anisotropic Body], Sbornik "Eksperimental'nyye metody opredeleniya napryazheniy i deformatsiy v uprugoy i plasticheskoy zonakh" [Collection "Experimental Methods of Determining the Stresses

and Strains in the Elastic and Plastic Zones"], ONTI [United Scientific and Technical Publishing Houses], 1935, pages 174-179. In this work the solution of the problem is carried out with all details, but with somewhat different denotations.

98 See our work mentioned in the preceding section where this problem is worked out in a detailed manner.

101 See our work mentioned in §20.

103* Abramov, V.M., Raspredeleniye napryazheniy v ploskom bezgranichnom kline pri proizvol'noy nagruzke [The Stress Distribution in a Plane Infinite Wedge in the Case of an Arbitrary Load], Trudy konferentsii po opticheskomu metodu izucheniya napryazheniy NIIMM LGU i NIIMekh MGU [Transactions of the Conference on the Optical Method of Studying Stresses of the Scientific Research Institute for Mathematics and Mechanics of the Leningrad State University and the Scientific Research Institute for Mechanics of the Moscow State University], ONTI, 1937.

103** Kufarev, P.P., Opredeleniye napryazheniy v anizotropnom kline [The Determination of Stresses in an Anisotropic Wedge], DAN SSSR, Vol. 32, No. 8, 1941.

105 The solution of this problem is carried out in the work: Lekhnitskiy, S.G., Ploskaya zadacha teorii uprugosti dlya te tela s tsilindricheskoy anizotropiye [The Plane Problem of the Theory of Elasticity for a Body with Cylindrical Anisotropoy], Uch. zap. Saratovskogo un-ta [Scientific Reports of the Saratov University], vol. 1 (14), Series for Physics and Mathematics, No. 2, 1938.

106 S.P. Timoshenko, Teoriya uprugosti, ONTI, 1937, page 72.

107* See our work mentioned in §23[in the formulas (7.8) of this work on page 151 there is a misprint].

107** If the section perpendicular to the line of action of the force leaves the limits of the girder (for small angles between the end sections) the maximum normal stresses will be obtained at the point of fixation.

108* In the textbook on the theory of elasticity by S.P. Timoshenko which we have mentioned several times the solution for the case of a radial force directed inward is given ($\omega = \pi$, pages 84-86).

108** Kosmodamianskiy, A.S., Izgib ploskogo krivolineynogo anizotropnogo brusa siloy prilozhennoy na kontse [The Bending of a Plane Curvilinear Anisotropic Girder by a Force Applied to the End], Prikladnaya matematika i mekhanika, Vol. 16, No. 2, 1952.

113 See the works: 1) de Saint-Venant, B., Memoire sur les divers genres d'homogeneite des corps solides [Report on the Various Kinds of Homogeneity of Solid Bodies], Journal de Math. pures

et appl. (Liouville) [Journal of Pure and Applied Mathematics (Liouville)], Vol. 10, 1865; 2) Voigt, W., Ueber die Elastizitaetsverhaeltnisse cylindrisch aufgebauter Koerper [On the Elastic Properties of Cylindrical Bodies], Nachrichten v.d. Koenigl. Gesellschaft der Wissenschaften und d. Georg-Augustin [Bulletin of the Royal Scientific Society and the Georg-Augustin University at Goettingen], 1886, No. 16

A.N. Mitinskiy determined the stresses in a wooden tube by regarding it as a body with cylindrical anisotropy (Mitinskiy, A.N., Raschet napryazheniy v derevyannoy sverlenoy trube [The Calculation of Stresses in a Bored Wooden Tube], Vestnik inzhenerov i tekhnikov, 1936, No. 5.

114 See our work mentioned in §23.

115 Timoshenko, S.P., Teoriya uprugosti, ONTI, 1937, page 69.

123 Zhitkov, P.N., Ploskaya zadacha teorii uprugosti neodnorodnogo ortotropnogo tela v polyarnykh koordinatakh [The Plane Problem of the Theory of Elasticity of an Inhomogeneous Orthotropic Body in Polar Coordinates], Trudy Voronezhskogo gos. un-ta [Transactions of the Voronezh State University], Vol. 27, Physical-Mathematical Collection, 1954.

Chapter 4

THE STRESS DISTRIBUTION IN AN INFINITE ELASTIC MEDIUM

§29. ELASTIC SEMIPLANE LOADED ALONG THE BOUNDARY

In this chapter we shall consider the stress distribution in a plane elastic medium with straight boundary loaded along the boundary ("elastic semiplane") as well as in an infinite medium whose section is limited by a parabola and a hyperbola. Besides, we shall consider the problem of the elastic equilibrium of an unlimited plane medium under the action of concentrated force and moment. For the sake of definiteness, we shall deal with a generalized plane stressed state; all results obtained are also transferred to the case of plane deformation.

Let us consider an elastic homogeneous anisotropic plate with a straight edge along which forces acting in the mid plane are distributed. If the dimensions of the plate in the direction of the straight edge and in other directions are great compared to the length of the loaded part of the edge then it is possible to get an idea of the stress distribution by simplifying the problem, i.e., considering the plate to be an infinite plane elastic medium with a straight boundary, in other words, an elastic semiplane.

There are several methods of solving the problem of stress distribution in an elastic semiplane. The first method which is based on the use of Fourier integrals is particularly convenient in the case of an orthotropic semiplane. We shall set forth briefly its principle and the way of its application.*

We shall choose the following restrictions: 1) the medium is orthotropic, in which case the directions parallel and perpendicular to the straight boundary are the principal directions; 2) the load is applied to the end part of the boundary, distributed symmetrically with respect to the center of the loaded part and results in a finite resultant.**

We shall choose the center of the loaded part of the boundary to be the origin of coordinates, place the y axis along the boundary and the x axis into the semiplane (Fig. 48). We shall designate the normal and tangential components of the load referred to unit length by $N(y)$ and $T(y)$; N will be an even function of y, and T an odd function. The stress function satisfies the equation

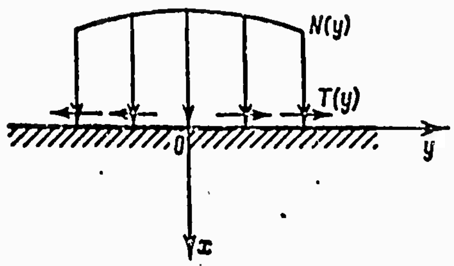

Fig. 48

$$\frac{1}{E_2}\cdot\frac{\partial^4 F}{\partial x^4}+\left(\frac{1}{G}-\frac{2\nu_1}{E_1}\right)\frac{\partial^4 F}{\partial x^2\,\partial y^2}+ +\frac{1}{E_1}\cdot\frac{\partial^4 F}{\partial y^4}=0 \qquad (29.1)$$

(volume forces are not considered).

We shall represent the functions N and T in terms of Fourier integrals; we obtain:*

$$N(y)=\frac{2}{\pi}\int_0^\infty \psi(\alpha)\cos\alpha y\,d\alpha,\quad T(y)=\frac{2}{\pi}\int_0^\infty \chi(\alpha)\sin\alpha y\,d\alpha, \qquad (29.2)$$

where

$$\psi(\alpha)=\int_0^\infty N(\eta)\cos\alpha\eta\,d\eta,\quad \chi(\alpha)=\int_0^\infty T(\eta)\sin\alpha\eta\,d\eta. \qquad (29.3)$$

With increasing distance from the boundary the stresses must tend to zero, and at the boundary itself the following conditions must be satisfied:

for $x = 0$
$$\sigma_x=-\frac{1}{h}N(y),\quad \tau_{xy}=-\frac{1}{h}T(y) \qquad (29.4)$$

(as always, h is the plate thickness).

The solution will be obtained with the help of a stress function of the form

$$F=\int^\infty \Phi(\alpha,\ x)\cos\alpha y\,d\alpha. \qquad (29.5)$$

The form of the function $\Phi(\alpha, x)$ depends on the roots of the equation

$$\frac{u^4}{E_2}-\left(\frac{1}{G}-\frac{2\nu_1}{E_1}\right)u^2+\frac{1}{E_1}=0. \qquad (29.6)$$

Denoting these roots by $\pm u_1, \pm u_2$ and comparing (29.6) with Eq. (7.5) which is satisfied by the complex parameters μ_1 and μ_2 we note that

$$u_1 = \frac{i}{\mu_1}, \quad u_2 = \frac{i}{\mu_2}, \tag{29.7}$$

and, therefore, u_1 and u_2 cannot be purely imaginary numbers.

The following three cases are possible:

Case 1. The roots of Eq. (29.6) are real and unequal:

$$\pm u_1, \ \pm u_2 \qquad (u_1 > 0, \ u_2 > 0).$$

Case 2. The roots are real and in pairs equal to one another:

$$\pm u \qquad (u > 0).$$

Case 3. The roots are complex

$$u \pm vi, \ -u \pm vi \qquad (u > 0, \ v > 0).$$

In Case 1

$$\Phi(\alpha, x) = Ae^{-u_1 \alpha x} + Be^{-u_2 \alpha x} \tag{29.8}$$

(A, B are arbitrary constants; terms becoming infinitely large with increasing x are discarded in this expression and in the two following ones).

In Case 2

$$\Phi(\alpha, x) = (A + Bx)e^{-u\alpha x}. \tag{29.9}$$

In Case 3

$$\Phi(\alpha, x) = (A \cos v\alpha x + B \sin v\alpha x)\, e^{-u\alpha x}. \tag{29.10}$$

The coefficients A and B depend on the parameter α. Fulfilling the boundary conditions (29.4) we obtain for Case 1:

$$\left.\begin{aligned}
\sigma_x &= \frac{2}{\pi h (u_1 - u_2)} \int_0^\infty [\psi(\alpha)(u_2 e^{-u_1 \alpha x} - u_1 e^{-u_2 \alpha x}) - \\
&\qquad - \chi(\alpha)(e^{-u_1 \alpha x} - e^{-u_2 \alpha x})] \cos \alpha y \, d\alpha, \\
\sigma_y &= \frac{2}{\pi h (u_1 - u_2)} \int_0^\infty [\psi(\alpha)\, u_1 u_2 (-u_1 e^{-u_1 \alpha x} + u_2 e^{-u_2 \alpha x}) + \\
&\qquad + \chi(\alpha)(u_1^2 e^{-u_1 \alpha x} - u_2^2 e^{-u_2 \alpha x})] \cos \alpha y \, d\alpha, \\
\tau_{xy} &= \frac{2}{\pi h (u_1 - u_2)} \int_0^\infty [\psi(\alpha)\, u_1 u_2 (e^{-u_1 \alpha x} - e^{-u_2 \alpha x}) - \\
&\qquad - \chi(\alpha)(u_1 e^{-u_1 \alpha x} - u_2 e^{-u_2 \alpha x})] \sin \alpha y \, d\alpha.
\end{aligned}\right\} \tag{29.11}$$

In order to be able to calculate stresses for a given load distribution with the help of these formulas we must calculate the integrals $\psi(\alpha)$ and $\chi(\alpha)$ and, on substitution of the values found into (29.11), carry out the integration. If the distribution law of the load is simple the calculation of the integrals does not encounter on particular difficulties.

The formulas for the two other cases of roots may be obtained from (29.11) by passing over to the limit, putting $u_1 \to u,\ u_2 \to u$ or (in Case 3) replacing u_1 and u_2 by the quantities $u + vi$ and $u - vi$.

The solution for the case of an antisymmetric load where $N(y)$ is an odd and $T(y)$ an even function of y is found in a completely analogous manner with the help of the stress function

$$F = \int_0^\infty \Psi(\alpha, x) \sin \alpha y \, d\alpha. \tag{29.12}$$

The function Ψ has exactly the same structure as the function Φ; according to which case of roots is under consideration it will be determined by Formula (29.8), (29.9) or (29.10).

Another method of solving the problem for the semiplane is based on the use of several properties of Cauchy integrals and is a generalization of the well-known method of Academician N.I. Muskhelishvili to the case of an anisotropic body. By this method we were the first to obtain the solution of the problem of stress distribution in an elastic anisotropic semispace in which the state of the so-called generalized plane deformation is realized (where the planes of elastic symmetry parallel to xy are absent). The solution of the plane problem is obtained automatically from the solution for the generalized plane deformation if the strain coefficients $a_{14}, a_{15}, a_{24}, a_{25}, a_{34}, a_{35}, a_{46}, a_{56}$ are put equal to zero in the latter. This solution will be presented without derivation.*

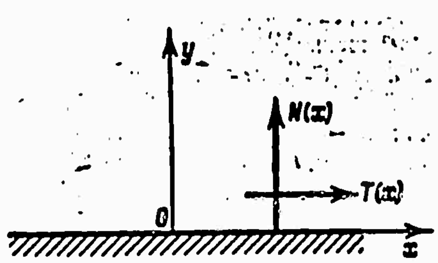

Fig. 49

Assuming that the elastic semiplane is not orthotropic in the general case we will refer it to a coordinate system in which the x axis is directed along the boundary, and the y axis outward, as shown in Fig. 49. We shall designate by $N(x)$ and $T(x)$ the normal and tangential components of the load (per unit length). We shall assume that the resulting vector of the forces distributed along an arbitrary section of the boundary is finite and tends to a certain limit if the ends of the section move to infinity; as to the rest, the force distribution may be completely arbitrary.

The functions $\Phi_1'(z_1)$ and $\Phi_2'(z_2)$ in terms of which the stresses are expressed [see (8.2)] will be determined from the formulas:

$$\left.\begin{aligned}\Phi_1'(z_1) &= \frac{1}{\mu_1-\mu_2}\cdot\frac{1}{2\pi ih}\int_{-\infty}^{\infty}\frac{\mu_2 N(\xi)+T(\xi)}{\xi-z_1}\,d\xi,\\ \Phi_2'(z_2) &= -\frac{1}{\mu_1-\mu_2}\cdot\frac{1}{2\pi ih}\int_{-\infty}^{\infty}\frac{\mu_1 N(\xi)+T(\xi)}{\xi-z_2}\,d\xi.\end{aligned}\right\} \quad (29.13)$$

G.N. Savin proposed another method of solving the problem considered based on the Schwartz formula which is known from the theory of functions of a complex variable; it expressed an analytic function in terms of its real part given on the contour.*

All methods indicated may be used to determine the stresses in a semiplane and in those cases where the displacements (second fundamental problem) are given at the boundary rather than the forces; in the general case, the procedure of solution remains the same as in the case of given forces.

Somewhat more complex is the situation in those cases where partly forces and partly displacements are given at the boundary (problems of mixed type). The problems of the action of one or several rigid stamps on an elastic semiplane, e.g., belong to them.

The solutions of a number of problems of this type were obtained by G.N. Savin** and L.A. Galin.***

The problem of the contact of an elastic semiplane and an elastic cylinder (as a special case of the problem of the contact of two elastic anisotropic bodies) without and with account taken of the frictional forces was considered by G.N. Savin and D.V. Grilitskiy.**** M. Sokolovskiy gave the solution of the problem of the contact of an infinitely long elastic band and an elastic anisotropic semiplane (an infinite beam on an elastic base, bent by a normal force). It is assumed that there are no frictional forces on the contact surface and the band cannot be detached from the semiplane. The solution is obtained with the help of Fourier integrals.*****

§30. THE ACTION OF A CONCENTRATED FORCE AND MOMENT APPLIED TO THE BOUNDARY

If we want to obtain the stress distribution due to a normal concentrated force P applied to the point O on a straight boundary of an infinite elastic medium (Fig. 50) we shall first consider a normal load distributed uniformly along a small section of the boundary having the length 2ε around the point O with a resultant equal to P.

We shall restrict ourselves to the case of an orthotropic semiplane in which the principal directions of elasticity are parallel and perpendicular to the boundary. Using the first method set forth in §29 we substitute the following values of the load components into Formulas (29.11):

$$N = \frac{P}{2\varepsilon} \quad \text{for} \quad -\varepsilon < y < \varepsilon,$$
$$N = 0 \quad \text{for} \quad -\infty \leqslant y < -\varepsilon \text{ и } \varepsilon < y \leqslant \infty;$$
$$T = 0.$$

All integrals entering (29.11) are easily computed; the results will depend on ε. Letting then ε pass to zero we obtain the following formulas for the stresses due to a concentrated force:*

$$\left.\begin{aligned} \sigma_x &= -\frac{P(u_1+u_2)}{\pi hk} \cdot \frac{x^3}{(y^2+u_1^2x^2)(y^2+u_2^2x^2)}, \\ \sigma_y &= -\frac{P(u_1+u_2)}{\pi hk} \cdot \frac{xy^2}{(y^2+u_1^2x^2)(y^2+u_2^2x^2)}, \\ \tau_{xy} &= -\frac{P(u_1+u_2)}{\pi hk} \cdot \frac{x^2y}{(y^2+u_1^2x^2)(y^2+u_2^2x^2)} \\ &\left(k = \sqrt{\frac{E_1}{E_2}}\right). \end{aligned}\right\} \quad (30.1)$$

This solution is derived for Case 1 where the roots of Eq. (29.6) are real and unequal. The solution for Case 2 is obtained from (30.1) by putting $u_1 = u_2 = u$, and the solution for Case 3 by putting $u_1 = \alpha + vi$, $u_2 = \alpha - vi$.

Passing over to polar coordinates we obtain:

$$\left.\begin{aligned} \sigma_r &= -\frac{P(u_1+u_2)}{\pi h\sqrt{E_1E_2}} \cdot \frac{\cos\theta}{rL(\theta)}, \\ \sigma_\theta &= \tau_{r\theta} = 0. \end{aligned}\right\} \quad (30.2)$$

Here

$$L(\theta) = \frac{\cos^4\theta}{E_1} + \left(\frac{1}{G} - \frac{2\nu_1}{E_1}\right)\sin^2\theta\cos^2\theta + \frac{\sin^4\theta}{E_2}. \quad (30.3)$$

As we have already remarked the function $L(\theta)$ has a certain physical meaning — it is a quantity which is the reciprocal value of Young's modulus for stretching (compression) in radial direction: $L = 1/E_r$.

Formulas (30.2) show that the stress distribution is "radial" or "ray-like"; this was, of course, to be expected since the semiplane may be regarded as a wedge with a top angle of π. The stress σ_r which is the principal one, decreases inversely proportionally to the distance r and for given r = const varies with varying angle θ according to a rather complex law.

The points at which the stress σ_r has the same value σ_0, positive and negative, lie on fourth-order curves; the equation of the family of these curves has the form:

$$\frac{x^4}{E_1} + \left(\frac{1}{G} - \frac{2\nu_1}{E_1}\right)x^2y^2 + \frac{y^4}{E_2} + \frac{P(u_1+u_2)}{\pi h\sigma_0\sqrt{E_1E_2}}\,x(x^2+y^2) = 0. \quad (30.4)$$

These curves are all closed, symmetric with respect to the line of action of the force (the x axis) and touch the boundary of the semiplane $x = 0$ at the point O where the force is applied.

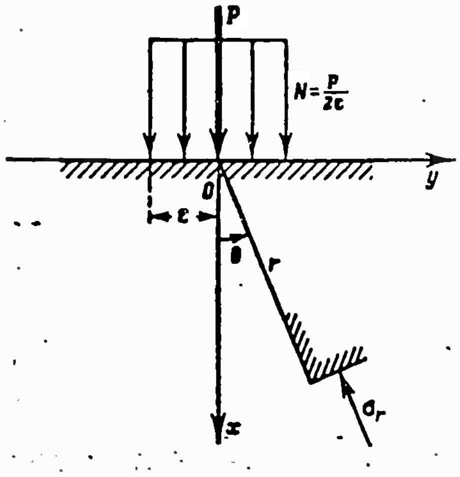

Fig. 50

The investigation of Eq. (30.2) and Eq. (30.4) leads to the following results:

The stress σ_r attains its maximum absolute value on the line of action of the force $\theta = 0$ if

$$\frac{1}{G}-\frac{2\nu_1}{E_1}>\frac{3}{2}\cdot\frac{1}{E_1} \qquad \text{(a)}$$

or if at the same time

$$\frac{1}{G}-\frac{2\nu_1}{E_1}=\frac{3}{2}\cdot\frac{1}{E_1} \qquad \text{(b)}$$

and

$$\frac{8}{E_2}-\frac{3}{E_1}\geqslant 0. \qquad \text{(b')}$$

If, besides Conditions (a) or (b) and (b') the medium constants satisfy the two other conditions:

$$\frac{1}{G}-\frac{2\nu_1}{E_1}>\frac{4}{E_2}, \qquad \text{(c)}$$

$$\left(\frac{1}{G}-\frac{4+2\nu_2}{E_2}\right)^2-\frac{4}{E_2}\left(\frac{2}{G}-\frac{3+4\nu_1}{E_1}\right)>0, \qquad \text{(c')}$$

then, besides the direction of the force action, there are two other directions for which $|\sigma_r|$ attains maximum values; the angles determining these "dangerous" directions in the medium will be found from the equation

$$\operatorname{tg}\theta=\pm\sqrt{\frac{E_2}{2}}\sqrt{\frac{1}{G}-\frac{4+2\nu_2}{E_2}+\sqrt{\left(\frac{1}{G}-\frac{4+2\nu_2}{E_2}\right)^2-\frac{4}{E_2}\left(\frac{2}{G}-\frac{3+4\nu_1}{E_1}\right)}}. \qquad (30.5)$$

If both Conditions (c) and (c'), or even one of them is not fulfilled the maximum of $|\sigma_r|$ on the line of action of the force will be the only one.

The two angles θ for which $|\sigma_r|$ attains its maximum value will be found from the equation

$$\operatorname{tg}\theta=\pm\sqrt{\frac{E_2}{2}}\sqrt{\frac{1}{G}-\frac{4+2\nu_2}{E_2}-\sqrt{\left(\frac{1}{G}-\frac{4+2\nu_2}{E_2}\right)^2-\frac{4}{E_2}\left(\frac{2}{G}-\frac{3+4\nu_1}{E_1}\right)}}. \qquad (30.6)$$

If the elastic constants of the medium fulfill the condition

$$\frac{1}{G}-\frac{2\nu_1}{E_1}<\frac{3}{2}\cdot\frac{1}{E_1} \qquad (d)$$

or simultaneously fulfill the two conditions:

$$\frac{1}{G}-\frac{2\nu_1}{E_1}=\frac{3}{2}\cdot\frac{1}{E_1}, \qquad (e)$$

$$\frac{8}{E_2}-\frac{3}{E_1}<0, \qquad (e')$$

then the stress σ_r on the line of action of the force attains a minimum rather than a maximum. In this case there are two angles corresponding to the maximum of $|\sigma_r|$; they are found from Eq. (30.5).

In accordance with these results three types of lines of equal stresses may be noted for different elastic orthotropic media. The curves of the first are obtained in a medium in which the elastic constants satisfy Conditions (a) or (b) and (b'), but do not satisfy Conditions (c) and (c'); they are shown in Fig. 51. The curves of the second type occur if Conditions (a) [or (b) and (b')], (c) and (c') are simultaneously satis ied (Fig. 52). The side branches correspond to the dangerous directions for which $|\sigma_r|$ attains additional maxima. The curves of the third type shown in Fig. 53 are obtained in a medium with elastic constants satisfying Condition (d) or Conditions (e) and (e').

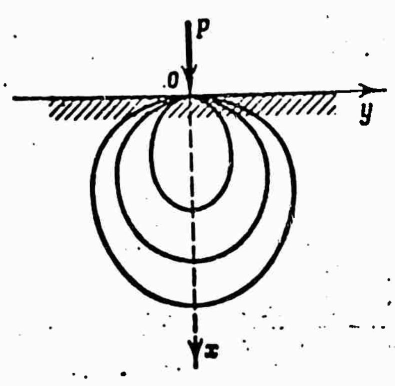

Fig. 51

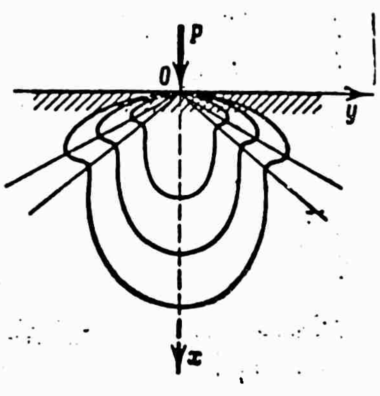

Fig. 52

The lines of equal stresses make it possible to get a rather clear notion on the way in which the stresses in a semiplane due to a concentrated force are distributed. In special cases of anisotropy the curves may degenerate to ellipses and even circles.

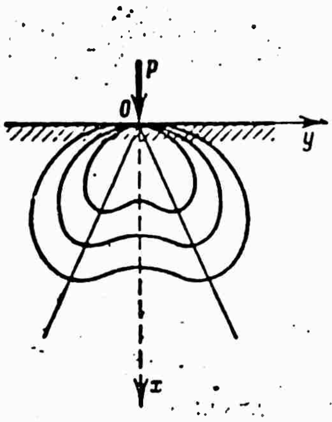

Fig. 53

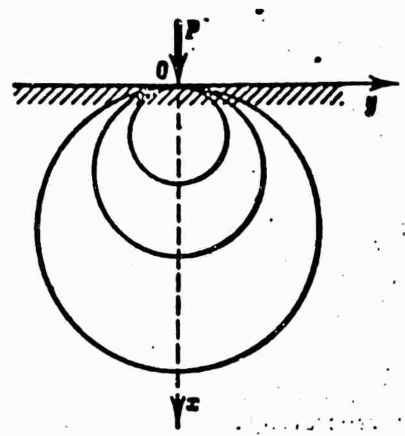

Fig. 54

In the case of an orthotropic medium $E_1 = E_2 = E$, $L = 1/E$, $u_1 = u_2 = 1$ and we obtain the well-known solution of Flaman:*

$$\sigma_r = -\frac{2P}{\pi h}\cdot\frac{\cos\theta}{r}, \quad \sigma_\theta = \tau_{r\theta} = 0; \tag{30.7}$$

the lines of equal stresses σ_r become circles (Fig. 54).

If the force acting on an orthotropic semiplane is directed arbitrarily in the xy plane then the general character of the stressed state will remain as before, but the stress σ_r will be determined from the formula

$$\sigma_r = -\frac{P(u_1+u_2)}{\pi h\sqrt{E_1E_2}}\cdot\frac{\cos\omega\cos\theta+\sqrt{\frac{E_1}{E_2}}\sin\omega\sin\theta}{rL(\theta)} \tag{30.8}$$

(ω is the angle formed by the line of action of the force with a normal toward the boundary, Fig. 55). In this case a neutral line (straight line) appears inside the semiplane, all stress components vanish on it; the angle of inclination it makes with the x axis will be determined from the equation

$$\operatorname{tg}\theta = -\sqrt{\frac{E_2}{E_1}}\operatorname{ctg}\omega. \tag{30.9}$$

In an anisotropic semiplane the neutral line is, in general, not perpendicular to the line of action of the force and forms an acute angle with it if $E_1 > E_2$ and an obtuse one of $E_1 < E_2$. On one side of the neutral line the medium is compressed, and on the other one it is stretched. The line of equal stresses are represented in the form of fourth-order curves similar to the curves (30.4).

It is easy to obtain the solution also for the case of a nonorthotropic semiplane deformed by a force applied to the boundary. For this purpose the second method mentioned in §29 may be used, or we can make use of the solution for a nonorthotropic wedge (see §20), putting the vertex angle equal to π. We shall not present

this method. We only note that the general character of the stress distribution will be the same as in the case of an orthotropic semiplane, merely the formulas for σ_r and the equations of the curves of equal stresses are somewhat more complex.

A concentrated moment M applied to the point O at the boundary of the semiplane (Fig. 56) will be considered to be the limiting case of two equal forces of opposite directions where the distance between the points of their application tends to zero, but the moment of the couple remains constant and equal to that given.

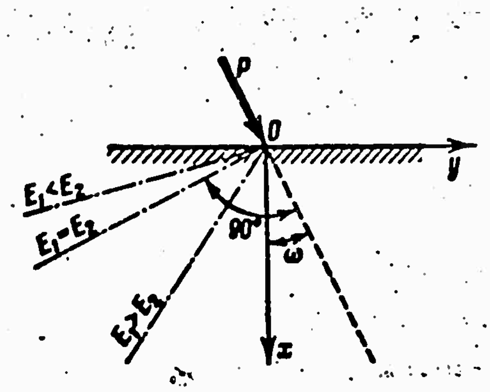

Fig. 55

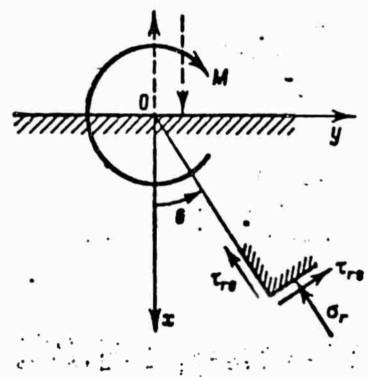

Fig. 56

Using the solution for the force we obtain the following final formulas for the stresses in an orthotropic semiplane:

$$\left.\begin{aligned}
\sigma_r &= -\frac{M(u_1+u_2)}{\pi h\sqrt{E_1E_2}}\cdot\frac{\sin 2\theta}{r^2[L(\theta)]^2}\left[\frac{1}{E_2}+\left(\frac{1}{G}-\frac{1+\nu_1}{E_1}-\frac{1+\nu_2}{E_2}\right)\cos^4\theta\right],\\
\sigma_\theta &= 0,\\
\tau_{r\theta} &= \frac{M(u_1+u_2)}{2\pi h\sqrt{E_1E_2}}\cdot\frac{1+\cos 2\theta}{r^2L(\theta)}.
\end{aligned}\right\} \qquad (30.10)$$

The same solution is obtained from the solution for a wedge set forth in §21, for $\psi = \pi/2$. In the case of an orthotropic semiplane

$$\left.\begin{aligned}
\sigma_r &= -\frac{2M}{\pi h}\cdot\frac{\sin 2\theta}{r^2},\\
\sigma_\theta &= 0,\\
\tau_{r\theta} &= \frac{M}{\pi h}\cdot\frac{1+\cos 2\theta}{r^2}.
\end{aligned}\right\} \qquad (30.11)$$

Among the other cases of loads only the simplest have been studied, as yet, where the load is applied to a finite section of the border and uniformly distributed along it, according to trapezoid and triangle laws.*

§31. THE ACTION OF A FORCE AND MOMENT APPLIED AT A POINT OF AN ELASTIC PLANE

The approach to the problem of stress distribution in an ani-

sotropic plane due to a concentrated force applied to an inside point which is far enough from the edge may be performed in the following manner. We shall regard the plate as an infinite plane medium — an elastic plane to one of whose points a concentrated force P is applied. This concentrated force may, in its turn, be regarded as the limiting case of a load distributed along the edge of an infinitely small opening and resulting in a resultant equal to P.*

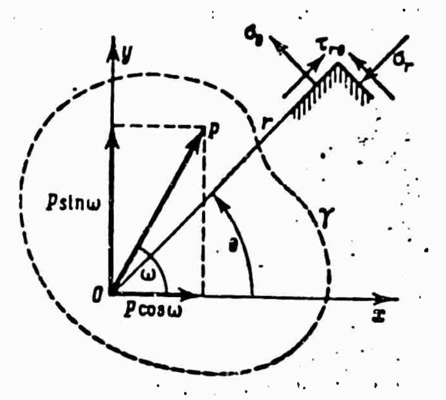

Fig. 57

We shall choose the point where the force is applied to be origin of coordinates, place the x and y axes arbitrarily (Fig. 57) and assume the elastic constants referring to these axes to be given; generally, we shall assume the plate to be nonorthotropic.

We shall make use of the complex representation of stresses and displacements in terms of two functions $\Phi_1(z_1)$ and $\Phi_2(z_2)$ (see §8). Since around the point O an opening, even one with infinitely small dimensions, is assumed the functions Φ_1 and Φ_2 must be multiple-valued and their increments if we pass along an arbitrary closed contour γ encircling the opening must satisfy Eqs. (8.10) where we must put $P_x = P\cos\omega$, $P_y = P\sin\omega$ (Fig. 57). If the distance from the point where the force is applied is increased the stress components must tend to zero. The functions of the form

$$\Phi_1(z_1) = A\ln z_1, \quad \Phi_2(z_2) = B\ln z_2, \tag{31.1}$$

allow all these conditions to be fulfilled, where A, B are constant, generally complex numbers which are determined from Eqs. (8.10). Designating by $\bar{A}$ and $\bar{B}$ we obtain the following equations on the basis of (8.10):

$$\left.\begin{aligned}
&A+B-\bar{A}-\bar{B}=\frac{P\sin\omega}{2\pi hl},\\
&\mu_1A+\mu_2B-\bar{\mu}_1\bar{A}-\bar{\mu}_2\bar{B}=-\frac{P\cos\omega}{2\pi hl},\\
&\mu_1^2A+\mu_2^2B-\bar{\mu}_1^2\bar{A}-\bar{\mu}_2^2\bar{B}=-\frac{a_{16}}{a_{11}}\cdot\frac{P\cos\omega}{2\pi hl}-\frac{a_{12}}{a_{11}}\cdot\frac{P\sin\omega}{2\pi hl}\\
&\frac{1}{\mu_1}A+\frac{1}{\mu_2}B-\frac{1}{\bar{\mu}_1}\bar{A}-\frac{1}{\bar{\mu}_2}\bar{B}=\frac{a_{12}}{a_{22}}\cdot\frac{P\cos\omega}{2\pi hl}+\frac{a_{26}}{a_{22}}\cdot\frac{P\sin\omega}{2\pi hl}
\end{aligned}\right\} \tag{31.2}$$

(the complex parameters μ_1 and μ_2 are the roots of Eq. (7.4) and supposed to be unequal).

The stress components are determined by the formulas:

$$\left.\begin{aligned}
\sigma_x&=2\mathrm{Re}\left(\frac{\mu_1^2A}{z_1}+\frac{\mu_2^2B}{z_2}\right),\\
\sigma_y&=2\mathrm{Re}\left(\frac{A}{z_1}+\frac{B}{z_2}\right),\\
\tau_{xy}&=-2\mathrm{Re}\left(\frac{\mu_1A}{z_1}+\frac{\mu_2B}{z_2}\right).
\end{aligned}\right\} \tag{31.3}$$

In the case of a nonorthotropic medium these expressions will be very cumbersome, after splitting of the real parts. We shall present here only the formulas for the stress components in polar coordinates in which these quantities are calculated in the case of an orthotropic plate when the x and y directions coincide with the principal ones, and the force acts in the direction of the x axis ($\omega = 0$):

$$\left.\begin{aligned}
\sigma_r&=\frac{P}{2\pi h(\mu_1^2-\mu_2^2)}\cdot\frac{\cos\theta}{r}\left[i\mu_1(1-\nu_2\mu_2^2)\frac{(1+\mu_1^2)\sin^2\theta+\mu_1^2}{\cos^2\theta-\mu_1^2\sin^2\theta}-\right.\\
&\qquad\left.-i\mu_2(1-\nu_2\mu_1^2)\frac{(1+\mu_2^2)\sin^2\theta+\mu_2^2}{\cos^2\theta-\mu_2^2\sin^2\theta}\right],\\
\sigma_\theta&=\frac{P}{2\pi hn}(1-\nu_2k)\frac{\cos\theta}{r},\\
\tau_{r\theta}&=\frac{P}{2\pi hn}(1-\nu_2k)\frac{\sin\theta}{r}
\end{aligned}\right\} \tag{31.4}$$

$$\left[k=-\mu_1\mu_2=\sqrt{\frac{E_1}{E_2}},\quad n=-i(\mu_1+\mu_2)\right].$$

In the given case the complex parameters μ_1 and μ_2 are roots of the equation

$$\frac{\mu^4}{E_1}+\left(\frac{1}{G}-\frac{2\nu_1}{E_1}\right)\mu^2+\frac{1}{E_2}=0. \tag{31.5}$$

Although the imaginary unit $i=\sqrt{-1}$, enters Expression (31.4) all three expressions are real: for an orthotropic plate k and n are always real numbers (in all three possible cases 1, 2, 3, mentioned in §7).

Attention must be paid to the very simple law of distribution of the stresses σ_θ and $\tau_{r\theta}$ and the very complex kind of dependence of the stress σ_r on the angle θ. As in the case of a semiplane

loaded by a force the stresses vary inversely proportionally to the distance r, but in our case the stress distribution will not be "radial" or "ray-like."

Putting $\mu_1 = \mu_2 = i$ we obtain the stress distribution in an isotropic medium (with a Poisson coefficient ν):*

$$\left.\begin{aligned} \sigma_r &= -\frac{P}{4\pi h}(3+\nu)\frac{\cos\theta}{r}, \\ \sigma_\theta &= \frac{P}{4\pi h}(1-\nu)\frac{\cos\theta}{r}, \\ \tau_{r\theta} &= \frac{P}{4\pi h}(1-\nu)\frac{\sin\theta}{r}. \end{aligned}\right\} \tag{31.6}$$

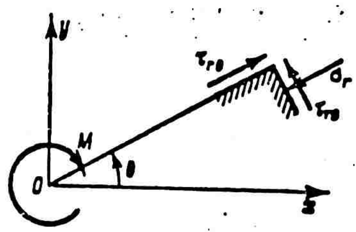

Fig. 58

If we know the stress distribution due to a single force (31.3) we may also obtain the stress distribution due to a moment M applied at the point O of an unlimited medium, but superposition and subsequently passing to the limit (Fig. 58). In this case, as in the case of the semiplane, the moment is regarded as the limiting case of two forces of opposite directions. In the case of an orthotropic medium the final formulas for the stresses have the form:

$$\left.\begin{aligned} \sigma_r &= \frac{M}{4\pi h(\mu_1-\mu_2)}\cdot\frac{\sin 2\theta}{r^2}\left[\frac{\mu_1(1+\mu_1^2)(-1-\mu_1\mu_2+i\mu_1-i\mu_2)}{(\cos^2\theta-\mu_1^2\sin^2\theta)^2}-\right. \\ &\qquad \left.-\frac{\mu_2(1+\mu_2^2)(-1-\mu_1\mu_2-i\mu_1+i\mu_2)}{(\cos^2\theta-\mu_2^2\sin^2\theta)^2}\right], \\ \sigma_\theta &= 0; \\ \tau_{r\theta} &= -\frac{M}{8\pi h E_1}\cdot\frac{1}{r^2 L(\theta)}[(1-k)^2+n(1+k)+ \\ &\qquad +(1-k)(1+k+n)\cos 2\theta]. \end{aligned}\right\} \tag{31.7}$$

L is here a quantity reciprocal to E_r [see Formula (30.3)]; the direction of the x axis from which the angles θ are counted coincides with the principal one.

A very simple stress distribution is obtained in an isotropic medium:

$$\sigma_r = \sigma_\theta = 0, \quad \tau_{r\theta} = -\frac{M}{2\pi h}\cdot\frac{1}{r^2}. \tag{31.8}$$

On the basis of the formulas given in this and the preceding sections it is easy to obtain by superposition the stress distribution in a semiplane under the action of force and moment whose points of application are inside the semiplane rather than on the boundary.

The solution of the problem for an orthotropic semiplane deformed by a force which is applied at some distance from the boundary is given in the works by Conway and M. Sokolowskiy.*

S.G. Mikhlin found the general solution of the problem of stress distribution in an elastic anisotropic plane with slits lying on one straight line, where the external forces are given at the boundaries of the slits.** Another method of solving this problem was proposed later on in a work by P.A. Zagubizhenko*** who also studied a special case: the compression of an anisotropic plane with one straight slit (thin slot).

§32. THE STRESS DISTRIBUTION IN A PLANE MEDIUM WITH PARABOLIC AND HYPERBOLIC BOUNDARIES

Let us consider the elastic equilibrium of an anisotropic plate with a concave boundary having the form of a parabola, under the action of a load applied to this boundary. If the length of the loaded section is small compared to the plate dimensions the latter may be regarded as an infinite plane elastic medium with hyperbolic boundary. To determine the stresses it is the simplest way to use a method analogous to the second method of solving the problem of an elastic semiplane, about which was discussed in §29.

Let us identify the origin of coordinates with the point of the concave boundary where the curvature is maximum, and place the x axis along the tangent, and the y axis outward (Fig. 59). In this system of coordinates the equation of the boundary line has the form:

$$y = ax^2 \quad (a > 0). \tag{32.1}$$

Let X_n, Y_n – the projections of the forces per unit length – be given functions of x; with respect to these forces we shall assume that their resultant for any finite or infinite section of the boundary is finite or zero. Generally we shall assume the material to be nonorthotropic.

We present here the final expressions for the functions $\Phi_1'(z_1)$, $\Phi_2'(z_2)$ determining the stresses [according to Formulas (8.2)]:****

$$\left.\begin{aligned}
\Phi_1'(z_1) &= \frac{1}{2\pi i(\mu_1-\mu_2)h}\cdot\frac{1}{\sqrt{1+4a\mu_1 z_1}}\int_{-\infty}^{\infty}\frac{X_n+\mu_2 Y_n}{\xi-\zeta_1(z_1)}\sqrt{1+4a^2\xi^2}\,d\xi,\\
\Phi_2'(z_2) &= -\frac{1}{2\pi i(\mu_1-\mu_2)h}\cdot\frac{1}{\sqrt{1+4a\mu_2 z_2}}\int_{-\infty}^{\infty}\frac{X_n+\mu_1 Y_n}{\xi-\zeta_2(z_2)}\sqrt{1+4a^2\xi^2}\,d\xi.
\end{aligned}\right\} \tag{32.2}$$

Here

$$\left.\begin{aligned}\zeta_1(z_1) &= \frac{\sqrt{1+4a\mu_1 z_1}-1}{2\mu_1 a},\\ \zeta_2(z_2) &= \frac{\sqrt{1+4a\mu_2 z_2}-1}{2\mu_2 a}\end{aligned}\right\} \tag{32.3}$$

are functions of the complex variables z_1 and z_2 which assume the same value x at the medium boundary $y = ax^2$.

For $a = 0$ we obtain from it the already known solution for the semiplane [Formulas (29.13)].

Formulas (32.2) make it possible to find the functions of the complex variables (and from them also the stresses) for any force distribution at the boundary if only these forces satisfy the above-mentioned conditions. No special cases of the elastic equilibrium of a plate with parabolic edge have been studied, as yet. Conversely, for an anisotropic plate whose edges are hyperbolic only the solution for one special case which we shall discuss in conclusion is known.

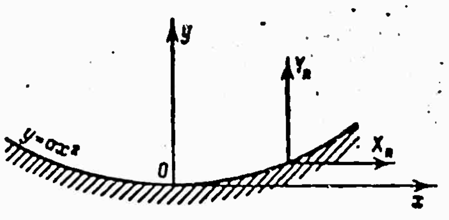

Fig. 59

Let be given an anisotropic plate whose region is bounded by two branches of a hyperbola and two equal straight segments (Fig. 60). Normal forces resulting in stretching axial forces P are distributed along the straight boundaries. The stresses are to be determined, and, in particular, the stresses in the narrowest section $x = 0$.

An approximate solution of this problem for an isotropic plate was obtained by Neyber,* and for an orthotropic one by Smith and Okubo.**

The results of Smith and Okubo are easy to generalize to the case where the plate is not orthotropic which we shall also suppose, at the beginning.

We shall consider the plate to be infinite and the region to be limited by two hyperbola branches. Placing the axes as is shown in Fig. 60 we may write the equation of the edge in the form

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1, \tag{32.4}$$

or in parametric representation

$$x = b\,\mathrm{sh}\,t,\quad y = \pm a\,\mathrm{ch}\,t \tag{32.5}$$

(the plus sign refers to the upper hyperbola branch, the minus sign to the lower one). Since the edge is not loaded the following conditions will be fulfilled on it:

$$\left.\begin{aligned} \sigma_x\,dy - \tau_{xy}\,dx &= 0, \\ \tau_{xy}\,dy - \sigma_y\,dx &= 0. \end{aligned}\right\} \tag{32.6}$$

The stresses in an arbitrary cross section $x = x_0$ must result in an axial force P from which three further conditions are obtained:

$$\int_{-y_0}^{y_0} \sigma_x\,dy = \frac{P}{h},\quad \int_{-y_0}^{y_0} \sigma_x y\,dy = 0,\quad \int_{-y_0}^{y_0} \tau_{xy}\,dy = 0, \tag{32.7}$$

where

$$y_0 = \frac{a}{b}\sqrt{b^2 + x_0^2}.$$

All conditions can be satisfied by choosing the functions Φ_1 and Φ_2 in the form:

$$\left.\begin{aligned} \Phi_1(z_1) &= A\ln\left(z_1 + \sqrt{z_1^2 + b^2 - \mu_1^2 a^2}\right), \\ \Phi_2(z_2) &= B\ln\left(z_2 + \sqrt{z_2^2 + b^2 - \mu_2^2 a^2}\right). \end{aligned}\right\} \tag{32.8}$$

From (32.6) and (32.7) we obtain the following equations for the constants A, B and the conjugate quantities:

$$\left.\begin{aligned} A + B + \bar{A} + \bar{B} &= 0, \\ A\mu_1 + B\mu_2 + \bar{A}\bar{\mu}_1 + \bar{B}\bar{\mu}_2 &= 0, \\ A\ln\frac{1+\mu_1 c}{1-\mu_1 c} + B\ln\frac{1+\mu_2 c}{1-\mu_2 c} &+ \\ + \bar{A}\ln\frac{1+\bar{\mu}_1 c}{1-\bar{\mu}_1 c} + \bar{B}\ln\frac{1+\bar{\mu}_2 c}{1-\bar{\mu}_2 c} &= 0, \\ A\mu_1\ln\frac{1+\mu_1 c}{1-\mu_1 c} + B\mu_2\ln\frac{1+\mu_2 c}{1-\mu_2 c} &+ \\ + \bar{A}\bar{\mu}_1\ln\frac{1+\bar{\mu}_1 c}{1-\bar{\mu}_1 c} + \bar{B}\bar{\mu}_2\ln\frac{1+\bar{\mu}_2 c}{1-\bar{\mu}_2 c} &= \frac{P}{h}. \\ \left(c = \frac{a}{b}\right). & \end{aligned}\right\} \tag{32.9}$$

Equations (32.9) simplify for an orthotropic plate with purely imaginary parameters $\mu_1 = \beta i$, $\mu_2 = \delta i$. Solving them we find:

$$A = \bar{A} = -\frac{P}{4hg},\quad B = \bar{B} = \frac{P}{4hg}, \tag{32.10}$$

where

$$g = \beta\,\mathrm{arctg}(\beta c) - \delta\,\mathrm{arctg}(\delta c); \tag{32.11}$$

$$\left.\begin{aligned}
\sigma_x &= \frac{P}{2hg}\,\mathrm{Re}\left(\frac{\beta^3}{\sqrt{z_1^2+b^2+\beta^2a^2}}-\frac{\delta^3}{\sqrt{z_2^2+b^2+\delta^2a^2}}\right),\\
\sigma_y &= \frac{P}{2hg}\,\mathrm{Re}\left(-\frac{1}{\sqrt{z_1^2+b^2+\beta^2a^2}}+\frac{1}{\sqrt{z_2^2+b^2+\delta^2a^2}}\right),\\
\tau_{xy} &= \frac{P}{2hg}\,\mathrm{Re}\left(\frac{\beta i}{\sqrt{z_1^2+b^2+\beta^2a^2}}-\frac{\delta i}{\sqrt{z_2^2+b^2+\delta^2a^2}}\right).
\end{aligned}\right\} \qquad (32.12)$$

In the narrowest section $x = 0$ the tangential stress vanishes, and the normal stress is determined by the formula:

$$(\sigma_x)_0 = \frac{P}{2hg}\left[\frac{\beta^3}{\sqrt{b^2+\beta^2(a^2-y^2)}}-\frac{\delta^3}{\sqrt{b^2+\delta^2(a^2-y^2)}}\right]. \qquad (32.13)$$

The maximum value of the stress is obtained at the ends of the narrow section, i.e., at the points $x=0$, $y=\pm a$; it is equal to:

$$\sigma_{max} = \frac{P}{2ah}K, \qquad (32.14)$$

where

$$K = \frac{(\beta^2-\delta^2)c}{g}. \qquad (32.15)$$

In Formula (32.14) the factor $P/2ah$ is the stress due to the stretching force in a prismatic rod in which the cross section is the same as the cross section of our plate in its narrowest part; according to G. Neuber, this stress may be called "nominal"; the factor K is the concentration factor; it indicates how many times the maximum stress in the plate with the hyperbolic edge is greater than the nominal one. We note that at the points $x = 0$, $y = \pm a$ the radius of curvature is $\rho = b/c$.

In the case of an isotropic plate

$$K = \frac{2c(1+c^2)}{c+(1+c^2)\,\mathrm{arctg}\,c}. \qquad (32.16)$$

The factor K grows with increasing curvature at the end points of the narrow section (or, which is the same, when the radius of curvature decreases). The stress obtained at the center of the section $x = 0$ is lower than the nominal one.

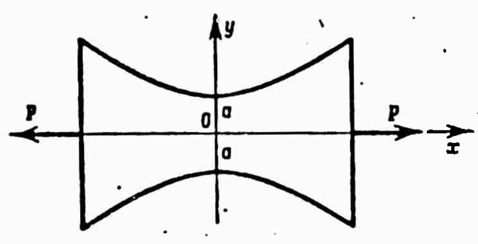

Fig. 60

In Table 1 the values of the concentration factor for different c for an isotropic plate and a plate of plywood whose elastic constants were given in §11 are presented. Two main cases must here be distinguished.

1) The plate is cut out such that the casing fibers are parallel to the axial direction (the plate is stretched in a direction for which Young's modulus is maximum: $E_1 = E_{max}$). In this case

$$\mu_1 = 4{,}11i, \quad \mu_2 = 0{,}343i, \quad \beta = 4{,}11, \quad \delta = 0{,}343.$$

2) The directions of the casing fibers are perpendicular to the axial direction (the plate is stretched in a direction for which Young's modulus is minimum: $E_1 = E_{min}$). For this case

$$\mu_1 = 0{,}243i, \quad \mu_2 = 2{,}91i, \quad \beta = 0{,}243, \quad \delta = 2{,}91.$$

TABLE 1

The Values of the Concentration Coefficient K for Different Values of the Ratio $c = a/b$.

$c = \frac{a}{b}$	1 Фанера		Изотропная пластинка 2
	$E_1 = E_{max}$	$E_1 = E_{min}$	
10	28,36	20,00	12,74
5	14,22	10,07	6,39
2	5,83	4,22	2,65
1	3,13	2,36	1,56
0,5	1,85	1,50	1,16
0,1	1,03	1,02	1,01

1) Plywood; 2) isotropic plate.

This table shows that the maximum stress in a plywood plate obtained is higher than the maximum stress in the same isotropic plate. If, however, the two cases of the plywood plate are compared, then, as is shown by Table 1, the concentration coefficient obtained is higher in the case where the plate is stretched in a direction for which Young's modulus is maximum.

The formulas and tables shown may be used to calculate approximately the stresses in stretched rectangular plates weakened by two identical side grooves.

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127* This method is set forth in greater detail in our abovementioned work "Nekotoryye sluchai ploskoy zadachi teorii uprugosti anizotropnogo tela" [Several Cases of the Plane Theory of the Theory of Elasticity of an Anisotropic Body], sbornik "Eksperimental'nyye metody opredeleniya napryazheniy i deformatsiy v uprugoy i plasticheskoy zonakh" [Collection "Experimental Methods of Determining Stresses and Strains in the Elastic and Plastic Zones"], ONTI, 1935, pages 164-173.

127** The problem may also be solved without these restrictions, but the solution obtained will be considerably more complex.

128 Smirnov, V.I., Kurs vysshey matematiki [Course on Higher Mathematics], vol. Gostekhizdat [State Publishing House on Technical and Theoretical Literature], Moscow-Leningrad, 1951, pages 464-468.

130 Lekhnitskiy, S.G., Nekotoryye sluchai uprugogo ravnovesiya odnorodnogo tsilindra s proizvol'noy anizotropiyey [Some Cases of Elastic Equilibrium of a Homogeneous Cylinder with Arbitrary Anisotropy], Prikladnaya matematika i mekhanika [Applied Mathematics and Mechanics], Vol. 3, No. 3, 1939, pages 359-361. See also the book: Lekhnitskiy, S.G., Teoriya uprugosti anizotropnogo tela [Theory of Elasticity of an Anisotropic Body], Gostekhizdat, Moscow-Leningrad, 1950, pages 115-117.

131* Savin, G.N., Nekotoryye zadachi teorii uprugosti anizotropnoy sredy [Some Problems of the Theory of Plasticity of an Anisotropic Medium], DAN SSSR [Proceedings of the Academy of Sciences of the USSR], new series, Vol. 23, 1939.

131** See the works by G.N. Savin: 1) Davleniye absolyutno zhestkogo shtampa na upruguyu anizotropnuyu srednu (ploskaya zadacha) [The Pressure of an Absolutely Rigid Stamp on an Elastic Anisotropic Medium (Plane Problem), DAN USSR [Proceedings of the Academy of Sciences of the UkrSSSR), 1939, No. 2; 2) Davleniye zhestkogo lentochnogo fundamenta na [illegible]goye anizotropnoye osnovaniye [The Pressure of a [illegible] Band Foundation on an Elastic Anisotropic Base], Ve[illegible]ik inzhenerov i tekhnikov [Herald of Engineers and Technicians], 1940, No. 5; 3) O dopolnitel'nom davlenii peredayushchemsya po podoshve absolyutno zhestkogo shtampa na uprugoye osnovaniye vyzvannom blizlezhashchey nagruzkoy [On the Additional Pressure Transferred at the Bottom of an Absolutely Rigid Stamp on the Elastic Base, Due to an Adjacent Load] (ploskaya zadacha) [Plane Problem], DAN USSR, 1940, No. 7; 4) Davleniye sistemy zhestkikh shtampov ny upruguyu

anizotropnuyu poluploskost' [The Pressure of a System of Rigid Stamps on an Elastic Anisotropic Semiplane], Soobshcheniya Gruzinskogo filiala AN SSSR [Communications of the Georgian Branch of the Academy of Sciences of the USSR], 1940, No. 10; 5) Smeshannaya zadacha dlya anizotropnoy poluploskosti [The Mixed Problem for an Anisotropic Semiplane], Uch. zap. L'vovskogo gos. un-ta im. I. Franko [Scientific Reports of the I. Franko L'vov State University], Vol. 5, Physical-Mathematical Series, No. 2, 1947.

131*** Galin, L.A., Kontaktnyye zadachi teorii uprugosti [Contact Problems of the Theory of Elasticity], Gostekhizdat, Moscow, 1953, see also the works of L.A. Galin mentioned in this book (pages 258-259).

131**** Savin, G.N. and Grilitskiy, D.V., Davleniye dvukh anizotropnykh tel (ploskaya zadacha) [The Pressure of Two Anisotropic Bodies (Plane Problem)], DAN USSR, 1952, No. 2; Grilitskiy, D.V., Szhatiye dvukh uprugokh anizotropnykh tel pri uchete sil treniya (ploskaya tadacha) [The Compression of Two Anisotropic Bodies Taking Account of the Frictional Forces (Plane Problem)], DAN USSR, 1953, No. 3.

131***** Sokolowski, M., Pewne zagadnienia plaskie teorii sprezystosci ciala ortotropowego [Some Problems of the Plane Theory of Elasticity of an Orthotropic Body], Arch. mech. stosowanej [Archive of Applied Mechanics], Vol. 6, No. 1, 1954.

132 See our work mentioned in the first footnote of the preceding section. This problem for media with an Anisotropy of a Special Kind was considered independently of us at different times by many foreign authors. The earliest investigations are due to Wolf and Okubo (see the works: 1) Wolf, K., Ausbreitung der Kraft in der Halbebene und im Halbraum bei anisotropem Material [Force Propagation in Semiplane and Semispace in the Case of Anisotropic Material], Zeitschrift f. Angew. Math. u. Mech., [Journal for Applied Mathematics and Mechanics], Vol. 15, No. 5, 1935; 2) Okubo, H., General Expression of Stress Components in Two Dimensions in an Aeolotzopic Substance, Sci. Rep. Tohoku Univ. 1, page 25, 1937).

135 See, e.g., Timoshenko, S.P., Teoriya uprugosti [Theory of Elasticity], ONTI [United Scientific and Technical Publishing Houses], 1937, page 97.

136 Savin, G.N., Napryazheniya v anizotropnom massive pri zadannoy nagruzke na poverkhnosti (ploskaya zadacha) [The Stresses in an Anisotropic Solid Body for Given Load on the Surface (Plane Problem)], Vestnik inzhenerov i tekhnikov [Herald of Engineers and Technicians], 1940, No. 3.

137 The solution of this problem is given in the work: Lekhnitskiy, S.G., Ploskaya staticheskaya zadacha teorii uprugosti anizotropnogo tela [The Plane Static Problem of the Theory of Elasticity of an Anisotropic Body], Prikladnaya matematika i mekhanika, New Series, Vol. 1, No. 1, 1937.

139 Timoshenko, S.P., Teoriya uprugosti, ONTI, 1937, page 126.

140* Conway, H.D., The Stress Distributions Induced by Concentrated Loads Acting in Isotropic and Orthotropic Half Planes, Journ. Appl. Mech., Vol. 20, No. 1, 1953; Sokolowski, M., Pewne zagadnienia plaskie teorii sprezystoscti ciala ortotropowego, Arch. mech. stosowanej, Vol. 6, No. 1, 1954.

140** Mikhlin, S.G., Ob odnoy chastnoy zadache teorii uprugosti [On One Special Solution of the Theory of Elasticity], DAN SSSR, New Series, Vol. 27, No. 6, 1940.

140*** Zagubizhenko, P.A., O napryazheniyakh anizotropnoy ploskosti oslablennoy pryamolineynymi shchelyami [On the Stresses of an Anisotropic Plane Weakened by Straight Slits], DAN SSSR, 1954, No. 6.

140*** Lekhnitskiy, S.G., Obobshchennaya ploskaya defomatsiya v bezkonechnom uprugom anizotropnom poluprostranstve ogranichennom poverkhnost'yu parabolicheskogo tsilindra [The Generalized Plane Deformation in an Infinite Elastic Anisotropic Semispace Limited by the Surface of a Parabolic Cylinder], DAN SSSR, Vol. 25, No. 3, 1939. In this work a detailed derivation of the solution of the posed problem is given.

141* Neuber, G., Kontsentratsiya napryazheniy [Stress Concentration], OGIZ [State United Publishing House], Moscow-Leningrad, 1947, Ch. 4.

141** Bassel Smith, C., Effect of Hyperbolic Notches on the Stress Distribution in a Wood Plate, Quarterly of Applied Mathematics, Vol. 6, No. 4, 1949; Okubo, H., On the Problem of a Notched Plate of an Aeolotropic Material, Philosophical Magazine, Vol. 40, Ser. 7, No. 308, 1949.

Chapter 5

STRESS DISTRIBUTION IN ELLIPTIC PLATE AND CIRCULAR DISK

§33. DISTRIBUTION OF STRESSES IN AN ELLIPTIC PLATE LOADED ALONG THE EDGE

In the present chapter we consider the problem of the stress distribution in a uniform elliptic plate loaded by forces applied to the edge, and that of such a plate which rotates at a constant angular velocity about an axis passing through its center. But here we consider the problem of a rotating round disk possessing cylindrical anisotropy, in both the case of a massive disk and that of a composite one.

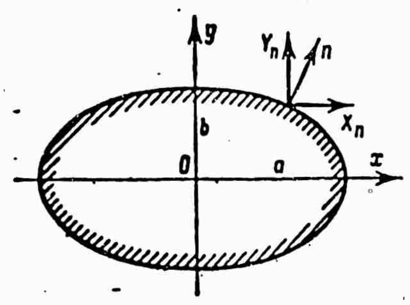

Fig. 61

Let us consider an elliptic plate of uniform anisotropic material, which is in equilibrium under the forces applied, when these forces are distributed along the edge according to an arbitrary law. In the general case we shall consider the plate to be nonorthotropic. We direct the axes x and y along the main axes of the ellipse (Fig. 61) so that the equations of the generalized Hooke's law, which link the mean values, with respect to the thickness, of the stress and strain components, can be written in the following form:

$$\left.\begin{aligned} \varepsilon_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{16}\tau_{xy}, \\ \varepsilon_y &= a_{12}\sigma_x + a_{22}\sigma_y + a_{26}\tau_{xy}, \\ \gamma_{xy} &= a_{16}\sigma_x + a_{26}\sigma_y + a_{66}\tau_{xy}. \end{aligned}\right\} \quad (33.1)$$

The constants $a_{11}, a_{12}, \ldots, a_{66}$ are assumed to be given. Let us denote by X_n, Y_n the projections of the external forces referred to unit area, and by a and b the semiaxes of the ellipse.

We assume absence of volume forces and the forces X_n, Y_n to be in equilibrium so that the principal and their principal moment are equal to zero.

For the ease of an isotropic plate this problem was solved by N.I. Muskhelishvili.* A general solution for the anisotropic plate by the method developed here was derived by the author,** and by means of another method, by P.P. Kufarev.***

The equation of the plate's contour is given in parametric form

$$x = a\cos\vartheta, \quad y = b\sin\vartheta. \tag{33.2}$$

The given forces are assumed to be functions of the variable ϑ (which varies from zero to 2π).

We express the stress components and the projections of the displacement in terms of functions of the variables $\Phi_1(z_1)$ and $\Phi_2(z_2)$, such that the boundary conditions for these functions can be written in the form of (8.7):

$$\left.\begin{aligned} 2\,\mathrm{Re}\,[\Phi_1(z_1)+\Phi_2(z_2)] &= -\int_0^s Y_n\,ds + c_1, \\ 2\,\mathrm{Re}\,[\mu_1\Phi_1(z_1)+\mu_2\Phi_2(z_2)] &= \int_0^s X_n\,ds + c_2. \end{aligned}\right\} \tag{33.3}$$

We expand the given forces X_n, Y_n in Fourier series and substitute them in the integral expressions of the right-hand sides of the conditions. In the general case of forces (being in equilibrium along the contour) we obtain:

$$\left.\begin{aligned} -\int_0^s Y_n\,ds + c_1 &= \alpha_0 + \sum_{m=1}^{\infty}(\alpha_m\sigma^m + \bar{\alpha}_m\sigma^{-m}), \\ \int_0^s X_n\,ds + c_2 &= \beta_0 + \sum_{m=1}^{\infty}(\beta_m\sigma^m + \bar{\beta}_m\sigma^{-m}), \end{aligned}\right\} \tag{33.4}$$

where $\sigma = e^{\vartheta i}$; α_m, β_m are given coefficients, which depend on the law of load distribution, $\bar{\alpha}_m$, $\bar{\beta}_m$ are conjugate quantities, α_0, β_0 are arbitrary constants. The coefficients α_1, β_1, $\bar{\alpha}_1$ and $\bar{\beta}_1$ must satisfy the equilibrium condition (principal moment vanishing):

$$\frac{\bar{\alpha}_1 - \alpha_1}{bi} = \frac{\beta_1 + \bar{\beta}_1}{a}. \tag{33.5}$$

The solution of the problem in the most general case of load distribution is obtained by means of the functions Φ_1 and Φ_2 in the form of the following series

$$\left.\begin{aligned}\Phi_1(z_1) &= A_0 + A_1 z_1 + \sum_{m=2}^{\infty} A_m P_{1m}(z_1),\\ \Phi_2(z_2) &= B_0' + B_1 z_2 + \sum_{m=2}^{\infty} B_m P_{2m}(z_2).\end{aligned}\right\} \tag{33.6}$$

P_{1m} and P_{2m} are here integral polynomials of power m with respect to the variables z_1 and z_2:*

$$\left.\begin{aligned}P_{1m}(z_1) &= -\frac{1}{(a-i\mu_1 b)^m}\left[\left(z_1+\sqrt{z_1^2-a^2-\mu_1^2 b^2}\right)^m +\right.\\ &\qquad \left.+\left(z_1-\sqrt{z_1^2-a^2-\mu_1^2 b^2}\right)^m\right],\\ P_{2m}(z_2) &= -\frac{1}{(a-i\mu_2 b)^m}\left[\left(z_2+\sqrt{z_2^2-a^2-\mu_2^2 b^2}\right)^m +\right.\\ &\qquad \left.+\left(z_2-\sqrt{z_2^2-a^2-\mu_2^2 b^2}\right)^m\right].\end{aligned}\right\} \tag{33.7}$$

On the plate's contour (33.2) the functions z_1, z_2, P_{1m} and P_{2m} take the following forms:

$$\left.\begin{aligned}z_1 &= \frac{a-i\mu_1 b}{2}\sigma + \frac{a+i\mu_1 b}{2}\cdot\frac{1}{\sigma},\\ z_2 &= \frac{a-i\mu_2 b}{2}\sigma + \frac{a+i\mu_2 b}{2}\cdot\frac{1}{\sigma};\end{aligned}\right\} \tag{33.8}$$

$$P_{1m} = -\sigma^m - t_1^m\sigma^{-m}, \quad P_{2m} = -\sigma^m - t_2^m\sigma^{-m}, \tag{33.9}$$

where

$$t_1 = \frac{a+i\mu_1 b}{a-i\mu_1 b}, \quad t_2 = \frac{a+i\mu_2 b}{a-i\mu_2 b}. \tag{33.10}$$

Substituting the boundary values of the functions Φ_1 and Φ_2 in the Conditions (33.3) we obtain equations for the determination of the coefficients A_m and B_m and the conjugate quantities $\bar{A}_m$ and $\bar{B}_m$:

$$\left.\begin{aligned}A_m + B_m + \bar{A}_m\bar{t}_1^m + \bar{B}_m\bar{t}_2^m &= -\alpha_m,\\ A_m\mu_1 + B_m\mu_2 + \bar{A}_m\bar{\mu}_1\bar{t}_1^m + \bar{B}_m\bar{\mu}_2\bar{t}_2^m &= -\beta_m,\\ A_m t_1^m + B_m t_2^m + \bar{A}_m + \bar{B}_m &= -\bar{\alpha}_m,\\ A_m\mu_1 t_1^m + B_m\mu_2 t_2^m + \bar{A}_m\bar{\mu}_1 + \bar{B}_m\bar{\mu}_2 &= -\bar{\beta}_m\end{aligned}\right\} \tag{33.11}$$

($m = 2, 3, 4, \ldots$; $\bar{t}_1$, $\bar{t}_2$ are constants adjoined to t_1 and t_2):

$$\left.\begin{aligned}A_1 + B_1 + \bar{A}_1 + \bar{B}_1 &= \frac{\alpha_1+\bar{\alpha}_1}{a},\\ A_1\mu_1 + B_1\mu_2 + \bar{A}_1\bar{\mu}_1 + \bar{B}_1\bar{\mu}_2 &= \frac{\bar{\alpha}_1-\alpha_1}{bi} = \frac{\beta_1+\bar{\beta}_1}{a},\\ A_1\mu_1^2 + B_1\mu_2^2 + \bar{A}_1\bar{\mu}_1^2 + \bar{B}_1\bar{\mu}_2^2 &= \frac{\bar{\beta}_1-\beta_1}{bi}.\end{aligned}\right\} \tag{33.12}$$

In the case of unequal complex parameters the system (33.11) always has a solution as its determinant is nonzero. For the four coefficients A_1, B_1, $\bar{A}_1$, $\bar{B}_1$ we have only three equations, but certain definite constant stresses correspond to the functions $A_1 z_1$ and $B_1 z_2$:

$$\sigma_x^0 = \frac{\bar{\beta}_1 - \beta_1}{bi}, \quad \sigma_y^0 = \frac{\alpha_1 + \bar{\alpha}_1}{a}, \quad \tau_{xy}^0 = \frac{\alpha_1 - \bar{\alpha}_1}{bi} = -\frac{\beta_1 + \bar{\beta}_1}{a}. \qquad (33.13)$$

The constants A_0 and B_0 remain arbitrary.

In this way we can obtain formulas for the stresses in a case of arbitrary distribution of the external forces. The series (33.6), after which both A_m and B_m can be obtained from Eqs. (33.11), prove to converge absolutely and steadily in both the case of distributed loads and the case of loads in the form of concentrated forces. A calculation of the stresses according to the functions Φ_1 and Φ_2 is generally connected with certain difficulties which cannot be avoided as yet.

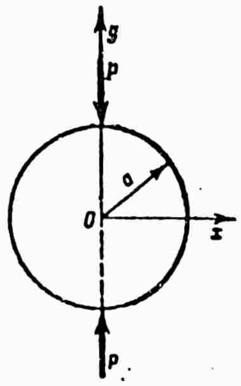

Fig. 62

It is interesting to note that in the case of an anisotropic round disk where $b = a$ and the parameter ϑ is equal to the polar angle θ we do not obtain any essential simplifications. The problems on the equilibrium of the elliptic plate and the round disk prove to be problems of almost the same difficulty. This does not hold true, however, for the case of the isotropic plate; the problem of the stress distribution in a round plate is much simpler than the same problem for an elliptic plate.

Of the particular cases of load (disregarding the trivial case of omnilateral compression or extension) only a single problem has so far been reduced to numerical results: the case of the compression of a round orthotropic disk by two equal forces P applied at the ends of the diameter (Fig. 62).

The solution to this particular case was obtained by Okubo (by a method which differs a little from that discussed in the given section, but which is also based on a complex representation of the stresses in terms of two functions of complex variables).

Okubo constructed a stress distribution diagram for a certain disk diameter at which the ratio of Young's moduli for the principal stresses is equal to 5.9 and the complex parameters are $\mu_1 = 2.307i$ and $\mu_2 = 1.053i$, for compressions in the principal direction of elasticity and at an angle of 45° to the principal direction.*

The graphs of distribution of normal str sses σ_y with respect to the diameter of this disk, which is perpendicular to the line of action of the force, are shown in Fig. 63. One of the graphs has been drawn for the case of compression in the direction of the diameter along which Young's modulus is highest ($E_2 > E_1$), the other for the compression along a diameter which corresponds to minimum modulis ($E_2 < E_1$). In the same figure the dashed line shows the distribution curve of σ_y in an isotropic disk. In all three cases the stress of maximum absolute magnitude acts at the

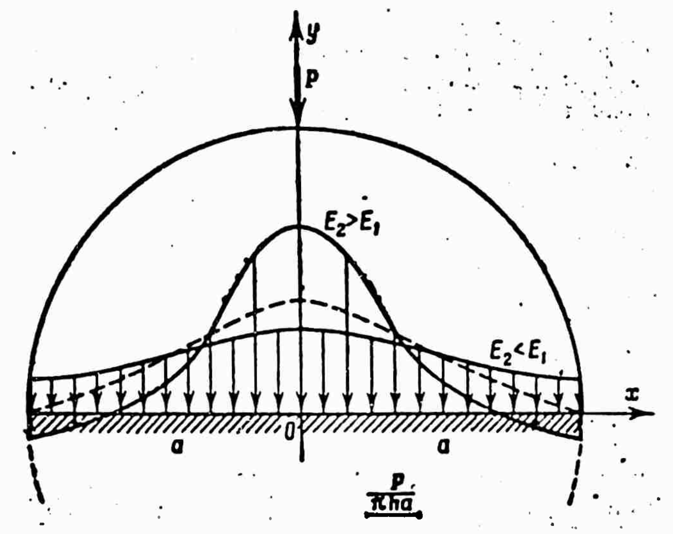

Fig. 63

point of intersection with the line of actic of the force. It is determined by the formula

$$\sigma_{max}=\frac{P}{\pi h a}K, \qquad (33.14)$$

where a and h denote radius and thickness of the disk. Approximate values of the coefficient K are equal: for $E_2>E_1$ $K=5$, for $E_2<E_1$ $K=2{,}2$, and in the case of isotropic material $(E_1=E_2)$ $K=3$. As regards the stress σ_x at points of the same diameter it is found to be much smaller than the stress σ_y.

§34. DISTRIBUTION OF STRESSES IN A ROTATING HOMOGENEOUS ELLIPTIC PLATE

Let us assume an elliptic homogeneous (rectilinear-anisotropic) plate rotating with constant angular velocity ω around an axis passing through its center, perpendicularly to the plane of the plate. The axis of rotation is considered to be a perfect mathematical straight line. The solution of this case has a very simple form and can be obtained by elementary means.*

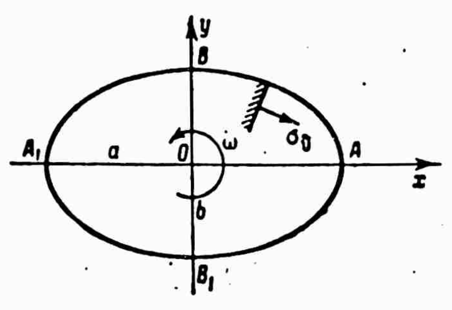

Fig. 64

We choose the directions of the axes x and y as shown in Fig. 64. For definiteness we shall assume that no external forces are applied to the edge of the plate and that displacements of the edge points are quite unrestricted. The plate is not supposed to be orthotropic and Eq. (33.1) applies to it.

The stress components, averaged with respect to the thickness, are determined according to the formulas

$$\left.\begin{aligned} \sigma_x &= \frac{\gamma\omega^2}{2g}\left[\frac{A}{b^2}\left(\frac{x^2}{a^2}+\frac{3y^2}{b^2}-1\right)+a^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\right],\\ \sigma_y &= \frac{\gamma\omega^2}{2g}\left[\frac{A}{a^2}\left(\frac{3x^2}{a^2}+\frac{y^2}{b^2}-1\right)+b^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\right],\\ \tau_{xy} &= -\frac{\gamma\omega^2}{2g}A\frac{2xy}{a^2b^2}. \end{aligned}\right\} \qquad (34.1)$$

γ is here the specific weight of the material and g the acceleration of gravity

$$\left.\begin{aligned} A &= b^4c^2\frac{a_{11}c^4+2a_{12}c^2+a_{22}}{3a_{11}c^4+(2a_{12}+a_{66})c^2+3a_{22}},\\ c &= \frac{a}{b}. \end{aligned}\right\} \qquad (34.2)$$

This stress distribution satisfies all equations of the plane problem where the volume forces and their potentials are equal:

$$X=\frac{\gamma\omega^2}{g}x, \quad Y=\frac{\gamma\omega^2}{g}y, \quad \bar{U}=-\frac{\gamma\omega^2}{2g}(x^2+y^2) \qquad (34.3)$$

and the boundary conditions $\sigma_n=0$, $\tau_n=0$.

The maximum stress σ_ϑ (tensile stress) in areas normal to the edge is obtained at the ends of the minor axis of the ellipse:

$$\sigma_\vartheta=\frac{\gamma\omega^2}{g}a^2\frac{a_{11}c^4+2a_{12}c^2+a_{22}}{3a_{11}c^4+(2a_{12}+a_{66})c^2+3a_{22}}. \qquad (34.4)$$

The stress at the center is equal to

$$\sigma_\vartheta=\frac{\gamma\omega^2}{g}a^2\frac{a_{11}c^4+0{,}5a_{66}c^2+a_{22}}{3a_{11}c^4+(2a_{12}+a_{66})c^2+3a_{22}}. \qquad (34.5)$$

When the elastic constants of the material satisfy the condition $a_{66}>4a_{12}$, the highest stress of the whole plate is reached at the center and it is determined by Eq. (34.5).

The stress distribution in a round disk of radius a rotating about an ideal axis passing through the center (Fig. 65) is obtained when we put $b = a$ and $c = 1$. In polar coordinates the stress components depend only on the distance r from the center of rotation:

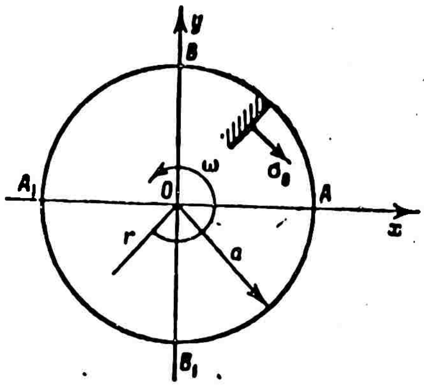

Fig. 65

$$\begin{rcases} \sigma_r = \frac{\gamma\omega^2}{2g}\left(1 - \frac{A}{a^4}\right)(a^2 - r^2), \\ \sigma_\theta = \frac{\gamma\omega^2}{2g}\left[\left(1 - \frac{A}{a^4}\right)(a^2 - r^2) + \frac{2A}{a^4} r^2\right], \\ \tau_{r\theta} = 0; \end{rcases} \tag{34.6}$$

$$A = a^4 \frac{a_{11} + 2a_{12} + a_{22}}{3a_{11} + 2a_{12} + a_{66} + 3a_{22}}. \tag{34.7}$$

In the case of an isotropic disk with the Poisson coefficient ν we obtain the well-known stress distribution:*

$$\begin{rcases} \sigma_r = \frac{\gamma\omega^2}{2g} \cdot \frac{3+\nu}{4}(a^2 - r^2), \\ \sigma_\theta = \frac{\gamma\omega^2}{2g}\left(\frac{3+\nu}{4} a^2 - \frac{1+3\nu}{4} r^2\right), \\ \tau_{r\theta} = 0. \end{rcases} \tag{34.8}$$

All the above formulas only permit the calculation of stresses averaged with respect to the thickness. In fact the stresses vary with the thickness, i.e., they also depend on the coordinate z directed perpendicularly to the plate with the origin in the mid-plane.

When we denote by $\sigma_x, \sigma_y, \tau_{xy}$ the true stresses in the plane and by $\bar{\sigma}_x, \bar{\sigma}_y, \bar{\tau}_{xy}$ their mean values with respect to the thickness, the formulas which take the stress variation with respect to thickness h into account can be rewritten as follows:

$$\begin{rcases} \sigma_x = \bar{\sigma}_x + \frac{\gamma\omega^2}{2g} B\left(\frac{h^2}{4} - 3z^2\right), \\ \sigma_y = \bar{\sigma}_y + \frac{\gamma\omega^2}{2g} C\left(\frac{h^2}{4} - 3z^2\right), \\ \tau_{xy} = \bar{\tau}_{xy} + \frac{\gamma\omega^2}{2g} D\left(\frac{h^2}{4} - 3z^2\right), \\ \sigma_z = \tau_{xz} = \tau_{yz} = 0. \end{rcases} \tag{34.9}$$

The constants B, C, D for the elliptic nonorthotropic plate are determined from the following equations:

$$\left.\begin{aligned} &Ba_{11}+Ca_{12}+Da_{16}=\frac{A}{3a^4}(a_{13}c^2+3a_{23})-\frac{1}{3}\left(a_{13}+\frac{a_{23}}{c^2}\right),\\ &Ba_{12}+Ca_{22}+Da_{26}=\frac{A}{3b^4}\left(3a_{13}+\frac{a_{23}}{c^2}\right)-\frac{1}{3}(a_{13}c^2+a_{23}),\\ &Ba_{16}+Ca_{26}+Da_{66}=-\frac{2A}{3a^2b^2}\cdot a_{36}\end{aligned}\right\} \qquad (34.10)$$

[a_{ij} are the elastic constants from the equations of Hooke's generalized law (2.5)].

In particular, for an elliptic isotropic plate we obtain

$$\left.\begin{aligned} &a_{11}=a_{22}=\frac{1}{E},\quad a_{12}=a_{23}=a_{13}=-\frac{\nu}{E},\\ &a_{66}=\frac{2(1+\nu)}{E},\quad a_{16}=a_{26}=a_{36}=0;\\ &B=\frac{2\nu}{3(1-\nu^2)}\cdot\frac{(c^4+c^2+2)(1+2\nu)+c^2(3c^2+1)\nu}{3c^4+2c^2+3},\\ &C=\frac{2\nu}{3(1-\nu^2)}\cdot\frac{(2c^4+c^2+1)(1+2\nu)+(c^2+3)\nu}{3c^4+2c^2+3},\\ &D=0.\end{aligned}\right\} \qquad (34.11)$$

§35. STRESS DISTRIBUTION IN ROTATING CURVILINEAR-ANISOTROPIC DISK

It is not difficult also to obtain the stress distribution in a rotating round disk whose anisotropy is not linear but cylindrical.

Let us assume that the disk represented in Fig. 65 possesses a cylindrical anisotropy with a pole of anisotropy located at the center and, moreover, that it is orthotropic so that every radial plane is a plane of elastic symmetry. The stress distribution in such a disk, for both the case of a massive disk and that of a disk possessing a round hole in the center, is obtained with the help of a stress function only depending on the distance r. This function is the solution of the nonhomogeneous equation (12.11) where

$$U=-\frac{\gamma\omega^2}{2g}r^2. \qquad (35.1)$$

It reads

$$F=f_0(r)=A+Br^2+Cr^{1+k}+Dr^{1-k}+\frac{\gamma\omega^2}{2g}(3-k^2-2\nu_\theta)r^2. \qquad (35.2)$$

Here

$$k=\sqrt{\frac{E_\theta}{E_r}}. \qquad (35.3)$$

E_r, E_θ are Young's moduli for the principal directions, the radial direction r and the tangential one θ; ν_r, ν_θ are Poisson's coefficients; γ is the specific gravity of the material and ω the angular

velocity. For the stress components and the projections of the displacements the following general formulas are obtained:

$$\left.\begin{aligned} \sigma_r &= C(1+k)r^{k-1} + D(1-k)r^{-k-1} - \frac{\gamma\omega^2}{g}\cdot\frac{3+\nu_\theta}{9-k^2}r^2, \\ \sigma_\theta &= C(1+k)kr^{k-1} - D(1-k)kr^{-k-1} - \frac{\gamma\omega^2}{g}\,\frac{k^2+3\nu_\theta}{9-k^2}r^2, \\ \tau_{r\theta} &= 0; \end{aligned}\right\} \qquad (35.4)$$

$$\left.\begin{aligned} E_\theta u_r &= C(1+k)(k-\nu_\theta)r^k - D(1-k)(k+\nu_\theta)r^{-k} - \frac{\gamma\omega^2}{g}\cdot\frac{k^2-\nu_\theta^2}{9-k^2}r^3, \\ u_\theta &= 0. \end{aligned}\right\}. \qquad (35.5)$$

We have here omitted the constant B as it corresponds to a many-valued displacement u_ϑ proportional to the angle θ; rigid displacements have also been neglected. The constants C and D are chosen such that at the rim and also at the edge of the opening the necessary conditions are satisfied.

A solution of the problem of the rotating disk, massive or weakened by a central opening, with free peripheral edge, has been found by G.S. Glushkov. Let us give the formulas for the stress in a massive disk:*

$$\left.\begin{aligned} \sigma_r &= \frac{\gamma\omega^2a^2}{g}\cdot\frac{3+\nu_\theta}{9-k^2}\left[\left(\frac{r}{a}\right)^{k-1} - \left(\frac{r}{a}\right)^2\right], \\ \sigma_\theta &= \frac{\gamma\omega^2a^2}{g}\cdot\frac{k}{9-k^2}\left[(3+\nu_\theta)\left(\frac{r}{a}\right)^{k-1} - k(1+3\nu_r)\left(\frac{r}{a}\right)^2\right], \\ \tau_{r\theta} &= 0. \end{aligned}\right\} \qquad (35.6)$$

With $k=1,\ \nu_\theta=\nu_r=\nu$ we obtain the stress distribution in an isotropic disk [see (34.8)]. With materials for which $k > 1$, i.e., $E_\theta > E_r$, the stress components in the center of the disk are vanishing and the stress becomes highest at the edge of the ring:

$$(\sigma_\theta)_a = \frac{\gamma\omega^2a^2}{g}\cdot\frac{k-\nu_\theta}{3+k}. \qquad (35.7)$$

With such materials for which $k < 1$, i.e., $E_\theta < E_r$, the stress must increase with decreasing distance to the center as this results from Eq. (35.6). In such cases the stresses must be concentrated around the pole of anisotropy, similarly as the concentration in a disk which is uniformly loaded at the rim (see §26 and Fig. 45).

When the peripheral edge of the disk is fixed to a perfectly rigid ring which cannot be deformed, the condition $u_r = u_\theta = 0$. must be satisfied on it. On the basis of the general formulas (35.4) and (35.5) we obtain the following stress distribution for a massive disk:

$$\left.\begin{aligned} \sigma_r &= \frac{\gamma\omega^2a^2}{g(9-k^2)}\left[(k+\nu_\theta)\left(\frac{r}{a}\right)^{k-1} - (3+\nu_\theta)\left(\frac{r}{a}\right)^2\right], \\ \sigma_\theta &= \frac{\gamma\omega^2a^2}{g(9-k^2)}\left[k(k+\nu_\theta)\left(\frac{r}{a}\right)^{k-1} - (k^2+3\nu_\theta)\left(\frac{r}{a}\right)^2\right], \\ \tau_{r\theta} &= 0. \end{aligned}\right\} \qquad (35.8)$$

On the surface of the contact between disk and stiff ring we obtain the following stresses (compressive):

$$(\sigma_r)_a = -\frac{\gamma\omega^2 a^2}{g} \cdot \frac{1}{3+k}, \quad (\sigma_\theta)_a = \nu_\theta (\sigma_r)_a. \tag{35.9}$$

Also in this case with $k < 1$ $(E_\theta < E_r)$ the stresses will be concentrated at the center.

§36. ROTATING NONHOMOGENEOUS CURVILINEAR-ANISOTROPIC DISK

Using the results of the preceding section and the method described in §27, we can obtain the stress distribution in a rotating disk consisting of a series of annular layers soldered or glued together, which possess cylindrical anisotropy and different elastic constants.

Let the disk shown in Fig. 46 (§27) rotate with constant angular velocity ω about an axis passing through the center. The center is assumed to have a round opening. We consider the case where the peripheral edge and the edge of the opening are free from external loads and their displacements are not subject to any limitations, i.e., $p = q = 0$. We again use the denotations of §27 and introduce additionally $\gamma_m (m = 1, 2, \ldots, n)$, the specific weight of the material constituting layer number m. In the case given the volume forces must be taken into account whose potential is for each layer equal to

$$\overline{U}_m = -\frac{\gamma_m \omega^2}{2g} r^2. \tag{36.1}$$

Denoting as before by q_{m-1} and q_m normal forces acting on the inner and the peripheral surfaces of layer number m $(q_0 = q_n = 0)$, we obtain the following formulas for the stresses and the radial displacements in this layer:

$$
\left.
\begin{aligned}
\sigma_r^{(m)} &= \frac{\gamma_m\omega^2}{g}\, a_m^2 \frac{3+\nu_\theta^{(m)}}{9-k_m^2}\left[\frac{1-c_m^{k_m+3}}{1-c_m^{2k_m}}\left(\frac{r}{a_m}\right)^{k_m-1} + \right.\\
&\quad \left. + \frac{1-c_m^{k_m-3}}{1-c_m^{2k_m}}\, c_m^{k_m+3}\left(\frac{a_m}{r}\right)^{k_m+1} - \left(\frac{r}{a_m}\right)^2\right] + \\
&\quad + \frac{q_{m-1}c_m^{k_m+1}}{1-c_m^{2k_m}}\left[\left(\frac{r}{a_m}\right)^{k_m-1} - \left(\frac{a_m}{r}\right)^{k_m+1}\right] + \\
&\quad + \frac{q_m}{1-c_m^{2k_m}}\left[-\left(\frac{r}{a_m}\right)^{k_m-1} + c_m^{2k_m}\left(\frac{a_m}{r}\right)^{k_m+1}\right],\\
\sigma_\theta^{(m)} &= \frac{\gamma_m\omega^2}{g}\cdot\frac{a_m^2}{9-k_m^2}\left\{(3+\nu_\theta^{(m)})\,k_m\left[\frac{1-c_m^{k_m+3}}{1-c_m^{2k_m}}\left(\frac{r}{a_m}\right)^{k_m-1} - \right.\right.\\
&\quad \left.\left. - \frac{1-c_m^{k_m-3}}{1-c_m^{2k_m}}\, c_m^{k_m+3}\left(\frac{a_m}{r}\right)^{k_m+1}\right] - (k_m^2+3\nu_\theta^{(m)})\left(\frac{r}{a_m}\right)^2\right\} + \\
&\quad + \frac{q_{m-1}c_m^{k_m+1}k_m}{1-c_m^{2k_m}}\left[\left(\frac{r}{a_m}\right)^{k_m-1} + \left(\frac{a_m}{r}\right)^{k_m+1}\right] - \\
&\quad - \frac{q_m k_m}{1-c_m^{2k_m}}\left[\left(\frac{r}{a_m}\right)^{k_m-1} + c_m^{2k_m}\left(\frac{a_m}{r}\right)^{k_m+1}\right],\\
\tau_{r\theta}^{(m)} &= 0;
\end{aligned}
\right\}
\qquad (36.2)
$$

$$
\left.
\begin{aligned}
u_r^{(m)} &= \frac{\gamma_m\omega^2 a_m^3}{gE_\theta^{(m)}(9-k_m^2)}\left\{(3+\nu_\theta^{(m)})\left[(k_m-\nu_\theta^{(m)})\frac{1-c_m^{k_m+3}}{1-c_m^{2k_m}}\left(\frac{r}{a_m}\right)^{k_m} - \right.\right.\\
&\quad \left. - (k_m+\nu_\theta^{(m)})\frac{1-c_m^{k_m-3}}{1-c_m^{2k_m}}\, c_m^{k_m+3}\left(\frac{a_m}{r}\right)^{k_m}\right] - \\
&\quad \left. - [k_m^2-(\nu_\theta^{(m)})^2]\left(\frac{r}{a_m}\right)^3\right\} + \frac{q_{m-1}a_m c_m^{k_m+1}}{E_\theta^{(m)}(1-c_m^{2k_m})}\times\\
&\quad \times\left[(k_m-\nu_\theta^{(m)})\left(\frac{r}{a_m}\right)^{k_m} + (k_m+\nu_\theta^{(m)})\left(\frac{a_m}{r}\right)^{k_m}\right] - \\
&\quad - \frac{q_m a_m}{E_\theta^{(m)}(1-c_m^{2k_m})}\left[(k_m-\nu_\theta^{(m)})\left(\frac{r}{a_m}\right)^{k_m} + \right.\\
&\quad \left. + (k_m+\nu_\theta^{(m)})\,c_m^{2k_m}\left(\frac{a_m}{r}\right)^{k_m}\right]\\
&\quad (m = 1,\ 2,\ \ldots,\ n, \quad q_0 = q_n = 0).
\end{aligned}
\right\}
\qquad (36.3)
$$

The unknown forces q_m are determined from the conditions of equality of the radial displacements of the points on the contact surfaces:

$$
\text{with} \quad r = a_{m-1} \quad u_r^{(m-1)} = u_r^{(m)}. \qquad (36.4)
$$

Hence we obtain the "equation of three forces" which differs from Eq. (27.6) only in its right-hand side:

$$q_{m+1}a_{m+1}\alpha_{m+1}+q_m a_m\beta_m+q_{m-1}a_{m-1}\alpha_m=\frac{\omega^2 a_m^3}{g}\delta_m \qquad (36.5)$$
$$(m=1,\ 2,\ \ldots,\ n-1).$$

The coefficients α_m and β_m are determined by Eqs. (27.7), δ_m by the formula

$$\delta_m=\frac{\gamma_m}{9-k_m^2}\left(\frac{1+3\nu_r^{(m)}}{E_r^{(m)}}-\frac{3+\nu_\theta^{(m)}}{E_\theta^{(m)}}k_m\frac{1-2c_m^{k_m+3}+c_m^{2k_m}}{1-c_m^{2k_m}}\right)-$$
$$-\frac{\gamma_{m+1}}{9-k_{m+1}^2}\left(\frac{1+3\nu_r^{(m+1)}}{E_r^{(m+1)}}-\frac{3+\nu_\theta^{(m+1)}}{E_\theta^{(m+1)}}k_{m+1}\frac{1-2c_{m+1}^{k_{m+1}-3}+c_{m+1}^{2k_{m+1}}}{1-c_{m+1}^{2k_{m+1}}}\right). \qquad (36.6)$$

Attributing the values 1, 2, ... up to $n-1$ to m, we can determine from Eq. (36.5) successively all unknown forces on the contact surfaces entering the stress formula (36.2). In particular, when the disk consists of two rings (see Fig. 47 where $p = 0$) we have

$$n=2,\quad q_1=\frac{\omega^2 a_1^2}{g}\cdot\frac{\delta_1}{\beta_1}. \qquad (36.7)$$

The stresses in a rotating disk or ring, displaying cylindrical anisotropy and being orthotropic but having variable moduli and Poisson coefficients and constant density γ, are determined by means of the functions $f_0(r)$ according to the formulas

$$\sigma_r=\frac{f_0'(r)}{r}-\frac{\gamma\omega^2}{2g}r^2,\quad \sigma_\theta=f_0''(r)-\frac{\gamma\omega^2}{2g}r^2,\quad \tau_{r\theta}=0. \qquad (36.8)$$

The function f_0 satisfies the inhomogeneous equation with variable coefficients

$$f_0'''+\left(\frac{1}{r}-\frac{E_\theta'}{E_\theta}\right)f_0''+\left(\frac{\nu_\theta E_\theta'}{rE_\theta}-\frac{\nu_\theta'}{r}-\frac{E_\theta}{r^2E_r}\right)f_0'=$$
$$=\frac{\gamma\omega^2}{2g}\left[\left(3-2\nu_\theta-\frac{E_\theta}{E_r}\right)r+\left(\frac{\nu_\theta-1}{E_\theta}E_\theta'-\nu_\theta'\right)r^2\right]. \qquad (36.9)$$

A general expression for f_0' can be written in the form

$$f_0'=\varphi_0(r)+A\varphi_1(r)+B\varphi_2(r), \qquad (36.10)$$

where φ_1 and φ_2 are linear-independent particular solutions of the homogeneous equation corresponding to (36.9) while φ_0 is an arbitrary particular solution of the inhomogeneous equation. The constants A and B are determined from the boundary conditions on the peripheral contour and the contour of the opening (if it exists).

In the simplest case Young's moduli are power functions of the distance and the Poisson coefficients are constants:

$$E_r=E_{rm}r^m,\quad E_\theta=E_{\theta m}r^m,\quad \nu_\theta=\text{const},\qquad \nu_r=\nu_\theta\frac{E_{rm}}{E_{\theta m}} \qquad (36.11)$$

(m is an arbitrary real number). In this case the particular solu-

tions $\varphi_0, \varphi_1, \varphi_2$ are also power functions of the distance r. Without discussing all possible cases in detail, we only give the solution for the massive disk with free peripheral edge:

$$\left.\begin{aligned} \sigma_r &= \frac{\gamma\omega^2a^2}{g}\frac{3+\nu_\theta-m}{9-k^2-(3-\nu_\theta)m}\left[\left(\frac{r}{a}\right)^{n_1-1}-\left(\frac{r}{a}\right)^2\right], \\ \sigma_\theta &= \frac{\gamma\omega^2a^2}{g}\frac{1}{9-k^2-(3-\nu_\theta)m}\times \\ &\quad\times\left[(3+\nu_\theta-m)n_1\left(\frac{r}{a}\right)^{n_1-1}-(k^2+3\nu_\theta-\nu_\theta m)\left(\frac{r}{a}\right)^2\right], \\ \tau_{r\theta} &= 0 \end{aligned}\right\} \tag{36.12}$$

$$\left(k=\sqrt{\frac{E_{\theta m}}{E_{rm}}},\quad n_1=\frac{1}{2}\left[\sqrt{m^2+4(k^2-\nu_\theta m)}+m\right]\right). \tag{36.13}$$

With $m = 0$ we hence obtain the well-known solution for a curvilinear-anisotropic disk with constant Young's moduli [Eq. (35.6)].

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[Footnotes]

149* Muskhelishvili, N.I., Nekotoryye osnovnyye zadachi matematicheskoy teorii uprugosti [Some Fundamental Problems of the Mathematical Theory of Elasticity]. Izd. AN SSSR, 1954, pages 238-244.

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150 Cf. our paper referred to in the second footnote of the foregoing page.

151 Okubo, H., The Stress Distribution in an Aeolotropic Circular Disk Compressed Diametrically, Journ. of Mathem. and Physics (Moore), Vol. XXXI, No. 1, 1952.

152 Lekhnitskiy, S.G., Raspredeleniye napryazheniy vo vrashchayushcheysya ellipticheskoy anizotropnoy plastinke [Stress Distribution in Rotating Elliptic Anisotropic Plate], Uch. zap. Leningr. gos. un-ta, seriya fiz.-matem. nauk, [Scient. Journal of the Leningrad State University, Physico-Mathematical Series], No. 13, 1944.

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156 Glushkov, G.S., Opredeleniye napryazheniy vo vrashchayushchemsya diske pri razlichnykh uprugikh svoystvakh materiala v dvukh napravleniyakh [Determination of Stresses in a Rotating Disk with Various Elastic Properties of the Material in Two Directions], Trudy Moskovskogo stanko-instrumental'nogo in-ta [Transactions of the Moscow Institute of Tools and Instruments], Collection III, 1939. We changed a little the denotations used by G.S. Glushkov.

Chapter 6

DISTRIBUTION OF STRESSES IN A PLATE WITH ELLIPTIC OR ROUND APERTURES

§37. DETERMINATION OF STRESSES IN A PLATE WITH ELLIPTIC APERTURE

In the present chapter we shall consider the problems connected with the determination of stresses in a plate weakened by an opening and deformed by forces acting on its midplane. It is well known that in an isotropic plate with an opening, which is not filled and not reinforced, the influence of the opening, compared with a massive, nonweakened plate, results in an increase of the stresses at certain points around the opening. This effect has been called the stress concentration. The problem of stress concentration in an isotropic plate has been developed rather completely, for various cases of apertures and loads.* In the case of an anisotropic material it is only the problem of the stress distribution in a plate with elliptic or round aperture which has been studied in sufficient detail; for a series of other cases of apertures we only have approximate solutions at our disposal. In the following we give the solutions for a series of problems of stress distributions in an anisotropic uniform plate with elliptic or circular aperture, not filled or filled with a rigid core or a core of elastic material with different elastic properties.

Let us consider an anisotropic plate which is homogeneous but generally nonorthotropic, of arbitrary form, weakened by an elliptic aperture and deformed by forces which are distributed along the edge of the aperture and act on the midplane. If the aperture dimensions are small compared with the plate's dimensions and if it is not near the edge of the plate, the problem can be simplified by assuming it infinitely large, thus neglecting the influence of the peripheral edge. With this statement we shall consider the problem.

To begin with, we consider the first fundamental problem with given external forces.

The directions of the axes x, y coincide with the principal axes of the ellipse (Fig. 66) and we use the denotations: a_{ij} are the elastic constants of the equations of the generalized Hooke's law (2.5) or (33.1) written for the given system of coordinates x, y (they are assumed known); a, b are the semiaxes of the ellipse; h is the thickness of the plate, X_n, Y_n are the projections of the forces acting on the edge of the aperture (per unit

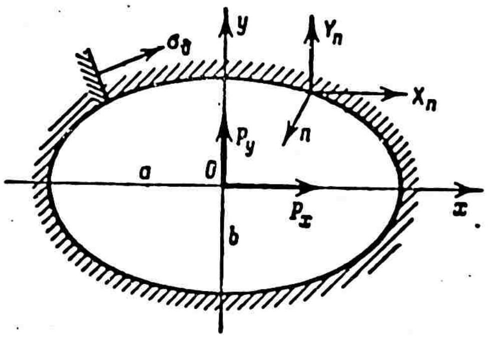

Fig. 66

area); P_x, P_y are the projections of the principal vector (resultant) of these forces. The contour equation of the opening reads in parametric form:

$$\begin{aligned} x &= a\cos\vartheta, \\ y &= b\sin\vartheta, \end{aligned} \tag{37.1}$$

and the given forces are taken to be functions of the parameter ϑ which varies from 0 to 2π with a full circumvention of the contour.

We use a complex representation of the stresses, by means of the functions $\Phi_1(z_1)$ and $\Phi_2(z_2)$ [see §8, Eqs. (8.2)-(8.6)]. The functions Φ_1 and Φ_2 must satisfy the boundary conditions (8.7). We expand the given forces X_n, Y_n in Fourier series. The boundary conditions (8.7) in the general case will then read

$$\left.\begin{aligned} 2\operatorname{Re}[\Phi_1(z_1)+\Phi_2(z_2)] &= \\ &= \frac{P_y}{2\pi h}\vartheta+\alpha_0+\sum_{m=1}^{\infty}(\alpha_m\sigma^m+\bar{\alpha}_m\sigma^{-m}), \\ 2\operatorname{Re}[\mu_1\Phi_1(z_1)+\mu_2\Phi_2(z_2)] &= \\ &= -\frac{P_x}{2\pi h}\vartheta+\beta_0+\sum_{m=1}(\beta_m\sigma^m+\bar{\beta}_m\sigma^{-m}). \end{aligned}\right\} \tag{37.2}$$

Here $\sigma=e^{i\vartheta}$; α_m, β_m are given coefficients, complex in general, and depending on the law of force distribution along the jet of the aperture; $\bar{\alpha}_m$, $\bar{\beta}_m$ are quantities which are adjoined to the former; α_0, β_0 are arbitrary constants. The stresses must tend to zero as the distance to the aperture increases.

The solution is obtained with the help of the function*

$$\left.\begin{aligned} \Phi_1(z_1) &= A_0+A\ln\zeta_1+\sum_{m=1}^{\infty}\frac{\bar{\beta}_m-\mu_2\bar{\alpha}_m}{\mu_1-\mu_2}\zeta_1^{-m}, \\ \Phi_2(z_2) &= B_0+B\ln\zeta_2-\sum_{m=1}^{\infty}\frac{\bar{\beta}_m-\mu_1\bar{\alpha}_m}{\mu_1-\mu_2}\zeta_2^{-m}. \end{aligned}\right\} \tag{37.3}$$

Here

$$\zeta_1=\frac{z_1+\sqrt{z_1^2-a^2-\mu_1^2b^2}}{a-i\mu_1 b},\quad \zeta_2=\frac{z_2+\sqrt{z_2^2-a^2-\mu_2^2b^2}}{a-i\mu_2 b} \tag{37.4}$$

are functions which assume one and the same value at the edge of the aperture, namely $\sigma=e^{i\theta}$; A_0, B_0 are arbitrary constants and A and B are constants determined from the equations

$$\left.\begin{aligned}
&A+B-\bar{A}-\bar{B}=\frac{P_y}{2\pi hi},\\
&\mu_1 A+\mu_2 B-\bar{\mu}_1\bar{A}-\bar{\mu}_2\bar{B}=-\frac{P_x}{2\pi hi},\\
&\mu_1^2 A+\mu_2^2 B-\bar{\mu}_1^2\bar{A}-\bar{\mu}_2^2\bar{B}=-\frac{a_{16}}{a_{11}}\cdot\frac{P_x}{2\pi hi}-\frac{a_{12}}{a_{11}}\cdot\frac{P_y}{2\pi hi},\\
&\frac{1}{\mu_1}A+\frac{1}{\mu_2}B-\frac{1}{\bar{\mu}_1}\bar{A}-\frac{1}{\bar{\mu}_2}\bar{B}=\frac{a_{12}}{a_{22}}\cdot\frac{P_x}{2\pi hi}+\frac{a_{26}}{a_{22}}\cdot\frac{P_y}{2\pi hi}.
\end{aligned}\right\} \tag{37.5}$$

As the explicit expressions for A and B are too complicate we do not give them here.

The derivatives of the functions Φ_1 and Φ_2 read

$$\left.\begin{aligned}
&\Phi_1'(z_1)=\frac{1}{\sqrt{z_1^2-a^2-\mu_1^2b^2}}\left(A-\sum_{m=1}^{\infty}m\,\frac{\bar{\beta}_m-\mu_2\bar{\alpha}_m}{\mu_1-\mu_2}\,\zeta_1^{-m}\right),\\
&\Phi_2'(z_2)=\frac{1}{\sqrt{z_2^2-a^2-\mu_2^2b^2}}\left(B+\sum_{m=1}^{\infty}m\,\frac{\bar{\beta}_m-\mu_1\bar{\alpha}_m}{\mu_1-\mu_2}\,\zeta_2^{-m}\right).
\end{aligned}\right\} \tag{37.6}$$

On the contour of the aperture they take the form

$$\left.\begin{aligned}
&\Phi_1'=\frac{1}{i(a\sin\theta-\mu_1 b\cos\theta)}\left(A-\sum_{m=1}^{\infty}m\,\frac{\bar{\beta}_m-\mu_2\bar{\alpha}_m}{\mu_1-\mu_2}\,\sigma^{-m}\right),\\
&\Phi_2'=\frac{1}{i(a\sin\theta-\mu_1 b\cos\theta)}\left(B+\sum_{m=1}^{\infty}m\,\frac{\bar{\beta}_m-\mu_1\bar{\alpha}_m}{\mu_1-\mu_2}\,\sigma^{-m}\right).
\end{aligned}\right\} \tag{37.7}$$

The stress components are determined from Eqs. (8.2) and the projections of the displacement from Eqs. (8.3).

Let us give yet another formula which will often be used in what follows. The stress σ_ϑ in areas normal to the edge of the aperture, at the edge of the aperture itself, is equal to*

$$\begin{aligned}
\sigma_\theta=&\frac{2}{a^2\sin^2\theta+b^2\cos^2\theta}\times\\
&\times\operatorname{Re}\left\{\frac{i(\mu_1 a\sin\theta+b\cos\theta)^2}{a\sin\theta-\mu_1 b\cos\theta}\left(-A+\sum_{m=1}^{\infty}m\,\frac{\bar{\beta}_m-\mu_2\bar{\alpha}_m}{\mu_1-\mu_2}\,\sigma^{-m}\right)-\right.\\
&\left.-\frac{i(\mu_2 a\sin\theta+b\cos\theta)^2}{a\sin\theta-\mu_2 b\cos\theta}\left(B+\sum_{m=1}^{\infty}m\,\frac{\bar{\beta}_m-\mu_1\bar{\alpha}_m}{\mu_1-\mu_2}\,\sigma^{-m}\right)\right\}.
\end{aligned} \tag{37.8}$$

If at the edge of the opening the given displacements u^*, v^* are

$$\left.\begin{aligned} u^* &= \alpha_0 + \sum_{m=1}^{\infty} (\alpha_m \sigma^m + \bar{\alpha}_m \sigma^{-m}), \\ v^* &= \beta_0 + \sum_{m=1}^{\infty} (\beta_m \sigma^m + \bar{\beta}_m \sigma^{-m}), \end{aligned}\right\} \tag{37.9}$$

and the projections of the resultant forces (whose distribution along the contour is unknown), the functions Φ_1, Φ_2 determined from the boundary conditions (8.8) have the form

$$\left.\begin{aligned} \Phi_1(z_1) &= A_0 + A \ln \zeta_1 + \left[\bar{\alpha}_1 q_2 - \bar{\beta}_1 p_2 + \frac{1}{2}\omega(ibq_2 + ap_2)\right]\frac{1}{D\zeta_1} + \\ &\quad + \frac{1}{D}\sum_{m=2}^{\infty} (\bar{\alpha}_m q_2 - \bar{\beta}_m p_2)\zeta_1^{-m}, \\ \Phi_2(z_2) &= B_0 + B \ln \zeta_2 - \left[\bar{\alpha}_1 q_1 - \bar{\beta}_1 p_1 + \frac{1}{2}\omega(ibq_1 + ap_1)\right]\frac{1}{D\zeta_2} - \\ &\quad - \frac{1}{D}\sum_{m=2}^{\infty} (\bar{\alpha}_m q_1 - \bar{\beta}_m p_1)\zeta_2^{-m}. \end{aligned}\right\} \tag{37.10}$$

Here A_0, B_0 are arbitrary constants; A, B are constants obtained from the same equations (37.5); p_1, p_2, q_1, q_2 are constants determined by Eqs. (8.4);

$$D = p_1 q_2 - p_2 q_1; \tag{37.11}$$

ω is a constant expressing the rotation of the plate in the xy-plane, for the determination of which we must formulate additional conditions of reinforcement (in all cases considered below $\omega = 0$).

The functions of the complex variables $\Phi_1(z_1)$ and $\Phi_2(z_2)$, which represent the solutions to the problem considered, are determined in the form of series. This method of solution is not the only one. G.N. Savin suggested another method of solving the problem of stress determination in an anisotropic plate with elliptic aperture, based on the application of Schwartz's formula, which is well known in the theory of functions of complex variables; the expressions for the functions Φ_1 and Φ_2 are obtained in the form of integrals taken along the contour of the unit circle.*

When the resultant vector of the forces given on the contour of the aperture is equal to zero, Eqs. (37.3) for the functions of complex variables may be given a new but also integral form:

$$\left.\begin{aligned} \Phi_1(z_1) &= \frac{1}{\mu_1 - \mu_2} \cdot \frac{1}{2\pi i} \int_{\gamma} \frac{f_1 - \mu_2 f_2}{\sigma - \zeta_1(z_1)}\, d\sigma + A_0, \\ \Phi_2(z_2) &= -\frac{1}{\mu_1 - \mu_2} \cdot \frac{1}{2\pi i} \int_{\gamma} \frac{f_1 - \mu_1 f_2}{\sigma - \zeta_2(z_2)}\, d\sigma + B_0. \end{aligned}\right\} \tag{37.12}$$

Here

$$f_1 = \int_0^s X_n\, ds, \quad f_2 = -\int_0^s Y_n\, ds, \tag{37.13}$$

ds is the arc element of the apertural contour, $\sigma = e^{\theta i}$ is a point on the contour γ of the unit circle $|\zeta| = 1$, ζ_1, ζ_2 determined by Eqs. (37.4), A_0, B_0 are arbitrary constants. These expressions for the functions Φ_1 and Φ_2 may be more convenient than the series expressions (37.3) (e.g., in the case where concentrated forces are applied to the contour or where part of the contour is free from loads while on the other part a uniformly distributed force is acting, etc.).

It is sometimes more convenient to use representations of stresses and displacements in terms of the functions $\varphi_1(z_1')$ and $\varphi_2(z_2')$ of the variables $z_1' = z + \lambda_1\bar{z}$, $z_2' = z + \lambda_2\bar{z}$ [see Eqs. (8.11) and (8.12)].

We give the form of the functions Φ_1 and Φ_2 determining the stresses and displacements in an anisotropic plate with elliptic aperture, at the edge of which the forces or displacements are given:

$$\left.\begin{aligned}\varphi_1(z_1') &= A_0 + A\ln t_1 + \sum_{m=1}^{\infty} A_m t_1^{-m},\\ \varphi_2(z_2') &= B_0 + B\ln t_2 + \sum_{m=1}^{\infty} B_m t_2^{-m}.\end{aligned}\right\} \tag{37.14}$$

Here

$$t_k = \frac{z_k' + \sqrt{z_k'^2 - 2(a^2+b^2)\lambda_k - (a^2-b^2)(1+\lambda_k^2)}}{a+b+(a-b)\lambda_k} \tag{37.15}$$

$$(k = 1,\ 2).$$

On the contour of the aperture $t_1 = t_2 = \sigma$. The constants A_0, B_0 are arbitrary; A, B depend on the vector sum of the external forces and A_m, B_m are determined from the boundary conditions (according to what is given at the boundary).

§38. PARTICULAR CASES OF LOAD

Let us give solutions for some particular cases of stress distribution in a plate with elliptic aperture.* We restrict ourselves to the functions Φ_1 and Φ_2, the coefficients $\bar{\alpha}_m$, $\bar{\beta}_m$ and the stress σ_ϑ on the whole contour of the aperture and in individual points of it; the formula for σ_ϑ is left in its complex form as the separation of the real part (i.e., the actual execution of operations represented by the symbol Re) results in very cumbersome expressions which take much place without being of particular interest. In all cases considered, each of the functions Φ_1 and Φ_2 is not represented by the series but only by its first or second term. The expressions for the complex parameters are generally different:

$$\mu_1 = \alpha + \beta i,\quad \mu_2 = \gamma + \delta i \quad (\beta > 0,\ \delta > 0).$$

In the formulas we use the abbreviated denotations

$$c = \frac{a}{b},\quad l = a^2\sin^2\theta + b^2\cos^2\theta. \tag{38.1}$$

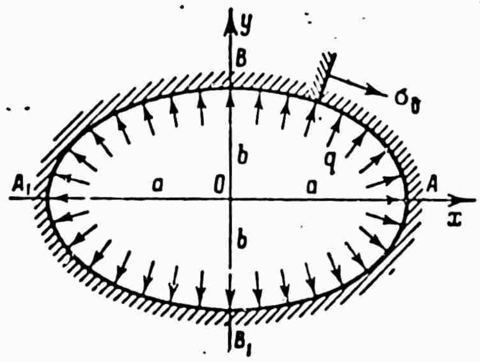

Fig. 67

1. Equilibrium pressure on the contour of the aperture. Normal forces q are assumed to act on the whole edge of the aperture in uniform distribution (per unit area, Fig. 67).

$$\Phi_1(z_1) = \frac{\bar{\beta}_1 - \mu_2\bar{\alpha}_1}{\mu_1 - \mu_2} \cdot \frac{1}{\zeta_1}, \quad \Phi_2(z_2) = -\frac{\bar{\beta}_1 - \mu_1\bar{\alpha}_1}{\mu_1 - \mu_2} \cdot \frac{1}{\zeta_2}; \tag{38.2}$$

$$\bar{\alpha}_1 = -\frac{qa}{2}, \quad \bar{\beta}_1 = -\frac{qbi}{2}; \tag{38.3}$$

$$\begin{aligned} \sigma_\theta = \frac{q}{i2}\,\mathrm{Re}\Big\{ & \frac{ie^{-\theta i}}{(a\sin\theta - \mu_1 b\cos\theta)(a\sin\theta - \mu_2 b\cos\theta)} \times \\ & \times [(\mu_1\mu_2 a - i\mu_1 b - i\mu_2 b)\,a^3\sin^3\theta + i(\mu_1\mu_2 - 2)\,a^2b^2\sin^2\theta\cos\theta + \\ & + (2\mu_1\mu_2 - 1)\,a^2b^2\sin\theta\cos^2\theta + (\mu_1 a + \mu_2 a - ib)\,b^3\cos^3\theta]\Big\}. \end{aligned} \tag{38.4}$$

At the points A and A_1 at the ends of the major axis (Fig. 67) where $\vartheta = 0$ and $\vartheta = \pi$,

$$\sigma_\theta = q\left[\frac{\alpha\gamma - \beta\delta}{(\alpha^2+\beta^2)(\gamma^2+\delta^2)} + c\left(\frac{\beta}{\alpha^2+\beta^2} + \frac{\delta}{\gamma^2+\delta^2}\right)\right]. \tag{38.5}$$

In an isotropic plate at the same points

$$\sigma_\theta = q(-1 + 2c). \tag{38.6}$$

At the points B and B_1, at the ends of the minor axis (Fig. 67), where $\theta = \frac{\pi}{2}, \theta = \frac{3\pi}{2}$,

$$\sigma_\theta = q\left(\alpha\gamma - \beta\delta + \frac{\beta+\delta}{c}\right). \tag{38.7}$$

In an isotropic plate at the same points

$$\sigma_\theta = q\left(-1 + \frac{2}{c}\right). \tag{38.8}$$

In a nonorthotropic plate the distribution of the stresses σ_ϑ along the edge of the aperture are found to be symmetric only with respect to its center O. In the case of an orthotropic plate in which the directions of the principal axes of the ellipse co-

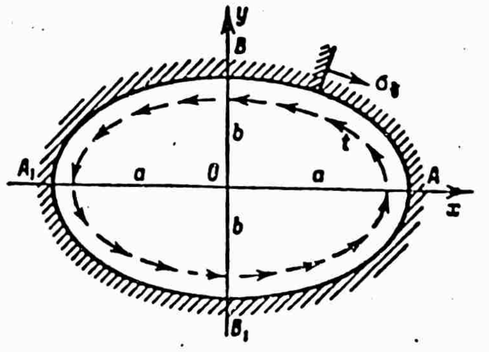

Fig. 68

incide with the principal directions of elasticity the σ_ϑ distribution also becomes symmetric with respect to these axes.

2. Tangential forces uniformly distributed along the edge of the aperture (Fig. 68). The functions Φ_1 and Φ_2 have the form (38.2), but in the given case

$$\bar{\alpha}_1 = \frac{tbi}{2}, \quad \bar{\beta}_1 = -\frac{ta}{2}, \tag{38.9}$$

where t is the intensity of the forces and

$$\begin{aligned}\sigma_\vartheta = -\frac{t}{l^2}\operatorname{Re}\Big\{&\frac{ie^{-\vartheta i}}{(a\sin\vartheta - \mu_1 b\cos\vartheta)(a\sin\vartheta - \mu_2 b\cos\vartheta)}\times \\ &\times[(\mu_1 a + \mu_2 a + i\mu_1\mu_2 b)a^3\sin^3\vartheta + (2 - \mu_1\mu_2)a^3 b\sin^2\vartheta\cos\vartheta + \\ &+ (2\mu_1\mu_2 - 1)ab^3 i\sin\vartheta\cos^2\vartheta + (a + i\mu_1 b + i\mu_2 b)b^3\cos^3\vartheta]\Big\}.\end{aligned} \tag{38.10}$$

3. Tension. A plate is extended by the forces p applied at a sufficiently large distance from the aperture (theoretically at infinity) and attack at an angle of φ relative to the major axis a (Fig. 69); the edge of the aperture is free from external forces.

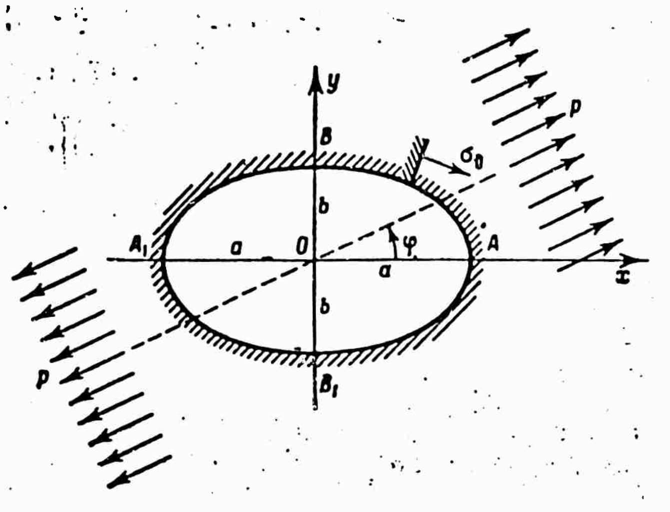

Fig. 69

The stress components are obtained by means of summation over the stresses in a massive and uniformly extended plate

$$\sigma_x^0 = p\cos^2\varphi, \quad \sigma_y^0 = p\sin^2\varphi, \quad \tau_{xy}^0 = p\sin\varphi\cos\varphi \tag{38.11}$$

and the stresses obtained by means of the functions Φ_1 and Φ_2 of the form (38.2), where

$$\left.\begin{aligned} \bar{\alpha}_1 &= -\frac{p\sin\varphi}{2}(a\sin\varphi - ib\cos\varphi), \\ \bar{\beta}_1 &= \frac{p\cos\varphi}{2}(a\sin\varphi - ib\cos\varphi). \end{aligned}\right\} \tag{38.12}$$

Let us give the expression for the stress σ_ϑ in the case where $\varphi = 0$, i.e., the plate is extended in the direction of the x-axis (Fig. 70):

$$\begin{gathered} \sigma_\vartheta = p\frac{a^2}{l^2}\sin^2\vartheta + \frac{pb}{l^2}\operatorname{Re}\left\{\frac{e^{-\vartheta i}}{(a\sin\vartheta - \mu_1 b\cos\vartheta)(a\sin\vartheta - \mu_2 b\cos\vartheta)}\times\right. \\ \left.\times\left[(\mu_1+\mu_2)a^3\sin^3\vartheta + (2-\mu_1\mu_2)a^2 b\sin^2\vartheta\cos\vartheta + b^3\cos^3\vartheta\right]\right\}. \end{gathered} \tag{38.13}$$

At the points A and A_1 (Fig. 70)

$$\sigma_\vartheta = p\frac{\alpha\gamma - \beta\delta}{(\alpha^2+\beta^2)(\gamma^2+\delta^2)}. \tag{38.14}$$

In an isotropic plate at the same points $\sigma_\vartheta = -p$.

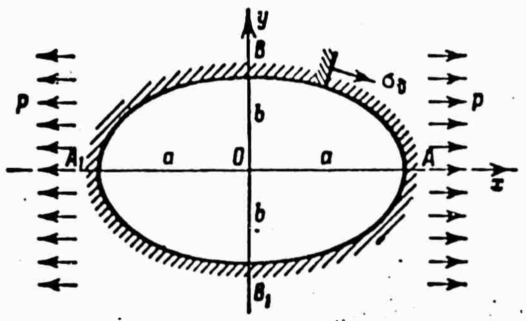

Fig. 70

At the points B and B_1 (at the ends of the diameter perpendicular to the tensile forces, Fig. 70)

$$\sigma_\vartheta = p\left(1 + \frac{\beta+\delta}{c}\right). \tag{38.15}$$

At the same points in an isotropic plate

$$\sigma_\vartheta = p\left(1 + \frac{2}{c}\right). \tag{38.16}$$

4. Shear. A rectangular plate with an elliptic aperture in its center is deformed by tangential forces of the intensity t, which are uniformly distributed with respect to the sides; the

major axis of the aperture makes an angle of φ with one of the sides.

When the dimensions of the aperture are small compared to the side dimensions, the plate may be considered infinitely large and the tensile forces t are applied at infinity (Fig. 71). The stress distribution is obtained by adding the stresses in the massive plate

$$\sigma_x^0 = -t\sin 2\varphi,\quad \sigma_y^0 = t\sin 2\varphi,\quad \tau_{xy}^0 = t\cos 2\varphi \tag{38.17}$$

and the stresses obtained by means of functions of the form (38.2) where

$$\left.\begin{aligned}\bar{\alpha}_1 &= \frac{t}{2}(-a\sin 2\varphi + ib\cos 2\varphi),\\ \bar{\beta}_1 &= \frac{t}{2}(a\cos 2\varphi + ib\sin 2\varphi).\end{aligned}\right\} \tag{38.18}$$

In particular, with φ = 0 (Fig. 72) we obtain the following law of stress distribution on the edge of the aperture:

$$\begin{aligned}\sigma_\theta = -t\frac{ab}{l^2}\sin 2\theta + \frac{t}{l^2}\mathrm{Re}\Big\{\frac{ie^{-\theta i}}{(a\sin\theta-\mu_1 b\cos\theta)(a\sin\theta-\mu_2 b\cos\theta)}\times\\ \times[(\mu_1 a+\mu_2 a-i\mu_1\mu_2 b)a^3\sin^3\theta+(2-\mu_1\mu_2)a^2 b\sin^2\theta\cos\theta+\\ +i(1-2\mu_1\mu_2)ab^2\sin\theta\cos^2\theta+(a-i\mu_1 b-i\mu_2 b)b^3\cos^3\theta]\Big\}.\end{aligned} \tag{38.19}$$

5. Bending of plate by moments. A rectangular beam-plate with an elliptic aperture in its center is bent by the moments M. The major axis of the aperture makes an angle of φ with the axis of the plate (Fig. 73).

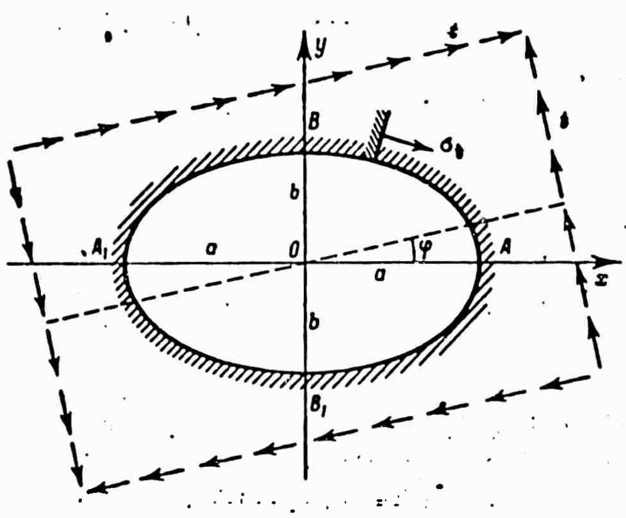

Fig. 71

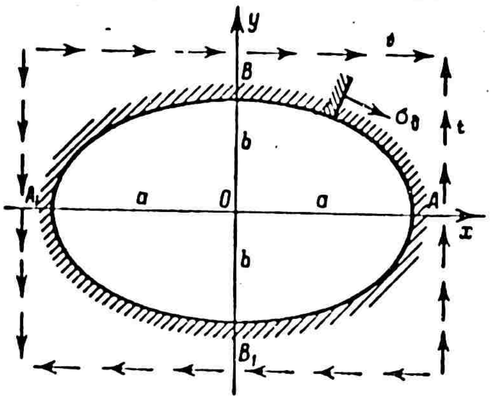

Fig. 72

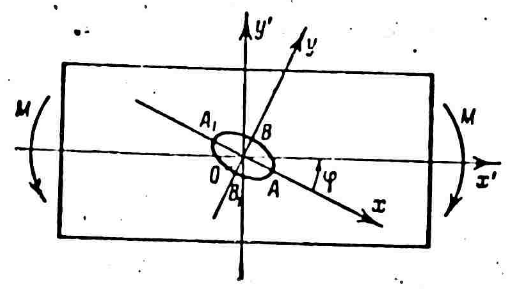

Fig. 73

In the case of small dimensions of the aperture the plate is assumed infinitely large. The stresses in it are obtained by adding the stresses in a massive plate-beam corresponding to pure bending*

$$\left.\begin{aligned} \sigma_x^0 &= \frac{M}{J}(y\cos\varphi - x\sin\varphi)\cos^2\varphi, \\ \sigma_y^0 &= \frac{M}{J}(y\cos\varphi - x\sin\varphi)\sin^2\varphi, \\ \tau_{xy}^0 &= \frac{M}{J}(y\cos\varphi - x\sin\varphi)\sin\varphi\cos\varphi, \end{aligned}\right\} \tag{38.20}$$

and the stresses obtained by means of functions of the form

$$\left.\begin{aligned} \Phi_1(z_1) &= \frac{\bar{\beta}_2 - \mu_2\bar{\alpha}_2}{\mu_1 - \mu_2}\cdot\frac{1}{\zeta_1^2}, \\ \Phi_2(z_2) &= -\frac{\bar{\beta}_2 - \mu_1\bar{\alpha}_2}{\mu_1 - \mu_2}\cdot\frac{1}{\zeta_2^2}, \end{aligned}\right\} \tag{38.21}$$

where

$$\left.\begin{aligned} \bar{\alpha}_2 &= -\frac{M}{8J}\sin\varphi\,(b^2\cos^2\varphi - a^2\sin^2\varphi + iab\sin 2\varphi), \\ \bar{\beta}_2 &= \frac{M}{8J}\cos\varphi\,(b^2\cos^2\varphi - a^2\sin^2\varphi + iab\sin 2\varphi). \end{aligned}\right\} \tag{38.22}$$

In particular, if the axis of the aperture is parallel to the sides of the plate, i.e., $\varphi = 0$ (Fig. 74), we obtain

$$\sigma_\vartheta = \frac{Mb}{J}\cdot\frac{a^2}{l^2}\sin^2\vartheta + \frac{Mb^2}{2Jl^2}\,\mathrm{Re}\left\{\frac{ie^{-2\vartheta i}}{(a\sin\vartheta - \mu_1 b\cos\vartheta)(a\sin\vartheta - \mu_2 b\cos\vartheta)}\times\right.$$
$$\left.\times\,[(\mu_1+\mu_2)a^3\sin^3\vartheta + (2-\mu_1\mu_2)a^2b\sin^2\vartheta\cos\vartheta + b^3\cos^3\vartheta]\right\}. \tag{38.23}$$

At the points A and A_1, at the ends of the diameter directed along the axis of the plate-beam (Fig. 74),

$$\sigma_\vartheta = \pm\frac{Mb}{2J}\cdot\frac{\beta\gamma + \alpha\delta}{(\alpha^2+\beta^2)(\gamma^2+\delta^2)}, \tag{38.24}$$

so that in an isotropic plate, at the same points, $\sigma_\vartheta = 0$.

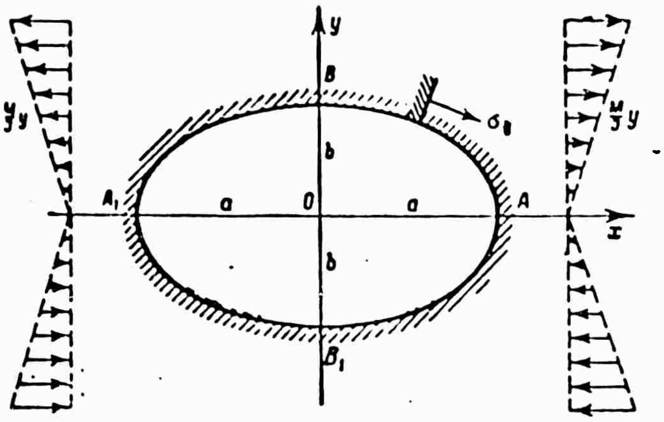

Fig. 74

At the points B and B_1 at the ends of a diameter perpendicular to the axis of the plate-beam (Fig. 74),

$$\sigma_\vartheta = \pm \frac{Mb}{J}\left(1 + \frac{\beta + \delta}{2c}\right), \tag{38.25}$$

and at the same points of an isotropic plate

$$\sigma_\vartheta = \pm \frac{Mb}{J}\left(1 + \frac{1}{c}\right). \tag{38.26}$$

G.N. Savin considered the case of bending of a plate with an elliptic aperture, with constant intersecting force and some other cases.*

In all the cases discussed the distribution of the stresses σ_ϑ along the edge of the aperture in an anisotropic and, moreover, nonorthotropic plate is governed by much more complicate laws than the distribution in an isotropic plate.

The stress distribution is symmetric with respect to the center of the aperture but in general nonsymmetric relative to its axes. The formulas for the stresses at the points A, A_1, B and B_1 at the ends of the axes of the ellipse give an idea on the concentration of stresses (at least for the orthotropic plate in which the principal directions of elasticity are parallel to the directions of the aperture's axes and $\varphi = 0$). All formulas remain valid in the case of a round aperture where $b = a$ and ϑ is the polar angle θ reckoned from the x-axis.

§39. STRESS DISTRIBUTION IN AN ORTHOTROPIC PLATE WITH A CIRCULAR APERTURE

In this and the following sections we shall consider some of the most interesting cases of stress distribution in an orthotropic plate which is weakened by a round aperture of radius a.**

In all cases the origin of coordinates is placed at the center of the aperture and the principal directions of elasticity are

taken according to the directions of the axes x and y. We use the denotations which were partly encountered previously, namely: E_1, E_2 are Young's moduli, ν_1, ν_2 are the Poisson coefficients and G is the modulus of shear (all for the principal directions), μ_1, μ_2 are complex parameters, solutions of the equation

$$\frac{\mu^4}{E_1}+\left(\frac{1}{G}-\frac{2\nu_1}{E_1}\right)\mu^2+\frac{1}{E_2}=0. \qquad (39.1)$$

Moreover, we introduce the notations

$$k=-\mu_1\mu_2=\sqrt{\frac{E_1}{E_2}}, \quad n=-i(\mu_1+\mu_2)=\sqrt{2\left(\frac{E_1}{E_2}-\nu_1\right)+\frac{E_1}{G}}; \qquad (39.2)$$

θ is the polar angle reckoned from the x-axis, E_θ is Young's modulus for extension (compression) in the direction of the tangent to the aperture's contour, connected with the elastic constants for the principal directions by the formula

$$\frac{1}{E_\theta}=\frac{\sin^4\theta}{E_1}+\left(\frac{1}{G}-\frac{2\nu_1}{E_1}\right)\sin^2\theta\cos^2\theta+\frac{\cos^4\theta}{E_2} \qquad (39.3)$$

Let us give by the way the formula for an orthotropic body which will be used in the following

$$\mu_1^2+\mu_2^2=2\nu_1-\frac{E_1}{G}. \qquad (39.4)$$

In all cases we point out the expressions for the stresses σ_θ acting on surfaces normal to the edge of the aperture, i.e., on the radial planes arranged at the edge of the aperture itself, and also formulas for σ_θ at individual points of the contour.

In order to illustrate this we give the results of calculations and stress distribution diagrams of a plate having the same elastic constants as a three-layer birch veneer (see §11).

Let us repeat the numerical values of the complex parameters

$$\mu_1=4{,}11i, \ \mu_2=0{,}343i, \ k=1{,}414, \ n=4{,}453,$$

when the x-axis is directed along the fibers of the sheet, and

$$\mu_1=0{,}243i, \ \mu_2=2{,}91i, \ k=0{,}707, \ n=3{,}153,$$

when the x-axis is perpendicular to the fibers of the sheet.

When we consider such a plate we shall call it simply "veneer,"* for the sake of brevity.

In the graphs the sections representing the magnitudes of the stresses σ_θ are plotted from the circles on the continuations of the radii; positive quantities are represented by arrows directed from the center outwardly, the negative ones by arrows pointing to the center. In each diagram we show in the upper right-hand corner a schematic diagram of the load; the dashed

lines represent the distribution of the stresses σ_θ in the isotropic plate, which is loaded by the same forces.

1. Normal pressure distributed uniformly on the edge of the aperture (Fig. 75).

$$\sigma_\theta = q\frac{E_\theta}{E_1}[-k + n(\sin^2\theta + k\cos^2\theta) + (1+\mu_1^2)(1+\mu_2^2)\sin^2\theta\cos^2\theta] \qquad (39.5)$$

q is the pressure per unit area.

For an isotropic plate $\sigma_\theta = q$. In an orthotropic plate the stress σ_θ is not distributed uniformly along the contour but according to a rather complex law, where the difference between the highest and lowest values of it may be very great.

At the points A and A_1 on the principal axis x (Fig. 75) where $\theta = 0$ and $\theta = \pi$,

$$\sigma_\theta = q\frac{n-1}{k}, \qquad (39.6)$$

and at the points B and B_1 $\left(\theta = \pm\frac{\pi}{2}\right)$ on the other principal axis

$$\sigma_\theta = q(n-k). \qquad (39.7)$$

A circular aperture, when deformed, becomes elliptic, with the semiaxes a' and b' = 2

$$\left.\begin{aligned} a' &= a\left[1 - q\left(\frac{1}{\sqrt{E_1E_2}} - \frac{n+\nu_1}{E_1}\right)\right], \\ b' &= a\left[1 - q\left(\frac{1-n}{\sqrt{E_1E_2}} - \frac{\nu_1}{E_1}\right)\right]. \end{aligned}\right\} \qquad (39.8)$$

In Fig. 75 we show the distribution of the stress σ_θ on the edge of the aperture of a veneer plate; the x-axis is parallel to the fibers of the sheet. The maximum value of the stress is equal to 3.04 q and is obtained at the points B and B_1. The minimum stress is small: it amounts to about 0.1 q.

2. Tangential forces distributed uniformly along the edge of the aperture (Fig. 76).

$$\sigma_\theta = t\frac{E_\theta}{2E_1}\sin 2\theta\,[(1-k)(n-k-1) + (1+\mu_1^2)(1+\mu_2^2)\cos 2\theta] \qquad (39.9)$$

(t is the force per unit area).

In an isotropic plate $\sigma_\theta = 0$.

Figure 76 gives the σ_θ distribution on the edge of an aperture in a veneer plate. The stress distribution along the contour is very irregular and changes of sign occur eight times. The maximum value exceeds t and is approximately equal to 1.5 t.

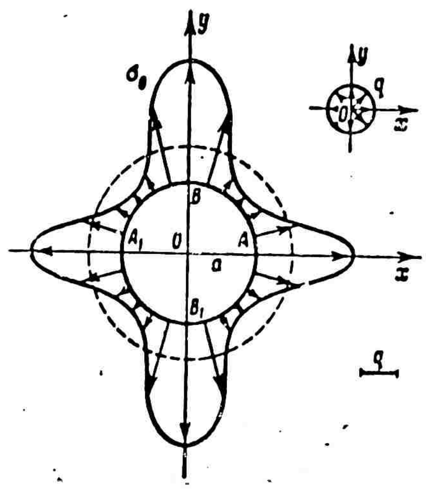

Fig. 75

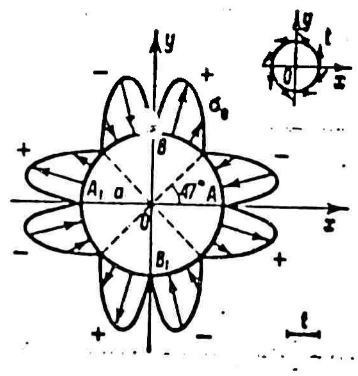

Fig. 76

3. Extension under an angle to the principal direction. For a plate extended by the forces p applied at a great distance from the aperture, which act under an angle of φ relative to the principal direction (Fig. 77), we obtain:

$$\sigma_\theta = p\frac{E_\theta}{E_1}\{[-\cos^2\varphi + (k+n)\sin^2\varphi]\,k\cos^2\theta + [(1+n)\cos^2\varphi - k\sin^2\varphi]\sin^2\theta - n(1+k+n)\sin\varphi\cos\varphi\sin\theta\cos\theta\}. \tag{39.10}$$

In an isotropic plate

$$\sigma_\theta = p[1 - 2\cos 2(\theta - \varphi)]. \tag{39.11}$$

The stress distribution in an orthotropic plate will not be symmetric with respect to the lines of action of the forces and the lines perpendicular to them; it is only symmetric relative to the center of the aperture. The maximum stress is not obtained at the ends of a diameter which is normal to the lines of action of the forces but at other points.

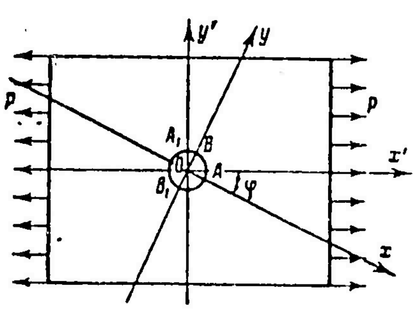

Fig. 77

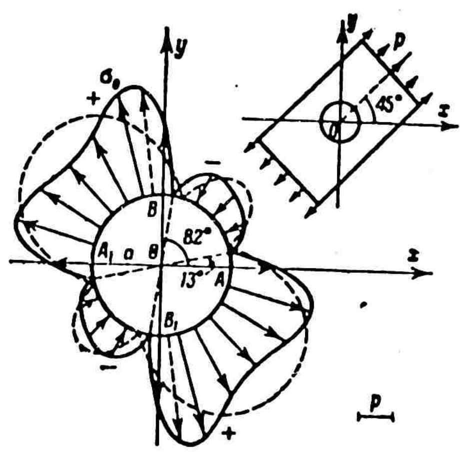

Fig. 78

Figure 78 shows the stress variation along the edge of the aperture of a veneer plate, extended at an angle of 45° with respect to the principal axes (the x-axis lies in the direction of the sheet fibers). The maximum stress was equal to 3.3 p whereas in the isotropic plate $\sigma_{max} = 3\, p$. In this case the coefficient of stress concentration for an orthotropic plate (K = 3.3) differs but slightly from the coefficient for the isotropic plate (K = 3). The stress vanishes at four points: θ = 13°, 82°, 193° and 262°.

4. Extension in the principal direction (Fig. 79). From Eq. (39.10) with φ = 0 we obtain

$$\sigma_\theta = p\frac{E_\theta}{E_1}[-k\cos^2\theta + (1+n)\sin^2\theta]. \tag{39.12}$$

The stress distribution will be symmetric relative to both principal axes x and y. At the points A and A_1 at the ends of a diameter parallel to the forces

$$\sigma_\theta = -\frac{p}{k}, \tag{39.13}$$

and at points B and B_1 at the ends of a diameter perpendicular to the forces,

$$\sigma_\theta = p(1+n). \tag{39.14}$$

One of these values will be highest in its absolute magnitude for the whole plate, but, without knowing the elastic constants, we cannot say which. It can be shown that not the stresses at the points B and B_1 will be highest in absolute value but the compressive stresses at points A and A_1.

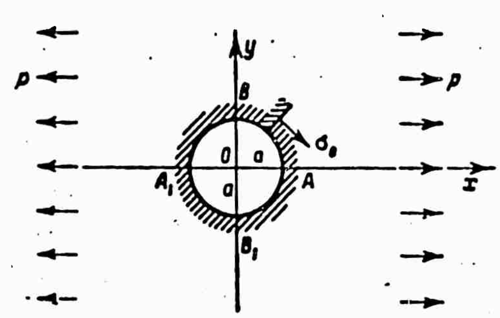

Fig. 79

A circular aperture is deformed to an elliptic one with the semiaxes a' and b' given by

$$\left.\begin{aligned} a' &= a\left[1+\frac{p}{E_1}(1+n)\right], \\ b' &= a\left(1-\frac{p}{\sqrt{E_1E_2}}\right). \end{aligned}\right\} \tag{39.15}$$

Figure 80 shows the variation of σ_θ along the contour of an

aperture in a venner plate extended in the direction of x for which Young's modulus is highest (i.e., along the fibers of the sheet).

At the points A and A_1

$$\sigma_\theta = -0{,}71p; \tag{39.16}$$

and at the points B and B_1

$$\sigma_\theta = 5{,}45p. \tag{39.17}$$

The points where $\sigma_\theta = 0$ are determined by the angles $\theta = \pm$ $\pm 27°$, $\pm 153°$.

The graph in Fig. 81 shows the distribution of σ_θ in a veneer plate extended in the direction of x for which Young's modulus is smallest (i.e., across the fibers of the sheet).

At the points A and A_1

$$\sigma_\theta = -1{,}41p; \tag{39.18}$$

at the points B and B_1

$$\sigma_\theta = 4{,}15p. \tag{39.19}$$

The stress vanishes at the points $\theta = \pm 22°30'$, $\pm 157°30'$.

In this case the concentration coefficient ($K = 4.15$) is smaller than in the case of a tension acting along the fibers of

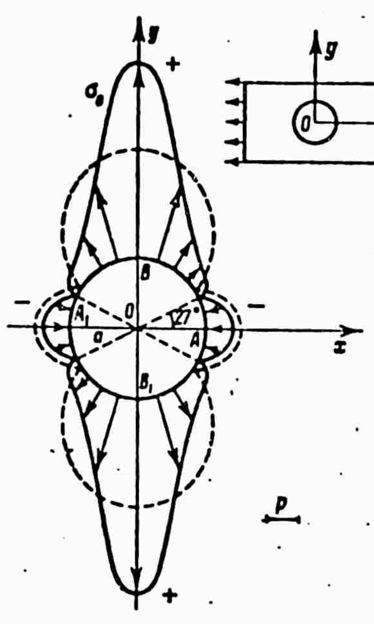

Fig. 80

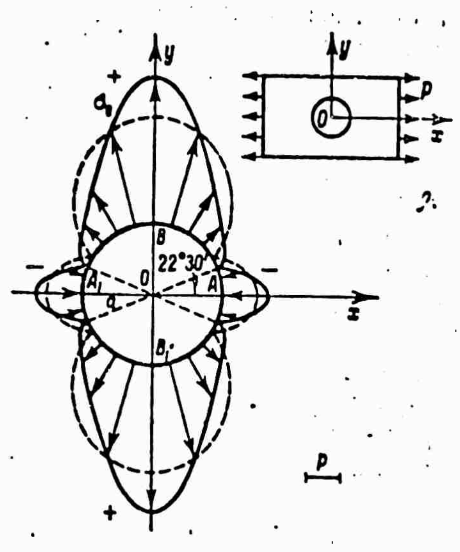

Fig. 81

the sheet (K = 5.45). In the case of an extension transverse to the fibers of the sheet the difference between the highest tensile stress and the highest compressive stress is not so considerable as in the former case. The ratio of the highest tensile stress to the highest compressive stress is 7.7 in the first case and only 2.95 in the second, i.e., it is here almost the same as in the case of the isotropic plate.

For a plate extended in the direction of the axis y $\left(\varphi = \frac{\pi}{2}\right)$, we obtain from (39.10):

$$\sigma_\theta = p\frac{E_\theta}{E_1}k[(k+n)\cos^2\theta - \sin^2\theta]. \tag{39.20}$$

5. Omnilateral extension of a plate. When a plate is extended in the two principal directions by equal forces p (this is equivalent to an omnilateral extension in the xy-plane) we have

$$\sigma_\theta = p\frac{E_\theta}{E_1}[-k + k(k+n)\cos^2\theta + (1+n)\sin^2\theta]. \tag{39.21}$$

In the case of an isotropic plate $\sigma_\theta = 2p$.

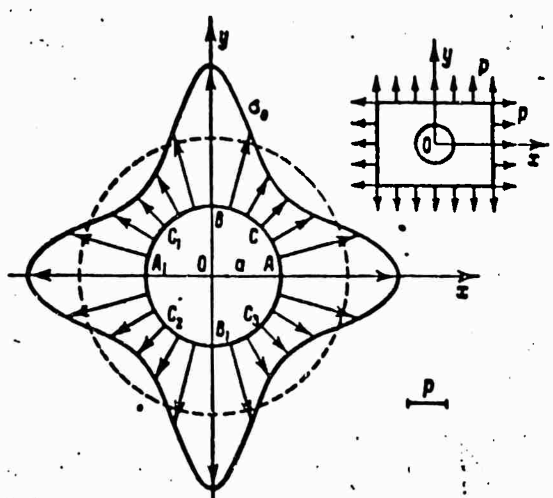

Fig. 82

In Fig. 82 we show the distribution of the stresses σ_θ along the edge of the aperture in a veneer plate; the x-axis is directed parallel to the fibers of the sheet. The maximum stress (at points B and B_1) is equal to

$$\sigma_\theta = 4{,}04p, \tag{39.22}$$

the minimum (at points C, C_1, C_2, C_3) is

$$\sigma_\theta = 1{,}09p. \tag{39.23}$$

6. Impeded compression of a plate. When a rectangular ortho-

tropic plate with a round aperture in the center is compressed in the principal direction but cannot expand in the transverse direction (owing to rigid walls, between which it is arranged, Fig. 83), this corresponds to a compression in the two principal directions of forces p and $\nu_2 p$.* We obtain

$$\sigma_\theta = p\frac{E_\theta}{E_1}\{k[1-\nu_2(k+n)]\cos^2\theta-(1-\nu_2 k+n)\sin^2\theta\}. \qquad (39.24)$$

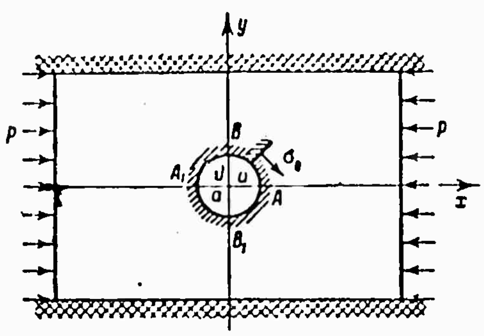

Fig. 83

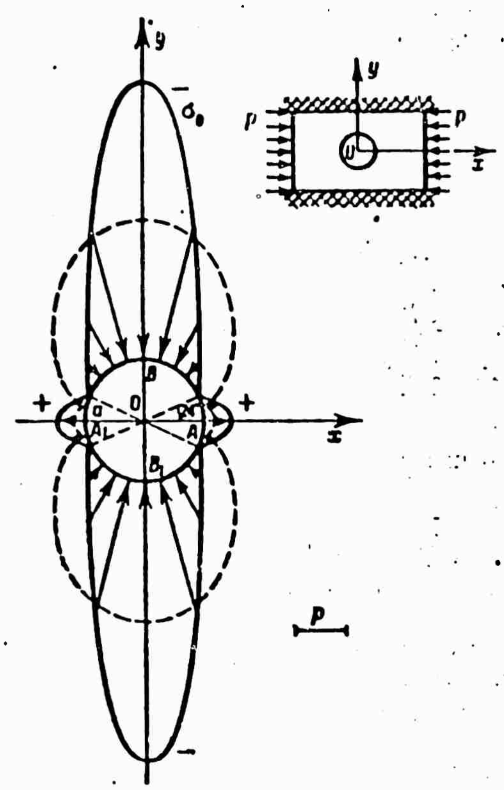

Fig. 84

In an isotropic plate

$$\sigma_\theta = -p[1+\nu-2(1-\nu)\cos 2\theta], \qquad (39.25)$$

where ν is Poisson's coefficient.

A graph of the distribution of the stress σ_θ along the edge of an aperture in a veneer plate compressed along the fibers of the sheet is shown in Fig. 84.

At the points A and A_1

$$\sigma_\theta = 0.56p. \qquad (39.26)$$

At the points B and B_1

$$\sigma_\theta = -5{,}40p. \tag{39.27}$$

The dashed lines in Fig. 84 show the variation of σ_θ in an isotropic plate with Poisson's coefficient $\nu = 0.25$; for such a plate at the points A and A_1

$$\sigma_\theta = 0{,}25p; \tag{39.28}$$

and at the points B and B_1

$$\sigma_\theta = -2{,}75p. \tag{39.29}$$

§40. DISTRIBUTION OF STRESSES IN AN ORTHOTROPIC PLATE WITH CIRCULAR APERTURE (CONTINUED)

7. Shear. A rectangular orthotropic plate with an aperture in its center is deformed by tangential forces t which are distributed uniformly along the sides; the principal axes x, y generally do not agree with the axes of symmetry of the plate (Fig. 85).

Considering the plate to be infinitely large we obtain a formula for the stress σ_θ near the aperture

$$\sigma_\theta = t\frac{E_\theta}{2E_1}(1+k+n)\{-n\cos 2\varphi \sin 2\theta + \\ + [(1+k)\cos 2\theta + k - 1]\sin 2\varphi\}. \tag{40.1}$$

In particular, for a plate on which forces parallel to the principal axes of elasticity ($\varphi = 0$) are acting, we have

$$\sigma_\theta = -t\frac{E_\theta}{2E_1}(1+k+n)\,n\sin 2\theta. \tag{40.2}$$

In an isotropic plate with $\varphi = 0$

$$\sigma_\theta = -4t\sin 2\theta. \tag{40.3}$$

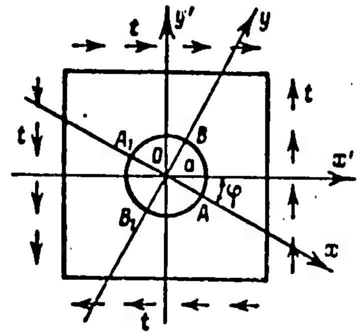

Fig. 85

The stress σ_θ on the contour of the aperture in an orthotropic plate vanishes at four points which, with $\varphi = 0$, coincide with the points of intersection of the contour and the principal directions.

Figure 86 shows the stress distribution in a veneer plate for the case where the forces t are parallel to the principal directions of elasticity. The direction of the x-axis corresponds to the direction for which Young's modulus is highest. The highest value of the stress σ_θ is obtained at four symmetrical points and is equal to

$$\sigma_{max} = 3,95t. \tag{40.4}$$

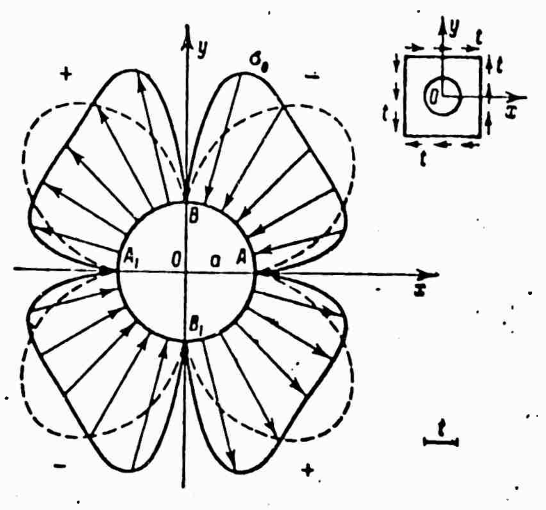

Fig. 86

In an isotropic plate we obtain $\sigma_{max} = 4\ t$, i.e., almost the same value.

Figure 87 shows a graph of the distribution of the stresses σ_θ along the contour of the aperture in a veneer plate for the case where the tangential forces act at an angle of 45° with respect to the principal directions $\left(\varphi = \frac{\pi}{4}\right)$. The values of the stress are highest at the points B and B_1 where

$$\sigma_\theta = -6,9t. \tag{40.5}$$

At the points A and A_1

$$\sigma_\theta = 4,9t. \tag{40.6}$$

A comparison of the values of the stresses obtained in the case of deformation by tangential forces attacking at various angles with respect to the principal directions, shows that the case where the forces have an angle of attack of 45° relative to

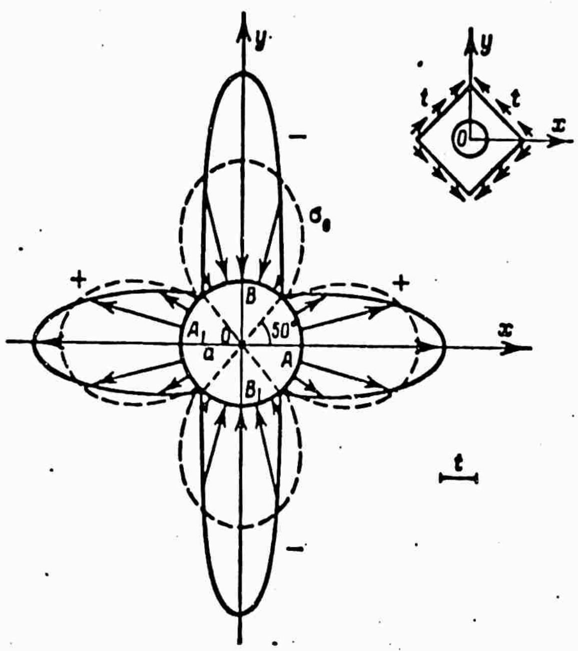

Fig. 87

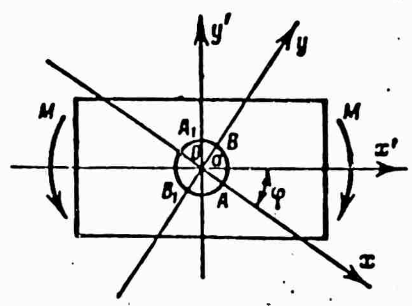

Fig. 88

the principal directions is least favorable since the coefficient of stress concentration is found to be the highest of all possible values ($K = 6.9$). On the other hand, the most favorable case is the case of deformation by forces parallel to the principal directions for which the coefficient of concentration is 3.95.*

8. Bending of a plate by moments. A rectangular orthotropic plate-beam with a round aperture in its center is bent by the moments M applied to two of its sides; the principal directions of elasticity are in general not coincident with the directions of the sides and their orientation is characterized by the angle φ (Fig. 88).

When the plate is considered to be infinitely large we obtain

$$\sigma_\theta = \frac{Ma}{2J} \cdot \frac{E_\theta}{E_1} \left(k \left[1 - k - (1 + k + n) \cos 2\theta \right] \sin^3 \varphi \cos \theta + \right.$$
$$+ \{ n^2 + k(k + 2n - 1) + [n(1 + n) + k(1 + k + 2n)] \cos 2\theta \} \times$$
$$\times \sin^2 \varphi \cos \varphi \sin \theta - [(1 + n)^2 - k - (k + n + 1)(1 + n) \cos 2\theta] \times \qquad (40.7)$$
$$\left. \times \sin \varphi \cos^2 \varphi \cos \theta + [1 - k - (1 + k + n) \cos 2\theta] \cos^3 \varphi \sin \theta \right).$$

When the direction of the axis of the plate-beam coincides with the principal axis ($\varphi = 0$, Fig. 89) we have

$$\sigma_\theta = \frac{Ma}{2J} \cdot \frac{E_\theta}{E_1} [1 - k - (1 + k + n) \cos 2\theta] \sin \theta. \qquad (40.8)$$

For an isotropic plate with $\varphi = 0$

$$\sigma_\theta = - \frac{2Ma}{J} \sin \theta \cos 2\theta. \qquad (40.9)$$

At the points B and B_1 (Fig. 89) of an orthotropic plate-beam

$$\sigma_\theta = \pm \frac{Ma}{J} \left(1 + \frac{n}{2} \right). \qquad (40.10)$$

In the case of an isotropic material for these points

$$\sigma_\theta = \pm \frac{2Ma}{J}; \qquad (40.11)$$

the maximum value of the stress σ_θ in the lateral sections is equal to $0{,}54 \frac{Ma}{J}$.

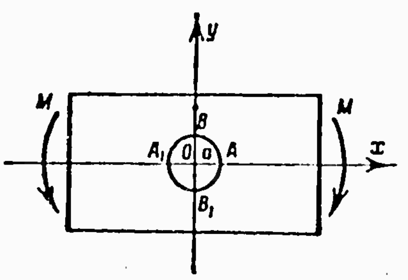

Fig. 89

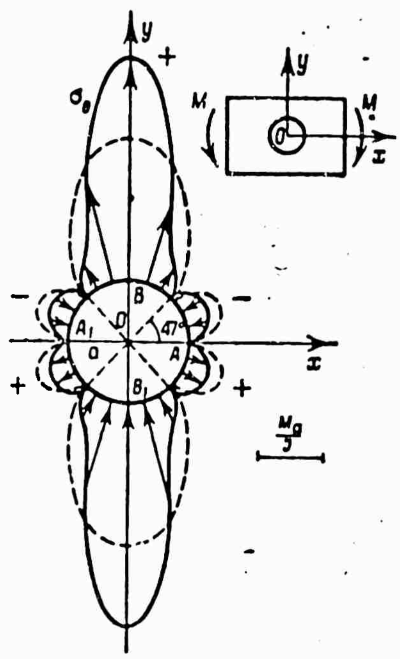

Fig. 90

When a veneer plate-beam is bent such that the direction of the x-axis agrees with the direction of the fibers in the sheet, the σ_θ stress distribution graph for the aperture's contour has the form shown in Fig. 90.

At the points B and B_1

$$\sigma_\theta = \pm 3{,}23 \frac{Ma}{J}, \tag{40.12}$$

and the stress reaches a maximum value of about $0{,}3 \frac{Ma}{J}$ in the lateral sections. The stress vanishes at six points: $\theta = 0°,\ 180°,\ \pm 47°$, $\pm 133°$.

With the bending of a plate-beam, which has been cut out of a sheet of the same veneer such that the direction of the fibers in the sheet are perpendicular to the x-axis, we obtain the σ_θ distribution on the contour shown in Fig. 91.

At the points B and B_1

$$\sigma_\theta = \pm 2{,}58 \frac{Ma}{J}. \tag{40.13}$$

In the lateral parts the stress does not exceed $0{,}6 \frac{Ma}{J}$.

In the case where the beam is cut out of a veneer sheet such that the direction of fibers in the sheet makes an angle of 45° with the axes of symmetry $\left(\varphi = \frac{\pi}{4}\right)$, we obtain a σ_θ distribution on the edge of the aperture as shown in Fig. 92. The stress reaches maximum values at the points θ = 110° and 290°; they are

$$\sigma_{max} = 1{,}64 \frac{Ma}{J}. \tag{40.14}$$

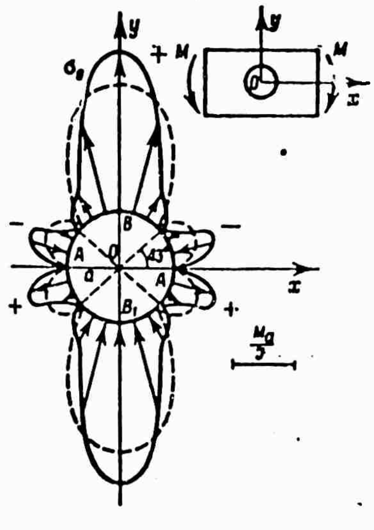

Fig. 91

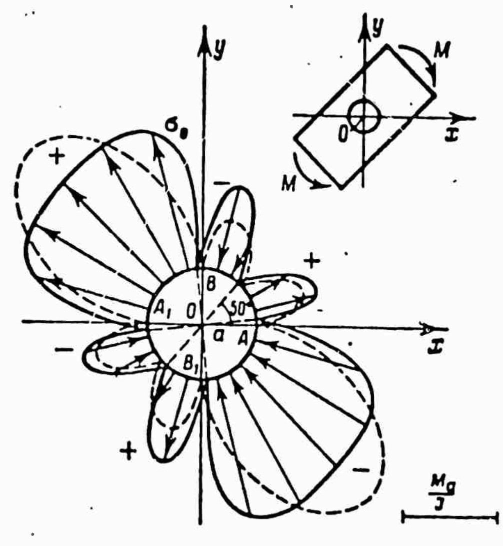

Fig. 92

Considering the bending of a veneer plate with various angles φ we can draw the following conclusions. The least favorable is the case where the fibers of the sheet are parallel to the free ends of the plate; in this case the coefficient of concentration obtained is highest: $K = 3.23$. The coefficient of concentration has its smallest value in the case where the fibers of the sheet make an angle of 45° with the axes of symmetry of the plate:

$$K = 1.64*$$

Let us also consider two cases where not the forces but the displacements are given for the contour of the aperture.

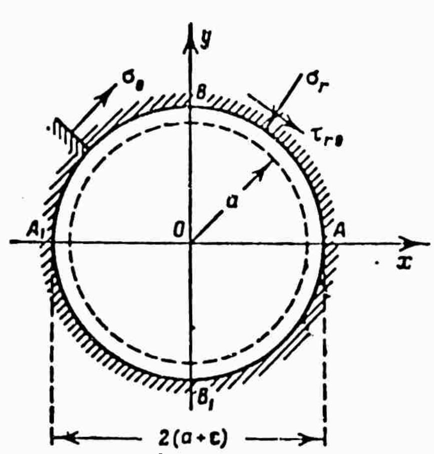

Fig. 93

9. Action of a rigid rod forced in the aperture with tension. In a circular aperture of the diameter $2a$ a rigid rod is forced in whose diameter $2\ (a + \varepsilon)$ is a little longer than that of the aperture. The rod's surface is rough so that the material of the plate

cannot slip on the material of the rod. In this case the edge of the aperture is displaced in a radial direction by an amount equal to ε (Fig. 93).

The solution is obtained by means of the functions Φ_1 and Φ_2 of the form*

$$\Phi_1(z_1)=\frac{\varepsilon}{2D}\cdot\frac{q_2-ip_2}{\zeta_1},\quad \Phi_2(z_2)=-\frac{\varepsilon}{2D}\cdot\frac{q_1-ip_1}{\zeta_2}$$
$$(D=p_1q_2-p_2q_1). \tag{40.15}$$

The normal pressure σ_r and the tangential stresses $\tau_{r\theta}$ at the surface of the aperture's edge are distributed according to the law

$$\left.\begin{aligned}\sigma_r&=-\frac{\varepsilon}{ag}[k-\nu_1+n(\sin^2\theta+k\cos^2\theta)],\\ \tau_{r\theta}&=-\frac{\varepsilon}{ag}n(1-k)\sin\theta\cos\theta,\end{aligned}\right\} \tag{40.16}$$

where

$$g=\frac{1-\nu_1\nu_2}{E_2}+\frac{k}{G}. \tag{40.17}$$

These formulas show that the rod transmits a compressive stress to the plate, which is distributed nonuniformly over the contact surface, but which is symmetric relative to the principal directions of elasticity. Moreover, frictional forces are generated in the contact surface, which reach their maximum values at the bisectrix of the angles between the principal directions. The maximum values of the normal pressure are obtained for either the points A and A_1 or B and B_1 (Fig. 93).

At the points A and A_1

$$\sigma_r=-\frac{\varepsilon}{ag}[k(1+n)-\nu_1],\quad \tau_{r\theta}=0. \tag{40.18}$$

At the points B and B_1

$$\sigma_r=-\frac{\varepsilon}{ag}(k+n-\nu_1),\qquad \tau_{r\theta}=0. \tag{40.19}$$

The maximum tangential strain is equal to

$$\tau_{max}=\frac{\varepsilon}{2ag}|1-k|n. \tag{40.20}$$

For the stress σ_θ in radial surfaces near the aperture the formula obtained is much more complex and we shall not give it here; we only give the values of σ_θ at the points A, A_1, B and B_1.

At the points A and A_1

$$\sigma_\theta=\frac{\varepsilon}{ag}\left[1+\frac{E_1}{Gk}-\frac{\nu_1(1+n)}{k}\right]; \tag{40.21}$$

at the points B and B_1

$$\sigma_1 = \frac{\varepsilon k}{ag}\left[k + \frac{E_1}{G} - \nu_1\left(1 + \frac{n}{k}\right)\right]. \tag{40.22}$$

In an isotropic plate with Young's modulus E and Poisson's coefficient ν we obtain for the edge of the aperture

$$\sigma_r = -\frac{E\varepsilon}{a(1+\nu)}, \quad \sigma_\theta = \frac{E\varepsilon}{a(1+\nu)}, \quad \tau_{r\theta} = 0. \tag{40.23}$$

For a veneer plate where the x-axis is parallel to the fibers of the sheet we obtain the following numerical values (in kg/cm^2):

at the points A and A_1

$$\sigma_r = -\frac{\varepsilon}{a}\cdot 0{,}349\cdot 10^5, \tag{40.24}$$

at the points B and B_1

$$\sigma_r = -\frac{\varepsilon}{a}\cdot 0{,}265\cdot 10^5; \tag{40.25}$$

$$\tau_{max} = \frac{\varepsilon}{a}\cdot 0{,}042\cdot 10^5. \tag{40.26}$$

The mean pressure at the edge of the aperture is equal to $\frac{\varepsilon}{a}\cdot 0{,}307\cdot 10^5$.

In the plate considered the maximum deviation of the pressure from the mean value amounts to 13.6%.

10. The torsion of a plate in its plane. The edge of the aperture is assumed to be rotated through a small angle α (or, what is the same, it undergoes a tangential displacement $a\alpha$); on its outer contour which, theoretically, is at infinity, the plate is assumed fixed so that it cannot be moved (Fig. 94). This case can be encountered when the edge of the aperture is held between two rigid round disks which are rotated through an angle of α or, when a rod is screwed in the aperture, which is then rotated so that it takes along the edge of the aperture.

The functions giving the stress distribution have the form

$$\Phi_1(z_1) = -\frac{a\alpha}{2D}\cdot\frac{p_2 + iq_2}{\zeta_1},$$
$$\Phi_2(z_2) = \frac{a\alpha}{2D}\cdot\frac{p_1 + iq_1}{\zeta_2}. \tag{40.27}$$

Both tangential $\tau_{r\theta}$ and normal forces (stresses) σ_r are assumed to act on the edge of the aperture; they are distributed nonuniformly:

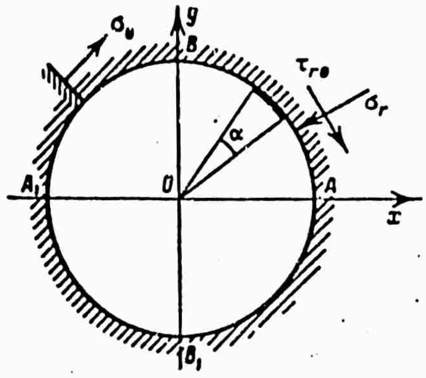

Fig. 94

$$\left.\begin{aligned}\sigma_r &= -\frac{\alpha}{g} n(1-k)\sin\theta\cos\theta,\\ \tau_{r\theta} &= -\frac{\alpha}{g}\left[k-\nu_1+n(\cos^2\theta+k\sin^2\theta)\right].\end{aligned}\right\} \tag{40.28}$$

The maximum values of the tangential strains are reached at the points of intersection of the contour with the principal directions, and the maximum normal stresses are obtained at the bisectrices of the angles between the principal directions. Comparing this case with the previous we see that the normal and tangential forces seem to have exchanged their positions.

In an isotropic plate at the edge of the aperture

$$\sigma_r = \sigma_\theta = 0, \quad \tau_{r\theta} = -\frac{E\alpha}{1+\nu}. \tag{40.29}$$

Some other cases of deformations of orthotropic plates with round holes were considered by I.I. Fayerberg.*

§41. DETERMINATION OF THE STRESSES IN A PLATE WITH ELLIPTIC CORE

With the help of the results obtained for the anisotropic elliptic plate and the plate with the elliptic aperture we can derive a solution of the more general problem of the stress distribution in an anisotropic plate with a sealed-in or glued-in core of an elastic or absolutely rigid material.

Let us consider an anisotropic plate of arbitrary shape with an elliptic aperture, in which, without having applied a previous tension, a core of the same thickness is soldered or glued in which consists of a different elastic material. The core dimensions are assumed to be small compared to the dimensions of the plate and far away from the edge. Arbitrary forces are distributed on the edge of the plate, which attack at the midplane; volume forces do not exist. We have to determine the stresses in plate and core which are caused by the external forces.**

The axes of coordinates are oriented according to the prin-

cipal axes of the ellipse (Fig. 95). All quantities which refer to the core, such as the stress components, the projections of the displacement, the elastic constants, etc. will be primed to distinguish them from the analogous quantities referring to the plate. The equation of the contour of the core is given in the form

$$x = a\cos\vartheta, \quad y = b\sin\vartheta \tag{41.1}$$

(a, b are the lengths of the semiaxes, $a \geqslant b$).

In the general case the equations of the generalized Hooke's law linking the stress and strain components in the plate averaged with respect to the thickness can be written in the form

$$\left.\begin{aligned} \varepsilon_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{16}\tau_{xy}, \\ \varepsilon_y &= a_{12}\sigma_x + a_{22}\sigma_y + a_{26}\tau_{xy}, \\ \gamma_{xy} &= a_{16}\sigma_x + a_{26}\sigma_y + a_{66}\tau_{xy}, \end{aligned}\right\} \tag{41.2}$$

which also hold true for the core when the a_{ij} in them are replaced by the constants a'_{ij} of the core material.

When, as in the previous cases, the strains are assumed small we can solve the problem approximately, by superposing the stresses in a plate without core and the stresses in an infinitely large plate with an elliptic aperture; the latter will be chosen such that at the contact surface between core and plate the necessary conditions are satisfied.

Denoting by F^0, σ_x^0, σ_y^0, τ_{xy}^0, u^0 and v^0 the functions of stresses, stress components and displacement projection in a plate without core which is exposed to the action of given external forces; all these quantities are assumed given. The formulas for stresses and displacements in a plate with core can then be written in the following form:

$$\left.\begin{aligned} \sigma_x &= \sigma_x^0 + 2\operatorname{Re}[\mu_1^2\Phi_1'(z_1) + \mu_2^2\Phi_2'(z_2)], \\ \sigma_y &= \sigma_y^0 + 2\operatorname{Re}[\Phi_1'(z_1) + \Phi_2'(z_2)], \\ \tau_{xy} &= \tau_{xy}^0 - 2\operatorname{Re}[\mu_1\Phi_1'(z_1) + \mu_2\Phi_2'(z_2)]; \end{aligned}\right\} \tag{41.3}$$

$$\left.\begin{aligned} u &= u^0 + 2\operatorname{Re}[p_1\Phi_1(z_1) + p_2\Phi_2(z_2)] - \omega y + u_0, \\ v &= v^0 + 2\operatorname{Re}[q_1\Phi_1(z_1) + q_2\Phi_2(z_2)] + \omega x + v_0. \end{aligned}\right\} \tag{41.4}$$

Here Φ_1, Φ_2 are functions for an infinite plate with aperture; ω, u_0, v_0 are constants characterizing "rigid" displacements;

$$\left.\begin{aligned} p_k &= a_{11}\mu_k^2 + a_{12} - a_{16}\mu_k, \\ q_k &= a_{12}\mu_k + \frac{a_{22}}{\mu_k} - a_{26} \\ &(k = 1, 2); \end{aligned}\right\} \tag{41.5}$$

μ_1, μ_2 are complex parameters of the plate, solutions of the equation

$$a_{11}\mu^4-2a_{16}\mu^3+(2a_{12}+a_{66})\mu^2-2a_{26}\mu+a_{22}=0, \qquad (41.6)$$

which are supposed to be nonequal.

The stresses in an elastic core are determined by means of the stress function F' which can also be written in terms of two functions of the complex variables $z_1'=x+\mu_1'y$ and $z_2'=x+\mu_2'y$ where μ_1' and μ_2' are complex parameters of the core.

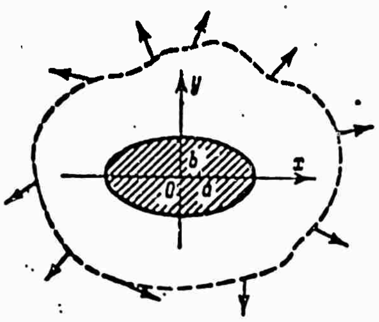

Fig. 95

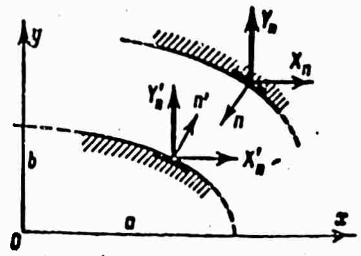

Fig. 96

The conditions at the points of the contact surface read as follows (Fig. 96):

$$\left.\begin{aligned} X_n&=-X_n', & Y_n&=-Y_n', \\ u&=u', & v&=v'. \end{aligned}\right\} \qquad (41.7)$$

After some transformations these conditions take the form

$$\left.\begin{aligned} 2\mathrm{Re}\,[\Phi_1(z_1)+\Phi_2(z_2)]&=\frac{\partial}{\partial x}(F'-F^0)+c_1, \\ 2\mathrm{Re}\,[\mu_1\Phi_1(z_1)+\mu_2\Phi_2(z_2)]&=\frac{\partial}{\partial y}(F'-F^0)+c_2, \\ 2\mathrm{Re}\,[p_1\Phi_1(z_1)+p_2\Phi_2(z_2)]&=u'-u^0+\omega y-u_0, \\ 2\mathrm{Re}\,[q_1\Phi_1(z_1)+q_2\Phi_2(z_2)]&=v'-v^0-\omega x-v_0. \end{aligned}\right\} \qquad (41.8)$$

The constants c_1, c_2, ω, u_0 and v_0 contained in them are determined from some simple additional conditions depending on the form of the plate and the distribution of the forces.

Conditions (41.8) written in this form are valid not only for a plate with an elliptic core but also for a plate with a core of any other form.

In the case of an elliptic core the form of the functions Φ_1 and Φ_2 is known to us (see §37):

$$\left.\begin{aligned} \Phi_1(z_1)&=A_0+A\ln\zeta_1+\sum_{m=1}^{\infty}A_m\zeta_1^{-m}, \\ \Phi_2(z_2)&=B_0+B\ln\zeta_2+\sum_{m=1}^{\infty}B_m\zeta_2^{-m}, \end{aligned}\right\} \qquad (41.9)$$

where

$$\zeta_k = \frac{z_k + \sqrt{z_k^2 - a^2 - \mu_k^2 b^2}}{a - i\mu_k b} \qquad (41.10)$$
$$(k = 1,\ 2).$$

§42. PARTICULAR CASES OF PLATES WITH ELLIPTIC CORE

The simplest particular case is that where the stresses in a plate without core are constant

$$\sigma_x^0 = p, \quad \sigma_y^0 = q, \quad \tau_{xy}^0 = t; \qquad (42.1)$$

$$F^0 = \frac{1}{2} q x^2 - t\,xy + \frac{1}{2} p y^2. \qquad (42.2)$$

Such stresses are obtained, for example, in a rectangular plate on the sides of which normal and tangential forces of the intensities p, q, t (see §13) are distributed uniformly.

Investigations showed that the stresses in the core will also be constant:

$$\sigma_x' = A, \quad \sigma_y' = B, \quad \tau_{xy}' = C; \qquad (42.3)$$

$$F' = \frac{1}{2} B x^2 - C\,xy + \frac{1}{2} A y^2, \qquad (42.4)$$

and the additional stresses in the plate, which represent the influence of the core, are determined by means of the functions Φ_1 and Φ_2 in the form*

$$\left.\begin{aligned} \Phi_1(z_1) &= \frac{1}{2(\mu_1 - \mu_2)} [(A - p) bi - (B - q)\mu_2 a + \\ &\qquad + (C - t)(i\mu_2 b - a)] \frac{1}{\zeta_1}, \\ \Phi_2(z_2) &= -\frac{1}{2(\mu_1 - \mu_2)} [(A - p) bi - (B - q)\mu_1 a + \\ &\qquad + (C - t)(i\mu_1 b - a)] \frac{1}{\zeta_2}. \end{aligned}\right\} \qquad (42.5)$$

From the boundary conditions (41.8) we obtain a system of four equations from which we determine the unknown stresses in the core A, B, C and the turn of the core, $\omega' - \omega$, relative to the plate. We give here the first two equations of this system:

$$
\left.\begin{aligned}
&A\left(i\,\frac{p_1-p_2}{\mu_1-\mu_2}\,b-a'_{11}a\right)+Ba\left(\frac{\mu_1p_2-\mu_2p_1}{\mu_1-\mu_2}-a'_{12}\right)-\\
&\quad -C\left[\frac{(p_1-p_2)\,a+i\,(\mu_1p_2-\mu_2p_1)\,b}{\mu_1-\mu_2}+a'_{16}a\right]+i\,(\omega'-\omega)\,b=\\
&\quad =p\left(i\,\frac{p_1-p_2}{\mu_1-\mu_2}\,b-a_{11}a\right)+qa\left(\frac{\mu_1p_2-\mu_2p_1}{\mu_1-\mu_2}-a_{12}\right)-\\
&\qquad\qquad -t\left[\frac{(p_1-p_2)\,a+i\,(\mu_1p_2-\mu_2p_1)\,b}{\mu_1-\mu_2}+a_{16}a\right],\\
&A\left(i\,\frac{q_1-q_2}{\mu_1-\mu_2}\,b-ia'_{12}b-a'_{16}a\right)+B\left(\frac{\mu_1q_2-\mu_2q_1}{\mu_1-\mu_2}\,a-ia'_{22}b-a'_{26}a\right)-\\
&\quad -C\left[\frac{(q_1-q_2)\,a+i\,(\mu_1q_2-\mu_2q_1)\,b}{\mu_1-\mu_2}+a'_{66}a+ia'_{26}b\right]-(\omega'-\omega)\,a=\\
&\quad =p\left(i\,\frac{q_1-q_2}{\mu_1-\mu_2}\,b-ia_{12}b-a_{16}a\right)+q\left(\frac{\mu_1q_2-\mu_2q_1}{\mu_1-\mu_2}\,a-ia_{22}b-a_{26}a\right)-\\
&\qquad\qquad -t\left[\frac{(q_1-q_2)\,a+i\,(\mu_1q_2-\mu_2q_1)\,b}{\mu_1-\mu_2}+a_{66}a+ia_{26}b\right].
\end{aligned}\right\}\qquad (42.6)
$$

The other two equations differ from (42.6) by the only fact that $i=\sqrt{-1}$ is replaced by $-i$.

Solving these equations, we obtain A, B, C and from them the functions Φ_1 and Φ_2 and their derivatives entering the stress formulas.

The next case, if we order them according to difficulty, is the case of a plate without core loaded by given external forces where we obtain the stresses as linear functions of the coordinates:

$$
\left.\begin{aligned}
\sigma_x^0&=m_2x+3m_3y,\\
\sigma_y^0&=3m_0x+m_1y,\\
\tau_{xy}^0&=-m_1x-m_2y;
\end{aligned}\right\}\qquad (42.7)
$$

$$
F^0=\frac{1}{2}(m_0x^3+m_1x^2y+m_2xy^2+m_3y^3).\qquad (42.8)
$$

A stress distribution of this type will be encountered, for example, in a rectangular plate (isotropic or anisotropic) with two or all four sides loaded by forces which are due to bending moments.

In this case we obtain the following results.

The stresses in the core will also be linear functions of the coordinates:

$$
\left.\begin{aligned}
\sigma'_x&=Cx+3Dy,\\
\sigma'_y&=3Ax+By,\\
\tau'_{xy}&=-Bx-Cy;
\end{aligned}\right\}\qquad (42.9)
$$

$$
F'=\frac{1}{2}(Ax^3+Bx^2y+Cxy^2+Dy^3).\qquad (42.10)
$$

The additional stresses in a plate with core are determined by means of the functions Φ_1 and Φ_2:*

$$
\left.\begin{aligned}
\Phi_1(z_1) = A_0 + \frac{1}{8(\mu_1-\mu_2)}\,[&-3(A-m_0)\mu_2 a^2 + \\
&+(B-m_1)(a-2i\mu_2 b)\,a + \\
&+(C-m_2)(2ia+\mu_2 b)\,b - 3(D-m_3)\,b^2]\,\frac{1}{\zeta_1^2},\\
\Phi_2(z_2) = B_0 - \frac{1}{8(\mu_1-\mu_2)}\,[&-3(A-m_0)\mu_1 a^2 + \\
&+(B-m_1)(a-2i\mu_1 b)\,a + \\
&+(C-m_2)(2ia+\mu_1 b)\,b - 3(D-m_3)\,b^2]\,\frac{1}{\zeta_2^2}.
\end{aligned}\right\}\tag{42.11}
$$

For the constants A, B, C, D from Conditions (41.8) at the contact surface we obtain a system of rather complex equations. The first two read as follows:

$$
\left.\begin{aligned}
&3A\left[\frac{\mu_1 p_2-\mu_2 p_1}{\mu_1-\mu_2}a^2 - a'_{12}a^2 - a'_{22}b^2\right] + 3D\left[-\frac{p_1-p_2}{\mu_1-\mu_2}b^2 -\right.\\
&\quad\left.-2ia_{11}ab + a'_{16}b^2\right] + B\left[\frac{p_1-p_2}{\mu_1-\mu_2}a^2 + 2i\frac{\mu_1 p_2-\mu_2 p_1}{\mu_1-\mu_2}\times\right.\\
&\quad\left.\times ab + a'_{16}a^2 - 2ia'_{12}ab + 2a'_{26}b^2\right] + C\left[2i\frac{p_1-p_2}{\mu_1-\mu_2}\times\right.\\
&\quad\times ab - \frac{\mu_1 p_2-\mu_2 p_1}{\mu_1-\mu_2}b^2 - a'_{11}a^2 + 2ia'_{16}ab -\\
&\quad\left.-(a'_{12}+a'_{66})\,b^2\right] = 3m_0\left[\frac{\mu_1 p_2-\mu_2 p_1}{\mu_1-\mu_2}a^2 -\right.\\
&\quad\left.-a_{12}a^2 - a_{22}b^2\right] + 3m_3\left[-\frac{p_1-p_2}{\mu_1-\mu_2}b^2 - 2ia_{11}ab +\right.\\
&\quad\left.+a_{16}b^2\right] + m_1\left[\frac{p_1-p_2}{\mu_1-\mu_2}a^2 + 2i\frac{\mu_1 p_2-\mu_2 p_1}{\mu_1-\mu_2}ab + a_{16}a^2 -\right.\\
&\quad\left.-2ia_{12}ab + 2a_{26}b^2\right] + m_2\left[2i\frac{p_1-p_2}{\mu_1-\mu_2}ab -\right.\\
&\quad\left.-\frac{\mu_1 p_2-\mu_2 p_1}{\mu_1-\mu_2}b^2 - a_{11}a^2 + 2ia_{16}ab - (a_{12}+a_{66})\,b^2\right],\\
&3A\left[\frac{\mu_1 q_2-\mu_2 q_1}{\mu_1-\mu_2}a^2 - a'_{26}a^2 - 2ia'_{22}ab\right] + 3D\left[-\frac{q_1-q_2}{\mu_1-\mu_2}\times\right.\\
&\quad\left.\times b^2 + a'_{11}a^2 + a'_{12}b^2\right] + B\left[\frac{q_1-q_2}{\mu_1-\mu_2}a^2 + 2i\times\right.\\
&\quad\left.\times\frac{\mu_1 q_2-\mu_2 q_1}{\mu_1-\mu_2}ab + (a'_{12}+a'_{66})\,a^2 + 2ia'_{26}ab + a'_{22}b^2\right] +\\
&\quad+C\left[2i\frac{q_1-q_2}{\mu_1-\mu_2}ab - \frac{\mu_1 q_2-\mu_2 q_1}{\mu_1-\mu_2}b^2 - 2a'_{16}a^2 -\right.\\
&\quad\left.-2ia'_{12}ab - a'_{26}b^2\right] = 3m_0\left[\frac{\mu_1 q_2-\mu_2 q_1}{\mu_1-\mu_2}a^2 - a_{26}a^2 -\right.\\
&\quad\left.-2ia_{22}ab\right] + 3m_3\left[-\frac{q_1-q_2}{\mu_1-\mu_2}b^2 + a_{11}a^2 + a_{12}b^2\right] +\\
&\quad+m_1\left[\frac{q_1-q_2}{\mu_1-\mu_2}a^2 + 2i\frac{\mu_1 q_2-\mu_2 q_1}{\mu_1-\mu_2}ab + (a_{12}+a_{66})\,a^2 +\right.\\
&\quad\left.+2ia_{26}ab + a_{22}b^2\right] + m_2\left[2i\frac{q_1-q_2}{\mu_1-\mu_2}ab -\right.\\
&\qquad\left.-\frac{\mu_1 q_2-\mu_2 q_1}{\mu_1-\mu_2}b^2 - 2a_{16}a^2 - 2ia_{12}ab - a_{26}b^2\right].
\end{aligned}\right\}\tag{42.12}
$$

The other two equations are obtained when i is everywhere replaced by $-i$.

From the solution for the plate with elastic core we obtain

by means of a simple substitutions or a limiting transition solutions for the limiting (extreme) cases where the core is perfectly stiff and cannot be deformed and where the core is lacking and no external forces act on the contour of the aperture. In the first case we must everywhere put $a'_{ij}=0$, and in the second, performing the limiting transition, all constants a'_{ij} are assumed infinitely large.

§43. EXTENSION OF AN ORTHOTROPIC PLATE WITH ROUND CORE IN ONE DIRECTION

Let us analyze in greater detail the fundamental cases of strains in a rectangular orthotropic plate with round core which is also orthotropic and, in particular, inflexible. In all cases considered in this section and in the following three sections, we assume the principal directions of elasticity of the materials of plate and core parallel to the axes of symmetry, which are taken as the axes x and y. For the elastic constants in most formulas we maintained the denotations a_{ij} and a'_{ij} as the simpler ones, and only some of them are written in terms of the "technical" constants. We also introduce the abbreviated denotations

$$\left.\begin{aligned} &k=-\mu_1\mu_2=\sqrt{\frac{a_{22}}{a_{11}}}=\sqrt{\frac{E_1}{E_2}}, \quad n=-i(\mu_1+\mu_2),\\ &\frac{E_1}{E_\theta}=\sin^4\theta+\frac{2a_{12}+a_{66}}{a_{11}}\sin^2\theta\cos^2\theta+\frac{a_{22}}{a_{11}}\cos^4\theta, \end{aligned}\right\} \tag{43.1}$$

where E_θ is Young's modulus for tension (compression) in a direction tangential to the contour of the aperture; E_1, E_2 are Young's moduli of the plate in the directions of x and y;

$$\left.\begin{aligned} &p_1=a_{11}\mu_1^2+a_{12}, \quad p_2=a_{11}\mu_2^2+a_{12},\\ &q_1=a_{12}\mu_1+\frac{a_{22}}{\mu_1}, \quad q_2=a_{12}\mu_2+\frac{a_{22}}{\mu_2}. \end{aligned}\right\} \tag{43.2}$$

The formulas for the case of equal complex parameters $\mu_1=\mu_2=\beta i$ are obtained from the formulas for unequal μ_1, μ_2 by means of a limiting transition.

We only give the final formulas for stresses in a plate near the core.* By way of illustration we consider a plate with given elastic constants, which are the same as in the case of the veneer sheet (see §39) with a core whose elastic constants are twice those of the plate $(a'_{ij}=2a_{ij})$, and we also consider the case where the core is perfectly inflexible $(a'_{ij}=0)$ and the case of no core $(a'_{ij}=\infty)$. For a complete representation of the influence of a core on the stress distribution in a plate we also give tables of the numerical values of the stresses σ_r, $\tau_{r\theta}$, σ_θ in a veneer plate near an elastic core and near a rigid core and the stress σ_θ near the edge of an aperture which is empty, and we also give graphs showing the stress distribution along the contour of the core (aperture).

Consider a plate with round core, which is extended in the principal direction x by the forces p which are uniformly distrib-

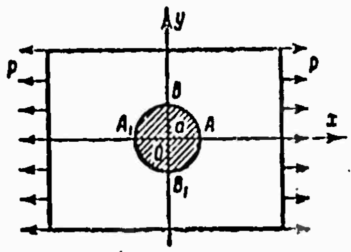

Fig. 97

In this we we obtain:*

$$\sigma'_x = A, \quad \sigma'_y = B, \quad \tau'_{xy} = 0, \tag{43.3}$$

where

$$\left.\begin{aligned} A &= \frac{p}{\Delta}[a_{11}a_{22}(k+n) + a_{11}a'_{22}k(1+n) + a_{22}(a_{12}+a_{66}+a'_{12})], \\ B &= \frac{p}{\Delta}[a_{22}(a_{11}-a'_{11}) + a_{11}(a_{12}-a'_{12})k(1+n)], \end{aligned}\right\} \tag{43.4}$$

$$\begin{aligned} \Delta = {} & (a_{11}a_{22}+a'_{11}a'_{22})k + a_{22}(a_{66}+2a'_{12}) + \\ & + (a_{11}a'_{22}k + a_{22}a'_{11})n - (a_{12}-a'_{12})^2 k; \end{aligned} \tag{43.5}$$

$$\left.\begin{aligned} \Phi_1(z_1) &= \frac{al}{2}\cdot\frac{A+B\mu_2 l - p}{\mu_1-\mu_2}\cdot\frac{1}{\zeta_1}, \\ \Phi_2(z_2) &= -\frac{al}{2}\cdot\frac{A+B\mu_1 l - p}{\mu_1-\mu_2}\cdot\frac{1}{\zeta_2}. \end{aligned}\right\} \tag{43.6}$$

The stresses $\sigma_r, \tau_{r\theta}$ at the edge of the aperture (and core) and σ_θ on the contour of the aperture (more exactly, in immediate proximity of the edge) are equal to

$$\left.\begin{aligned} \sigma_r &= \frac{p}{2\Delta}[\Delta + a_3 + a_2 + (\Delta + a_3 - a_2)\cos 2\theta], \\ \tau_{r\theta} &= -\frac{p}{2\Delta}(\Delta + a_3 - a_2)\sin 2\theta; \end{aligned}\right\} \tag{43.7}$$

$$\begin{aligned} \sigma_\theta = \frac{p}{\Delta}\cdot\frac{E_\theta}{E_1}\{ & (\Delta - a_1)\sin^6\theta + [\Delta(n^2-2k) + (k+n)a_1 + \\ & + (1+2k)a_2 - (2+k)(1+n)a_3]\sin^4\theta\cos^2\theta + [k^2\Delta - \\ & - (1+2k)(k+n)a_2 + k(2+k)a_3 + k(1+n)a_4] \times \\ & \times \sin^2\theta\cos^4\theta - k^2 a_4\cos^6\theta\}. \end{aligned} \tag{43.8}$$

Here a_1, a_2, a_3, a_4 are coefficients depending on the elastic constants of plate and core:

$$\left.\begin{aligned} a_1 &= (a_{11}-a'_{11})a_{22}(n^2-k) + [(a_{11}-a'_{11})a'_{22} + \\ &\qquad + (a_{12}-a'_{12})^2]kn - a_{22}(a_{12}-a'_{12}), \\ a_2 &= (a_{11}-a'_{11})a_{22} + a_{11}(a_{12}-a'_{12})k(1+n), \\ a_3 &= (a_{11}-a'_{11})(a_{22}n + a'_{22}k) + a_{22}(a_{12}-a'_{12}) + (a_{12}-a'_{12})^2 k, \\ a_4 &= -(a_{11}-a'_{11})a'_{22} + (a_{12}-a'_{12})[2a_{12} + a_{66} + \\ &\qquad + a_{11}(k+n)] - (a_{12}-a'_{12})^2. \end{aligned}\right\} \tag{43.9}$$

In particular, for an isotropic plate with core of isotropic material we obtain:

$$\left.\begin{aligned}
\sigma_r &= \frac{p}{E\Delta}\left[\frac{3-\nu}{E}+\frac{1+\nu'}{E'}+2\left(\frac{1+\nu}{E}+\frac{1-\nu'}{E'}\right)\cos 2\theta\right],\\
\tau_{r\theta} &= -\frac{2p}{E\Delta}\left(\frac{1+\nu}{E}+\frac{1-\nu'}{E'}\right)\sin 2\theta,\\
\sigma_\theta &= \frac{p}{\Delta}\left\{\frac{(3-\nu)\nu}{E^2}+\frac{3-3\nu'+2\nu\nu'}{EE'}+\frac{1-\nu'^2}{E'^2}+\right.\\
&\left.+2\left[\frac{(1+\nu)\nu}{E^2}-\frac{1+\nu'+2\nu\nu'}{EE'}-\frac{1-\nu'^2}{E'^2}\right]\cos 2\theta\right\};
\end{aligned}\right\} \tag{43.10}$$

$$\Delta = \frac{3+2\nu-\nu^2}{E^2}+2\,\frac{2-\nu'+\nu\nu'}{EE'}+\frac{1-\nu'^2}{E'^2} \tag{43.11}$$

(E, ν denote Young's modulus and Poisson's coefficient for the plate, E', ν' those of the core).

The stress distribution in a plate with inflexible core is obtained on the assumption that $a'_{ij}=0$ in Eqs. (43.3)-(43.9) and $E'=\infty$ in Eqs. (43.10)-(43.11).

At the points A and A_1 of an orthotropic plate (Fig. 97)

$$\sigma_r = \frac{p}{g\sqrt{E_1E_2}}\left(k+n-\nu_1+\frac{E_1}{G}\right), \quad \sigma_\theta=\nu_2\sigma_r, \quad \tau_{r\theta}=0, \tag{43.12}$$

where

$$g=\frac{1-\nu_1\nu_2}{E_2}+\frac{k}{G}. \tag{43.13}$$

At the points B and B_1 (Fig. 97)

$$\sigma_r = \frac{p}{gE_1}[k-\nu_1(1+n)], \quad \sigma_\theta=\nu_1\sigma_r, \quad \tau_{r\theta}=0. \tag{43.14}$$

The stresses in an isotropic plate with a rigid core are determined according to the formulas:

$$\left.\begin{aligned}
\sigma_r &= p\left(\frac{1}{1+\nu}+\frac{2}{3-\nu}\cos 2\theta\right),\\
\sigma_\theta &= \nu\sigma_r, \quad \tau_{r\theta}=-p\frac{2}{3-\nu}\sin 2\theta.
\end{aligned}\right\} \tag{43.15}$$

Assuming all coefficients a'_{ij} infinitely large, we obtain by means of a limiting transition from (43.8) Eq. (39.12) which we know already for the extended plate with empty aperture, and from (43.7) we obtain $\sigma_r=\tau_{r\theta}=0$, which is evident.

In Table 2 we have compiled the numerical values (in fractions of p) of the stresses in the first quadrant of the aperture's contour, taken at every 15°, for a veneer plate with stiff core and with elastic core, where $a'_{ij}=2a_{ij}$, and for a plate with an aperture without core. The tensile forces attack in the direction of x in which Young's modulus is highest, or, briefly, in the direction of the sheet's fibers.

Comparing the values given in this table we note first of all that in the case where the aperture contains a core, elastic or rigid, not only the value of the maximum stress is considerably lower, but also the general pattern of stress distribution is

TABLE 2

Stress Components at the Points of the Aperture's Contour

$\theta°$	1 Жесткое ядро			2 Упругое ядро ($a'_{ij}=2a_{ij}$)			3 Без ядра
	σ_r	$\tau_{r\theta}$	σ_θ	σ_r	$\tau_{r\theta}$	σ_θ	σ_θ
0	1,24	0	0,04	0,84	0	—0,06	—0,71
15	1,16	—0,30	0,09	0,78	—0,21	0,03	—0,34
30	0,94	—0,52	0,27	0,63	—0,37	0,23	0,07
45	0,64	—0,60	0,52	0,41	—0,43	0,49	0,40
60	0,34	—0,52	0,70	0,20	—0,37	0,78	0,96
75	0,20	—0,30	0,56	0,04	—0,21	1,19	2,57
90	0,04	0	0	—0,02	0	1,68	5,45

1) Inflexible core; 2) elastic core; 3) without core.

changed qualitatively. This also becomes obvious from the graphs attached.

In Fig. 98 we show the distribution of the stresses σ_r on the contour of the aperture (core) with the veneer plate extended along the fibers of the sheet. The solid line represents the graph of the stresses in the case of a core where $a'_{ij}=2a_{ij}$ and the dashed line the same for the case of a rigid core. In Fig. 99 we show the graphs of the distributions of stresses σ_θ along the contour of the aperture in the cases of an elastic core (solid line), a rigid core and no core in the aperture (dashed lines).

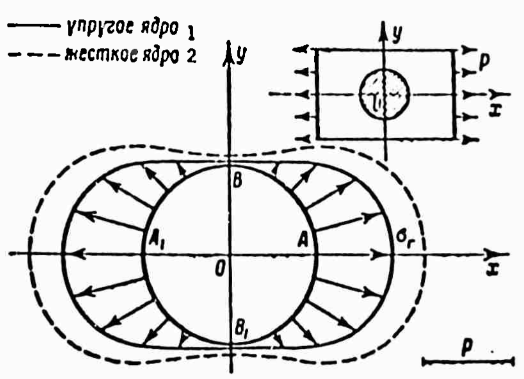

Fig. 98. 1) Elastic core; 2) rigid core.

When the core is rigid, the stress σ_r at the ends of the diameter parallel to the forces (at points A and A_1 in Fig. 97) is the highest in the entire plate; it is equal to 1.24 p. In a plate without core the stress σ_θ is highest at the ends of the diameter perpendicular to the forces (at points B and B_1); it amounts to 5.45 p.

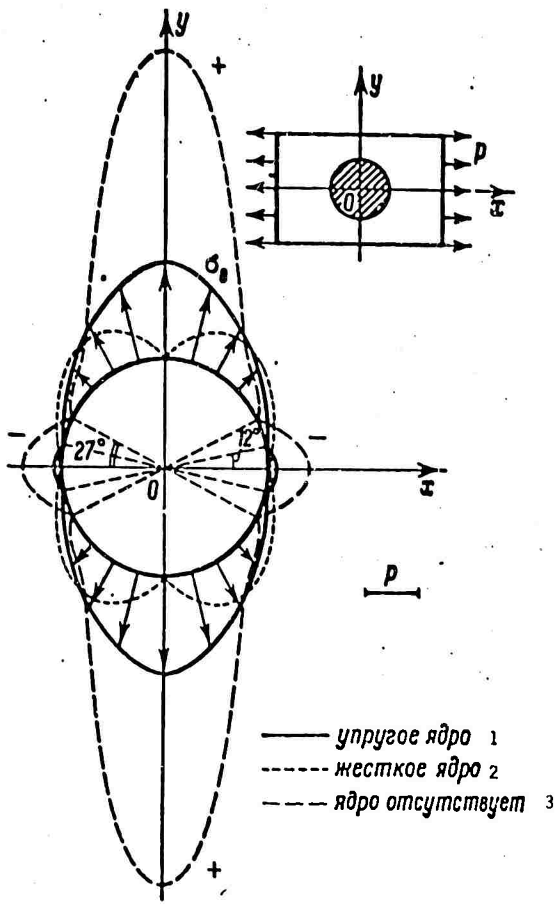

Fig. 99. 1) Elastic core; 2) rigid core; 3) no core.

The case of the elastic core is an intermediate case between these two; the maximum stress (σ_θ) is equal to 1.68 p. If we consider cores of various materials for which $a'_{ij} = ma_{ij}$, the stress pattern approaches the pattern of a plate with empty aperture as m increases and to the graph for a plate with rigid core as m decreases.

When the plate is extended in the direction of x, for which Young's modulus is smallest (transverse to the fibers of the sheet), the general character of the graphs of stress variation on the contour of the aperture is maintained; the magnitudes of the stresses at the corresponding points vary, the points at which the stresses vanish are shifted but the position of the points where the stresses reach their highest values remain unchanged. The maximum stress is equal to: 1) in the case of a rigid core 1.34 p, 2) in the case of an elastic core ($a'_{ij} = 2a_{ij}$) — 1,60p and 3) in a plate without core 4.15 p.

Hence we can draw the simple conclusion that a plate with an elastic core in which $a'_{ij} = 2a_{ij}$, just as a plate without core, is favorable to extend such that the strains act in a direction for which Young's modulus has the lowest value. Vice versa, in the

case of a rigid core the maximum stress is found to be smaller in the case of a tension in the direction of maximum Young's modulus.

§44. EXTENSION OF A PLATE WITH ROUND CORE IN TWO DIRECTIONS

For an orthotropic plate extended in two directions by equal forces p (Fig. 100) we obtain the following results:*

$$\sigma'_x = A, \quad \sigma'_y = B, \quad \tau'_{xy} = 0; \tag{44.1}$$

$$\left.\begin{aligned} A = \frac{p}{\Delta}[&a_{11}a_{22}(k+n) + a_{11}a'_{22}k(1+n) + \\ &+ a_{12}a_{22}(1+k+n) + a_{22}a'_{12}(1-k-n) + \\ &+ a_{22}(a_{22}+a_{66}-a'_{22})], \\ B = \frac{p}{\Delta}[&a_{11}a_{22}k(1+n) + a_{22}a'_{11}(k+n) + \\ &+ a_{12}a_{11}k(1+k+n) + a_{11}a'_{12}k(k-n-1) + \\ &+ a_{22}(a_{11}+a_{66}-a'_{11})], \end{aligned}\right\} \tag{44.2}$$

where Δ is the expression of (43.5);

$$\left.\begin{aligned} \Phi_1(z_1) &= \frac{al}{2} \cdot \frac{A + B\mu_2 l - p(1+\mu_2 l)}{\mu_1 - \mu_2} \cdot \frac{1}{\zeta_1}, \\ \Phi_2(z_2) &= -\frac{al}{2} \cdot \frac{A + B\mu_1 l - p(1+\mu_1 l)}{\mu_1 - \mu_2} \cdot \frac{1}{\zeta_2}. \end{aligned}\right\} \tag{44.3}$$

The stress components σ_r, $\tau_{r\theta}$, σ_θ on the contour of the aperture in the plate are determined by means of the formulas

$$\left.\begin{aligned} \sigma_r &= \frac{p}{2\Delta}[2\Delta + b_3 + b_2 + \\ &\qquad + (b_3 - b_2)\cos 2\theta], \\ \tau_{r\theta} &= -\frac{p}{2\Delta}(b_3 - b_2)\sin 2\theta; \end{aligned}\right\} \tag{44.4}$$

$$\begin{aligned} \sigma_\theta = \frac{p}{\Delta} \cdot \frac{E_\theta}{E_1} \{&(\Delta - b_1)\sin^6\theta + [\Delta(1-2k+n^2) + \\ &+ (k+n)b_1 + (1+2k)b_2 - (2+k) \times \\ &\times (1+n)b_3]\sin^4\theta\cos^2\theta + [\Delta(k^2-2k+n^2) - \\ &- (k+n)(1+2k)b_2 + k(2+k)b_3 + \\ &+ k(1+n)b_4]\sin^2\theta\cos^4\theta + \\ &+ k^2(\Delta - b_4)\cos^6\theta\} \end{aligned} \tag{44.5}$$

(the expression of Δ has the form (43.5) as in the case of the unilateral extension).

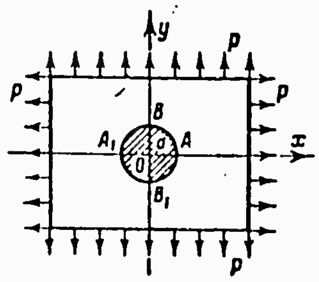

Fig. 100

The coefficients b_1, b_2, b_3 and b_4 are given in terms of the elastic constants as follows:

$$\left.\begin{aligned}
b_1 &= (a_{11}-a'_{11})[a_{22}(n^2-k)+a'_{22}kn]-(a_{22}-a'_{22})a'_{11}k^2+\\
&\quad +(a_{12}-a'_{12})a_{22}(n-1)(1+k+n)+\\
&\qquad\qquad +(a_{12}-a'_{12})^2k(n-k),\\
b_2 &= (a_{11}-a'_{11})a_{22}+(a_{22}-a'_{22})(a_{11}n+a'_{11})k+\\
&\quad +(a_{12}-a'_{12})[a_{22}+a_{11}k(1+n)]+(a_{12}-a'_{12})^2k,\\
b_3 &= (a_{11}-a'_{11})(a_{22}n+a'_{22}k)+(a_{22}-a'_{22})a_{22}+\\
&\quad +(a_{12}-a'_{12})a_{22}(1+k+n)+(a_{12}-a'_{12})^2k,\\
b_4 &= -(a_{11}-a'_{11})a'_{22}+(a_{22}-a'_{22})(2a_{12}+a_{66}+a_{11}k+a'_{11}n)+\\
&\quad +(a_{12}-a'_{12})[2a_{12}+a_{66}-a_{22}+a_{11}(k+n)]+\\
&\qquad\qquad +(a_{12}-a'_{12})^2(n-1).
\end{aligned}\right\} \qquad (44.6)$$

In an isotropic plate with an isotropic core the stresses are independent of the polar angle θ and on the contour of the aperture they are equal to

$$\sigma_r = p\left(1+\frac{b_1}{\Delta}\right),\quad \sigma_\theta = p\left(1-\frac{b_1}{\Delta}\right),\quad \tau_{r\theta}=0, \qquad (44.7)$$

where

$$b_1 = \frac{3-4\nu+\nu^2}{E^2}-2\,\frac{1-2\nu'+\nu\nu'}{EE'}-\frac{1-\nu'^2}{E'^2}, \qquad (44.8)$$

and Δ is Expression (43.11).

If in an isotropic plate a rigid core has been soldered in, for it

$$\sigma_r = \frac{2p}{1+\nu},\quad \sigma_\theta = \nu\sigma_r,\quad \tau_{r\theta}=0. \qquad (44.9)$$

With $a'_{ij} = \infty$ Eq. (44.5) goes over to Eq. (39.21) given previously for a plate with an aperture without core.

Calculations for a veneer plate with rigid core show that the stress σ_r distributed along its contour is almost uniform, as it varies only between 1.32 p and 1.37 p, and the tangential stress is small, it does not exceed 0.03 p; the maximum σ_r, which is equal to 1.37 p, is at the same time the maximum stress for the whole plate. In the case of an elastic core where $a'_{ij} = 2a_{ij}$, the distribution of the stress σ_r on the contour is also almost uniform; the value of $(\sigma_r)_{max}$ is equal to 0.81 p and $\tau_{r\theta}$ does not exceed 0.01 p.

The σ_θ stress distribution pattern along the edge of the aperture is a veneer plate with rigid and elastic core and without core is shown in Fig. 101; the x-axis is directed along the fibers of the sheet.

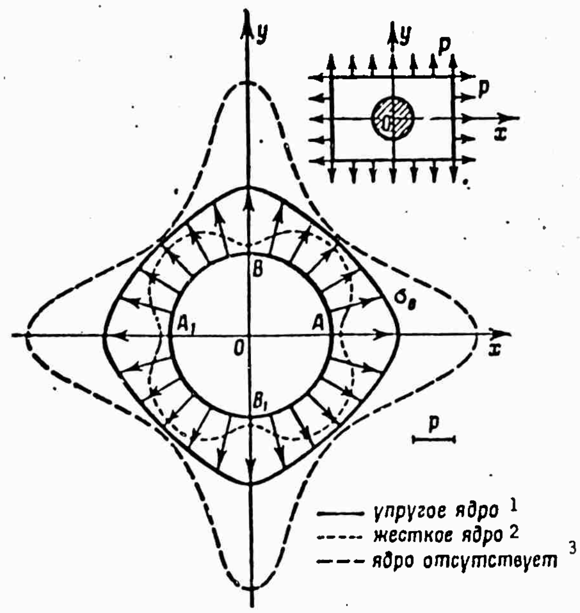

Fig. 101. 1) Elastic core; 2) rigid core; 3) no core.

The numerical values of σ_θ (in fractions of p) are compiled in Table 3.

TABLE 3

Stresses σ_θ at the Points of the Contour of the Aperture

θ°	σθ (в долях p) 1		
	2 жесткое ядро	упругое ядро 3	без ядра 4
0	0,05	1,54	3,44
15	0,46	1,31	2,38
30	0,84	1,09	1,41
45	0,96	1,02	1,09
60	0,93	1,04	1,23
75	0,65	1,22	2,18
90	0,10	1,56	4,04

1) (in fractions of p); 2) rigid core; 3) elastic core; 4) no core.

The maximum stress in a plate with elastic core is equal to 1.56 p and in a plate with an aperture and no core in it, it is equal to 4.04 p.

A more complex stress distribution pattern is obtained when the plate is extended or compressed in two directions by forces of different intensities.

Let us give formulas for the stresses σ_r and $\tau_{r\theta}$ on the contour of a rigid core in a plate which is compressed in the principal direction of x by the forces p, but when it cannot expand in the transverse direction (Fig. 102):

$$\left.\begin{aligned}\sigma_r &= -\frac{p}{2}\Big\{1+\nu_2+\frac{\lambda}{g}(kn+k-\nu_1)+\\&\quad+\Big[1-\nu_2+\frac{\lambda}{g}(kn-k+\nu_1)\Big]\cos 2\theta\Big\},\\ \tau_{r\theta} &= \frac{p}{2}\Big[1-\nu_2+\frac{\lambda}{g}(kn-k+\nu_1)\Big]\sin 2\theta.\end{aligned}\right\} \qquad (44.10)$$

Here g is a quantity determined according to Eq. (43.13) and

$$\lambda=\frac{1-\nu_1\nu_2}{E_1}. \qquad (44.11)$$

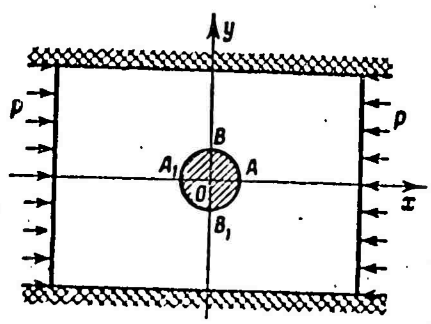

Fig. 102

§45. PLATE WITH ROUND CORE UNDER THE ACTION OF TANGENTIAL FORCES

An orthotropic rectangular plate with a round core is deformed by tangential forces t, which are distributed uniformly over all four sides (Fig. 103).

In this case*

$$\left.\begin{aligned}\sigma'_x&=\sigma'_y=0,\\ \tau'_{xy}&=\frac{t}{\Delta_1}[a_{11}kn+(2a_{12}+a_{66})k+a_{22}(2+n)],\end{aligned}\right\} \qquad (45.1)$$

where

$$\Delta_1=a_{11}kn+(2a_{12}+a'_{66})k+a_{22}(2+n); \qquad (45.2)$$

$$\left.\begin{aligned}\Phi_1(z_1)&=-ta(a_{66}-a'_{66})\frac{k(1-\mu_2 l)}{2(\mu_1-\mu_2)\Delta_1}\cdot\frac{1}{\zeta_1},\\ \Phi_2(z_2)&=ta(a_{66}-a'_{66})\frac{k(1-\mu_1 l)}{2(\mu_1-\mu_2)\Delta_1}\cdot\frac{1}{\zeta_2}.\end{aligned}\right\} \qquad (45.3)$$

For the stress components in the plate on the contour of the aperture we obtain the formulas

$$\left.\begin{aligned}\sigma_r &= \frac{t}{\Delta_1}[a_{11}kn + (2a_{12} + a_{66})k + a_{22}(2+n)]\sin 2\theta,\\ \tau_{r\theta} &= \sigma_r \operatorname{ctg} 2\theta;\end{aligned}\right\} \qquad (45.4)$$

$$\sigma_\theta = -t\left\{1 + \frac{a'_{66} - a_{66}}{2\Delta_1}\cdot\frac{E_\theta}{E_1}[(1+k)n + (n^2-2)\sin^4\theta + \right.$$
$$\left. + 2k\sin^2\theta\cos^2\theta + (n^2 - 2k^2)\cos^4\theta]\right\}. \qquad (45.5)$$

For an isotropic plate with isotropic core these formulas take the form

$$\left.\begin{aligned}\sigma_r &= \frac{8t}{E\Delta_1}\sin 2\theta, \quad \tau_{r\theta} = \frac{8t}{E\Delta_1}\cos 2\theta,\\ \sigma_\theta &= \frac{8t}{\Delta_1}\left(\frac{\nu}{E} - \frac{1+\nu'}{E'}\right)\sin 2\theta;\end{aligned}\right\} \qquad (45.6)$$

$$\Delta_1 = 2\left(\frac{3-\nu}{E} + \frac{1+\nu'}{E'}\right). \qquad (45.7)$$

As previously the letters E and ν denote Young's modulus and Poisson's coefficient of the plate material and E' and ν' these quantities for the core.

In order to obtain expressions for the stresses in a plate with rigid core we must everywhere put $a'_{66} = 0$ or $G' = \infty$. In particular, in an isotropic plate with rigid core

$$\left.\begin{aligned}\sigma_r &= \frac{4t}{3-\nu}\sin 2\theta,\\ \tau_{r\theta} &= \frac{4t}{3-\nu}\cos 2\theta,\\ \sigma_\theta &= \nu\sigma_r.\end{aligned}\right\} \qquad (45.8)$$

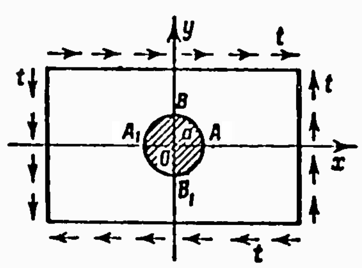

Fig. 103

When a'_{66} is allowed to tend to infinity in Eq. (45.5) we obtain Eq. (40.2) for the stress σ_θ in the orthotropic plate with the empty aperture.

Table 4 (page 206) contains the results of calculations for a veneer plate, i.e., the numerical values of the stresses (in fractions of t) at points in the first quadrant of the aperture's contour. The x-axis is supposed to be directed along the fibers of the sheet.

In Fig. 104 we show the graphs of the distribution of σ_r

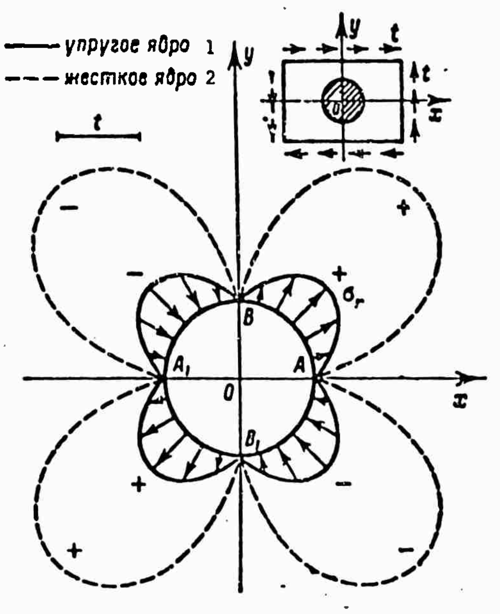

Fig. 104. 1) Elastic core; 2) rigid core.

along the contour of an elastic core for which $a'_{ij} = 2a_{ij}$ (solid line) and a rigid core (dashed line). The maximum value of σ_r in the case of a rigid core is equal to 2.28 t and in the case of an elastic core it is about one fourth as high.

In Fig. 10t we show graphs of the distribution of the stress σ_θ on the contour of an aperture for four cases. The innermost dashed line is the σ_θ distribution for a veneer plate with a rigid core. The stresses are positive in the first and third quadrants and negative in the second and fourth; its maximum value amounts to about 3.9 t. The outer dashed line represents the distribution of the stress σ_θ in the plate with the empty aperture. In this case the highest value of σ_θ differs but slightly from 3.9 t but here the stress will be positive not in the first and third quadrants but in the second and fourth. The solid line with the four zeros shows the distribution of the stress σ_θ in a veneer plate with elastic core for which $a'_{ij} = 2a_{ij}$. Just as in the case of the empty aperture, this stress is positive in the second and fourth quadrants and negative in the first and third ones; its maximum value amounts to 1.65 t.

When we consider a veneer plate with cores of various elastic materials for which $a'_{ij} = ma_{ij}$, m being a positive number, integral fractional, we may note the following. With high values of m the σ_θ graph will resemble the graphs for plates without core (outer dashed line). As m decreases from two downwardly, the stress σ_θ on the contour will first decrease in magnitude, remaining positive in the second and fourth quadrants and negative in the first and third quadrants, the graph representing its variation will gener-

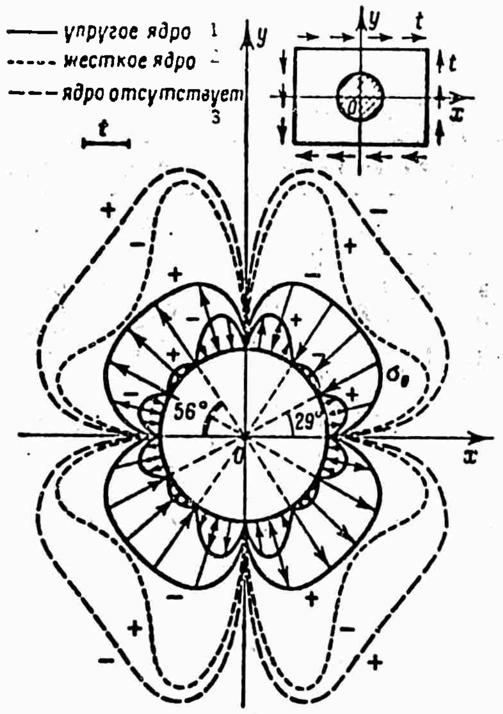

Fig. 105. 1) Elastic core; 2) rigid core; 3) no core.

ally resemble the graph for $m = 2$.

When the coefficient m decreases further, besides the points $\theta = 0°$, $90°$, $180°$ and $270°$ new points will appear on the contour of the aperture where the stress is vanishing; in th first and third quadrants we have marked the sections with tensile stresses.

Thus, for example, for the case where $m = 0.5$ the distribution of the stresses σ_θ on the contour is represented by the innermost solid line in Fig. 105. On the contour there are altogether 12 points at which $\sigma_\theta = 0$. The maximum stress values are not high, namely equal to $0.85\ t$.

With low values of m the number of points at which the stress is vanishing again drops to four and the curve of variation of σ_θ approaches the inner dashed curve with which it coincides with $m = 0$.

TABLE 4

Comparison of the Stresses at Points on the Contour of the Aperture

$\theta°$	1 Жесткое ядро			2 Упругое ядро ($a'_{ij}=2a_{ij}$)			3 Без ядра
	σ_r	$\tau_{r\theta}$	σ_θ	σ_r	$\tau_{r\theta}$	σ_θ	σ_θ
0	0	2,28	0	0	0,54	0	0
15	1,14	1,97	2,34	0,27	0,47	—1,18	—2,74
30	1,97	1,14	1,89	0,47	0,27	—1,53	—3,03
45	2,28	0	1,63	0,54	0	—1,63	—3,06
60	1,97	—1,14	2,39	0,47	—0,27	—1,64	—3,42
75	1,14	—1,97	3,88	0,27	—0,47	—1,55	—3,94
90	0	—2,28	0	0	—0,54	0	0

1) Rigid core; 2) elastic core; 3) no core.

§46. BENDING OF A PLATE WITH ROUND CORE BY MOMENTS

On two sides of a rectangular orthotropic plate with a round core forces are distributed such that they generate bending moments M (Fig. 106).

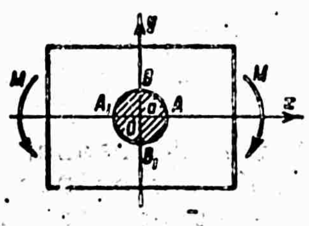

Fig. 106

On the basis of the general formulas and the equations of §42 we obtain:*

$$\sigma'_x = 3Dy, \quad \sigma'_y = By, \quad \tau'_{xy} = -Bx; \tag{46.1}$$

$$\left.\begin{aligned} B &= \frac{M}{Jd}[a_{11}(a_{12} - a'_{12})k(2+n) + \\ &\qquad + (a_{11} - a'_{11})(2a_{22} + a_{11}kn)], \\ 3D &= \frac{M}{Jd}[a_{11}(a_{11} + a_{12} + a'_{22} + a'_{12} + a'_{66})kn + \\ &\qquad + 2a_{11}(a_{22} + 2a_{12} + a'_{22} + a'_{66})k + 4a_{11}a_{22}(1+n) + \\ &\qquad + 2a_{22}(a_{12} + a_{66} + a'_{12})]; \end{aligned}\right\} \tag{46.2}$$

$$\begin{aligned} d &= 2a_{22}(a_{66} + 2a'_{11} + 2a'_{12}) + 2[a_{11}a_{22} + a'_{11}(2a_{12} + a'_{22} + a'_{66})]k + \\ &+ 4a_{22}a'_{11}n + 4a_{11}(a'_{11} + a'_{22} + 2a'_{12} + a'_{66})kn - 2(a_{12} - a'_{12})^2 k \end{aligned} \tag{46.3}$$

(J is the moment of inertia of the cross section);

$$\left.\begin{aligned}\Phi_1(z_1) &= -\frac{a^2}{8(\mu_1-\mu_2)}\left[B(1-2\mu_2 l)-3D+\frac{M}{J}\right]\cdot\frac{1}{\zeta_1^2},\\ \Phi_2(z_2) &= -\frac{a^2}{8(\mu_1-\mu_2)}\left[B(1-2\mu_1 l)-3D+\frac{M}{J}\right]\cdot\frac{1}{\zeta_2^2}.\end{aligned}\right\} \quad (46.4)$$

On the contour of the aperture (core)

$$\left.\begin{aligned}\sigma_r &= \frac{Ma}{2Jd}\left[d+2c_3+(d-2c_2+2c_3)\cos 2\theta\right]\sin\theta,\\ \tau_{r\theta} &= \frac{Ma}{2Jd}\left[-(d+2c_3)+(d-2c_2+2c_3)\cos 2\theta\right]\cos\theta;\end{aligned}\right\} \quad (46.5)$$

$$\begin{aligned}\sigma_\theta = \frac{Ma}{Jd}\cdot\frac{E_\theta}{E_1}\sin\theta\,\{&(d-c_1)\sin^6\theta+[d(n^2-2k)+c_1(1+k+2n)+\\ &+c_2(1+2k)-c_3(2+k)(2+n)]\sin^4\theta\cos^2\theta+[dk^2-c_1k-\\ &-c_2(1+2k)(1+k+2n)+c_3(2+k)(2k+n)+\\ &+c_4(2+n)]\sin^2\theta\cos^4\theta+[c_2k(1+2k)-c_4(2k+n)]\cos^6\theta\}.\end{aligned} \quad (46.6)$$

Here

$$\left.\begin{aligned}c_1 &= (a_{11}-a_{11}')[2a_{22}(n^2-k)+(2a_{12}+a_{22}'+a_{66}')\,kn]-\\ &\quad-(a_{12}-a_{12}')(a_{11}kn+2a_{22})+(a_{12}-a_{12}')^2\,kn,\\ c_2 &= (a_{11}-a_{11}')(a_{11}kn+2a_{22})+a_{11}(a_{12}-a_{12}')\,k(2+n),\\ c_3 &= (a_{11}-a_{11}')[a_{22}(1+2n)+(2a_{12}+a_{12}'+a_{60}')\,k]+\\ &\quad+(a_{12}-a_{12}')(a_{22}-a_{11}k)+(a_{12}-a_{12}')^2k,\\ c_4 &= (a_{11}-a_{11}')[a_{22}+(a_{66}-a_{22}'-a_{66}')\,k]+(a_{12}-a_{12}')\times\\ &\quad\times[a_{22}+(a_{11}+2a_{12}+a_{66}')\,k+2a_{11}kn]-(a_{12}-a_{12}')^2k.\end{aligned}\right\} \quad (46.7)$$

For an isotropic plate with isotropic core we obtain

$$\left.\begin{aligned}\sigma_r &= \frac{4Ma}{EJd}\left[2\left(\frac{1}{E}+\frac{1}{E'}\right)+\left(\frac{1+\nu}{E}+\frac{3-\nu'}{E'}\right)\cos 2\theta\right]\sin\theta,\\ \tau_{r\theta} &= \frac{4Ma}{EJd}\left[-2\left(\frac{1}{E}+\frac{1}{E'}\right)+\left(\frac{1+\nu}{E}+\frac{3-\nu'}{E'}\right)\cos 2\theta\right]\cos\theta,\\ \sigma_\theta &= \frac{4Ma}{Jd}\left\{\frac{2}{E}\left(\frac{\nu}{E}+\frac{2-\nu'}{E'}\right)+\right.\\ &\quad\left.+\left[\frac{\nu(1+\nu)}{E^2}+\frac{2\nu(1-\nu')-1-\nu'}{EE'}-\frac{3+2\nu'-\nu'^2}{E'^2}\right]\cos 2\theta\right\}\sin\theta;\end{aligned}\right\} \quad (46.8)$$

$$d=\frac{3+2\nu-\nu^2}{E^2}+2\,\frac{5-\nu-\nu'+\nu\nu'}{EE'}+\frac{3+2\nu'-\nu'^2}{E'^2}. \quad (46.9)$$

In the case of an isotropic plate and a rigid core ($E' = \infty$)

$$\left.\begin{aligned}\sigma_r &= \frac{2Ma}{J(3-\nu)(1+\nu)}\left[2+(1+\nu)\cos 2\theta\right]\sin\theta,\\ \tau_{r\theta} &= \frac{2Ma}{J(3-\nu)(1+\nu)}\left[-2+(1+\nu)\cos 2\theta\right]\cos\theta,\\ \sigma_\theta &= \nu\sigma_r.\end{aligned}\right\} \quad (46.10)$$

In the limiting case of (46.6) we rearrive at Eq. (40.8) for the plate without core.

The results of calculations for a veneer plate have been compiled in Table 5 (cf. page 209) which contains the numerical values of the stresses in fractions of Ma/J. The x-axis is assumed

parallel to the fibers of the sheet (load applied to the sides perpendicular to these fibers).

On the basis of Table 5 the graphs of distributions of σ_r (Fig. 107) and σ_θ (Fig. 108) on the contour of the aperture containing a core and empty.

The maximum value of the stresses σ_θ in the case of a rigid core is equal to $0.76\ Ma/J$ and is obtained near the point $\theta = 60°$ and the points symmetrical to it.

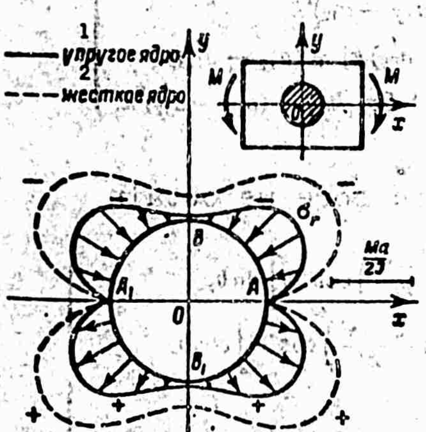

Fig. 107. 1) Elastic core; 2) rigid core.

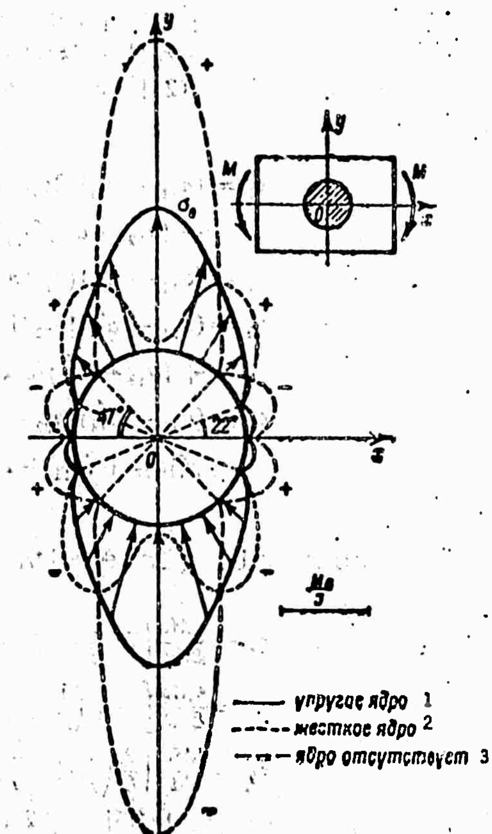

Fig. 108. 1) Elastic core; 2) rigid core; 3) no core.

In a plate without core the maximum stress is equal to 3.23 Ma/J as we know already from §40.

The case of an elastic core with the constants $a'_{ij} = 2a_{ij}$ takes an intermediate position and for it

$$\sigma_{max} = 1{,}50\frac{Ma}{J}. \qquad (46.11)$$

A comparison of the graphs represented in Figs. 107 and 108 by the solid and dashed lines illustrates in sufficient clearness the influence of elastic and rigid cores on the stress distribu-

TABLE 5

Comparison of Stresses at the Points of the Aperture's Contour

$\theta°$	1 Жесткое ядро			2 Упругое ядро			3 Без ядра
	σ_r	$\tau_{r\theta}$	σ_θ	σ_r	$\tau_{r\theta}$	σ_θ	σ_θ
0	0	−0,13	0	0	0,02	0	0
15	0,34	−0,21	−0,09	0,19	−0,04	−0,02	−0,29
30	0,54	−0,39	0,10	0,29	−0,16	0,07	−0,22
45	0,54	−0,54	0,44	0,27	−0,27	0,28	−0,03
60	0,39	−0,54	0,75	0,16	−0,29	0,58	0,34
75	0,21	−0,34	0,68	0,04	−0,19	1,01	1,38
90	0,13	0	0,01	−0,02	0	1,50	3,23

1) Rigid core; 2) elastic core; 3) no core.

tion in an anisotropic plate with round aperture.

If the forces are applied to the sides of the plate which are parallel to the fibers of the sheet, the graphs of the stress distribution on the contour will generally resemble those shown in Figs. 107 and 108.

The maximum values of the stresses are obtained as follows: for a plate with rigid core

$$\sigma_{max} = 0,67 \frac{Ma}{J}; \qquad (46.12)$$

for a plate with elastic core $(a'_{ij} = 2a_{ij})$

$$\sigma_{max} = 1,41 \frac{Ma}{J}, \qquad (46.13)$$

while in the case of no core the maximum stress is equal to 2.58 *Ma/J*.

Thus, for the veneer plate considered, with core or without core, it is favorable to apply the forces on the sides which are parallel to the fibers of the sheet since the resulting stresses are then smaller than the stresses in such a plate loaded on the sides perpendicular to the fibers. We must, however, make the reservation that this conclusion (as that at the end of §43) need not apply to an anisotropic plate with other elastic constants.

Manu-script Page No.	[Footnotes]
162	Savin, G.N., Kontsentratsiya napryazheniy okolo otverstiy [Stress Concentration Around an Aperture] Gostekhizdat [State Technical Publishers] Moscow-Leningrad, 1951.
163	See our paper: "Napryazheniya v neogranichennoy anizotropnoy plastinke, oslablennoy ellipticheskim otverstiyem" [Stresses in an Unlimited Anisotropic Plate Weakened by an Elliptic Aperture] DAN SSSR, Vol. IV (XII), No. 3 (107), 1936. All formulas given in this section were obtained for the case of unequal complex parameters. In the case of equal complex parameters the formulas for the stress can be obtained from formulas corresponding to $\mu_1 \neq \mu_2$, by means of a limiting transition.
164	See first Eq. (8.6). In the case given $\cos(n, x) = \pm \frac{a \sin \vartheta}{\sqrt{a^2 \sin^2 \vartheta + b^2 \cos^2 \vartheta}}, \quad \cos(n, y) = \mp \frac{b \cos \vartheta}{\sqrt{a^2 \sin^2 \vartheta + b^2 \cos^2 \vartheta}}.$
165	Savin, G.N., Nekotoryye zadachi teorii uprugosti anizotropnoy sredy [Some Problems of the Theory of Elasticity of Anisotropic Media] DAN SSSR, New Series, Vol. XXIII, No. 3, 1939; see also his book: Kontsentratsiya napryazheniy okolo otvertiy [Stress Concentration About an Aperture], Gostekhizdat, Moscow-Leningrad, 1951, pages 185-190.
166	These cases were considered in our papers: 1) Teoreticheskoye issledovaniye napryazhennogo sostoyaniya anizotropnoy plastinki, oslablennoy ellipticheskim ili krugovym otverstiyem [Theoretical Investigation of the State of Stress in an Anisotropic Plate Weakened by an Elliptic or Circular Aperture], Trudy konferentsii po opticheskomu metodu izucheniya napryazheniy [Transactions of the Conference on Optical Methods of Stress Investigation], NIIMM LGU and NIIMekh MGU, ONTI, 1937; 2) Kontsentratsiya napryazheniy vblizi ellipticheskogo i krugovogo otverstiya v rastyagivayemoy anizotropnoy plastinke [Stress Concentrations Near an Elliptic or Circular Aperture in an Extended Anisotropic Plate], Vestnik inzhenerov i tekhnikov [Herald of Engineers and Technicians], 1936, No. 5.
171	J is the moment of inertia of a cross section of a massive nonweakened plate (we understand, of course, an initially given plate which, when we want to obtain an approximate solution, is assumed to be infinitely large).
172*	Savin, G.N., Izgib anizotropnoy balki postoyannoy pererezyvayushchey siloy, oslablennoy ellipticheskim i krugovym otverstiyem [The Bending of an Anisotropic Beam

by a Constant Crosscut Force, Which is Weakened by an Elliptic or Round Aperture], Vestnik inzhenerov i tekhnikov [Herald of Engineers and Technicians] 1938, No. 4; see also: G.N. Savin, Kontsentratsiya napryazheniy okolo otverstiy [Concentration of Stresses Around an Aperture] GTTI, Moscow-Leningrad, 1951, pages 212-234.

172** A great part of these cases has been dealt with in our papers: 1) Teoreticheskoye issledovaniye napryazhennogo sostoyaniya anizotropnoy plastinki, oslablennoy ellipticheskim ili krugovym otverstiyem [Theoretical Investigation of the State of Stress of an Anisotropic Plate Weakened by an Elliptic or Circular Aperture], Trudy konferentsii po opticheskomu metodu izucheniya napryazheniy [Transactions of the Conference on Optical Methods of Investigating Stresses], NIIMM LGU and NIIMekh MGU, ONTI, 1937; 2) Nekotoryye sluchai raspredeleniya napryazheniy v anizotropnoy plastinke s krugovym otverstyem [Some Cases of Stress Distribution in an Anisotropic Plate with Circular Aperture], Uch. zap. LGU [Scient. Journal of Leningrad State University] Series of Mathematical Sciences (Mechanics) No. 8, 1939.

173 In the first edition of our book we gave the results of calculations and graphs for an anisotropic plate with other values of the elastic constants and the complex parameters.

179 This follows from the equations of the generalized Hooke's law for an orthotropic plate, (9.8). If $\sigma_x = -p$ and $\varepsilon_y = 0$ we have, obviously, $\sigma_y = \frac{E_2 \nu_1}{E_1} \sigma_x = -\nu_2 p$.

182 Investigations of the stresses in orthotropic plates deformed by tangential forces attacking at a certain angle with respect to the principal directions, and calculations of the stresses for various materials and angles of φ were carried out vy A.S. Dorogobed (see his paper: "Raspredeleniye napryazheniy v ortotropnoy plastinke s krugovym otverstiyem pri chistom sdvige" [Stress Distribution in an Orthotropic Plate with Circular Hole Under Pure Torsion] Inzhenernyy sbornik [Engineering Collection] Vol. XXI, 1955).

185 The bending of an orthotropic plate-beam with an aperture, with various orientations of the principal directions of elasticity was studied by V.B. Lipkin in his diplomate thesis "Raspredeleniye napryazheniy v ortotropnoy plastinke s krugovym otverstiyem pod deystviyem izgibayushchikh momentov" [Stress Distribution in an Orthotropic Plate with Circular Aperture Under the Action of Bending Moments] (Saratov State University, Saratov, 1951). In this paper we also find results of calculations for various materials.

186 See §37, Eqs. (37.9)-(37.10).

188* Fayerberg, I.I., Kontsentratsiya napryazheniy v aniso- tropnoy plastinke s krugovym otverstiyem [Stress Concentration in an Anisotropic Plate with Circular Aperture] Ministry of Aviation Industry USSR, Transactions No. 674, 1948.

188** A general solution to this problem and also solutions for other cases are considered in the paper: S.G. Lekhnitskiy, Raspredeleniye napryazheniy v anizotropnoy plastinke s ellipticheskim uprugim yadrom [Stress Distribution in an Anisotropic Plate with Elliptic Elastic Core] (ploskaya zadacha) [Plane Problem], Inzhenernyy sbornik, Vol. XIX, Moscow, 1954.

191 See pages 87-88 of our paper referred to in the preceding section.

192 See paper referred to, pages 99-101.

194 This formula was derived in our paper, referred to in §41 and §42.

195 See our paper, referred to in §41 and §42.

199 See our paper referred to in §§41-43, pages 95-96.

202 See our paper mentioned in the preceding sections, pages 97-99.

206 See our paper mentioned repeatedly in §§41-45, pages 10[illegible]-105.

Chapter 7

APPROXIMATION METHOD IN ORDER TO DETERMINE THE STRESSES IN A SLIGHTLY ANISOTROPIC PLATE

§47. THE PLANE PROBLEM FOR A SLIGHTLY ANISOTROPIC PLATE

The solutions derived in Chapters 3-6 for the two-dimensional problem of a homogeneous anisotropic body show that the complex parameters μ_1 and μ_2 are the fundamental quantities on which the stresses depend. The complex parameters enter many formulas for the stresses on which the maximum stress, the stress concentration, etc. depend. In an isotropic plate $\mu_1 = \mu_2 = i$. When the complex parameters of a homogeneous anisotropic plate are represented in the form

$$\left.\begin{aligned} \mu_1 &= i(1+\alpha_1), & \mu_2 &= i(1+\alpha_2), \\ \bar{\mu}_1 &= -i(1+\bar{\alpha}_1), & \bar{\mu}_2 &= -i(1+\bar{\alpha}_2), \end{aligned}\right\} \tag{47.1}$$

the quantities α_1 and α_2 are in general complex and can be considered as quantities characterizing the deviations of the plate from the isotropic one. The smaller $|\alpha_1|$ and $|\alpha_2|$ compared to unity the more similar is the plate in its properties to the isotropic plate.*

We shall call a plate "slightly anisotropic" when for it $|\alpha_1|$ and $|\alpha_2|$ are so small compared to unity that higher powers and products of α_1 and α_2 can be neglected when they exceed the first or second powers. We restrict ourselves to investigations of the stresses in slightly anisotropic orthotropic plates in a second approximation, seeking the stress function in the form of a series in powers of α_1 and α_2, neglecting higher powers of these quantities beginning with the third. This enables us to reduce the problem of the anisotropic plate to some plane problems for the isotropic plate and makes it possible to solve them approximately with the help of the well-known methods of two-dimensional problems of the theory of elasticity of the isotropic body.

Note that many authors apply a similar method of power expansions with respect to small parameters, and a cutoff at higher powers (the "method of the small parameter") in order to develop approximate solutions of various problems of the theory of elasticity (e.g., G.Yu. Dzhanelidze, N.V. Zvolinskiy, A.I. Lur'ye, D.Yu. Panov, P.M. Riz and others).**

Let us consider the case where volume forces are absent.***

The stress components are given in terms of the stress func-

tion F:

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x\,\partial y}, \tag{47.2}$$

which, in the case of an orthotropic body, satisfies the equation

$$\frac{1}{E_2}\cdot\frac{\partial^4 F}{\partial x^4} + \left(\frac{1}{G} - \frac{2\nu_1}{E_1}\right)\frac{\partial^4 F}{\partial x^2\,\partial y^2} + \frac{1}{E_1}\cdot\frac{\partial^4 F}{\partial y^4} = 0. \tag{47.3}$$

Expressing the coefficient of this equation in terms of the complex parameters $\mu_1 = i(1+\alpha_1)$ and $\mu_2 = i(1+\alpha_2)$, we omit powers of α_1 and α_2 higher than the second. Equation (47.3) will then read

$$\nabla^2\nabla^2 F + (2\alpha_1 + 2\alpha_2 + \alpha_1^2 + \alpha_2^2)\frac{\partial^2}{\partial x^2}\nabla^2 F + 4\alpha_1\alpha_2\frac{\partial^4 F}{\partial x^4} = 0, \tag{47.4}$$

where ∇^2 is the Laplace operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

In the case of an orthotropic plate the quantities $\alpha_1 + \alpha_2$, $\alpha_1^2 + \alpha_2^2$ and $\alpha_1\alpha_2$ are always real numbers.

We seek the function F in the form

$$F = F_{00} + (\alpha_1 + \alpha_2)F_{10} + (\alpha_1^2 + \alpha_2^2)F_{20} + \alpha_1\alpha_2 F_{11}, \tag{47.5}$$

where F_{ij} is independent of α_1 and α_2 and obtain the equations

$$\left.\begin{aligned} &\nabla^2\nabla^2 F_{00} = 0,\\ &\nabla^2\nabla^2 F_{10} = -2\frac{\partial^2}{\partial x^2}\nabla^2 F_{00},\\ &\nabla^2\nabla^2 F_{20} = -\frac{\partial^2}{\partial x^2}\nabla^2(2F_{10} + F_{00}),\\ &\nabla^2\nabla^2 F_{11} = -4\frac{\partial^2}{\partial x^2}\left(\nabla^2 F_{10} + \frac{\partial^2 F_{00}}{\partial x^2}\right). \end{aligned}\right\} \tag{47.6}$$

Integrating these equations one after the other we obtain the following general expressions:*

$$\left.\begin{aligned} &F_{00} = \mathrm{Re}\,[\bar{z}\varphi_{00}(z) + \chi_{00}(z)],\\ &E_{10} = \mathrm{Re}\left[\bar{z}\varphi_{10}(z) + \chi_{10}(z) - \frac{1}{4}\bar{z}^2\varphi'_{00}(z)\right],\\ &F_{20} = \mathrm{Re}\left[\bar{z}\varphi_{20}(z) + \chi_{20}(z) - \frac{1}{4}\bar{z}^2\varphi'_{10}(z) + \frac{1}{24}\bar{z}^3\varphi''_{00}(z) + \frac{1}{8}\bar{z}^2\varphi'_{00}(z)\right],\\ &F_{11} = \mathrm{Re}\left[\bar{z}\varphi_{11}(z) + \chi_{11}(z) - \frac{1}{2}\bar{z}^2\varphi'_{10}(z) + \frac{1}{24}\bar{z}^3\varphi''_{10}(z) - \frac{1}{8}\bar{z}^2\chi''_{00}(z)\right]. \end{aligned}\right\} \tag{47.7}$$

Here

$$z = x + iy, \quad \bar{z} = x - iy,$$

$\varphi_{ij}(z)$, $\chi_{ij}(z)$ are arbitrary analytical functions of the complex variable z. In the following we shall use abbreviations omitting the argument:

$$\varphi_{ij} = \varphi_{ij}(z), \quad \psi_{ij} = \chi'_{ij}(z), \quad \bar{\varphi}_{ij} = \bar{\varphi}_{ij}(\bar{z}), \quad \bar{\psi}_{ij} = \bar{\chi}'_{ij}(\bar{z})$$

etc.

On the basis of Expressions (47.7) we obtain general formulas for the first derivatives of F and the stress components which are convenient to represent in a complex form:

$$\begin{aligned}\frac{\partial F}{\partial x}+i\frac{\partial F}{\partial y}={}&z\overline{\varphi'_{00}}+\overline{\psi_{00}}+\varphi_{00}+(\alpha_1+\alpha_2)\Big[z\overline{\varphi'_{10}}+\overline{\psi_{10}}+\varphi_{10}-\\&-\frac{1}{4}\left(z^2\overline{\varphi''_{00}}+2\bar{z}\varphi'_{00}\right)\Big]+(\alpha_1^2+\alpha_2^2)\Big[z\overline{\varphi'_{20}}+\overline{\psi_{20}}+\varphi_{20}-\\&-\frac{1}{4}\left(z^2\overline{\varphi''_{10}}+2\bar{z}\varphi'_{10}\right)+\frac{1}{8}\left(\frac{z^3}{3}\overline{\varphi'''_{00}}+z^2\overline{\varphi''_{00}}+\bar{z}^2\varphi''_{00}+2\bar{z}\varphi'_{00}\right)\Big]+\\&+\alpha_1\alpha_2\Big[z\overline{\varphi'_{11}}+\overline{\psi_{11}}+\varphi_{11}-\frac{1}{2}\left(z^2\overline{\varphi''_{10}}+2\bar{z}\varphi'_{10}\right)+\\&+\frac{1}{8}\left(\frac{z^3}{3}\overline{\varphi'''_{00}}-z^2\overline{\psi''_{00}}+\bar{z}^2\varphi''_{00}-2\bar{z}\psi'_{00}\right)\Big];\end{aligned}\tag{47.8}$$

$$\begin{aligned}\sigma_y-\sigma_x+2i\tau_{xy}={}&2\left(\bar{z}\varphi''_{00}+\psi'_{00}\right)+(\alpha_1+\alpha_2)\Big[2\left(\bar{z}\varphi''_{10}+\psi'_{10}\right)-\\&-\frac{\bar{z}^2}{2}\varphi'''_{00}-\overline{\varphi'_{00}}\Big]+(\alpha_1^2+\alpha_2^2)\Big[2\left(\bar{z}\varphi''_{20}+\psi'_{20}\right)-\frac{\bar{z}^2}{2}\varphi'''_{10}-\overline{\varphi'_{10}}+\\&+\frac{\bar{z}^3}{12}\varphi^{IV}_{00}+\frac{\bar{z}^2}{4}\varphi'''_{00}+\frac{z}{2}\overline{\varphi''_{00}}+\frac{1}{2}\overline{\varphi'_{00}}\Big]+\alpha_1\alpha_2\Big[2\left(\bar{z}\varphi''_{11}+\psi'_{11}\right)-\\&-\bar{z}^2\varphi'''_{10}-2\overline{\varphi'_{10}}+\frac{\bar{z}^3}{12}\varphi^{IV}_{00}-\frac{\bar{z}^2}{4}\psi'''_{00}+\frac{z}{2}\overline{\varphi''_{00}}-\frac{1}{2}\overline{\psi'_{00}}\Big];\end{aligned}\tag{47.9}$$

$$\begin{aligned}\sigma_x+\sigma_y=\operatorname{Re}\Big[&4\varphi'_{00}+(\alpha_1+\alpha_2)\left(4\varphi'_{10}-2\bar{z}\varphi''_{00}\right)+(\alpha_1^2+\alpha_2^2)\times\\&\times\left(4\varphi'_{20}-2\bar{z}\varphi''_{10}+\frac{\bar{z}^2}{2}\varphi'''_{00}+\bar{z}\varphi''_{00}\right)+\\&+\alpha_1\alpha_2\left(4\varphi'_{11}-4\bar{z}\varphi''_{10}+\frac{\bar{z}^2}{2}\varphi'''_{00}-\bar{z}\psi''_{00}\right)\Big].\end{aligned}\tag{47.10}$$

The displacements are also given in terms of functions of the complex variables, but we shall not explicate them here.

When the external forces X_n, Y_n acting on the boundary are given, the boundary conditions can be written in the following form:

$$\frac{\partial F}{\partial x}+i\frac{\partial F}{\partial y}=\pm\int_0^s(iX_n-Y_n)\,ds+c.\tag{47.11}$$

We consider, as previously, the anticlockwise direction of circumvention around the contour to be positive; we must take the upper sign when we consider the outer contour of the domain occupied by the body, the lower sign when we consider the contour of the aperture.

In this case where the forces given for the contour are independent of the elastic constants of the material, we obtain on the basis of (47.8) and (47.11) the following boundary conditions for the functions φ_{ij} and ψ_{ij}:*

$$
\left.\begin{aligned}
z\overline{\varphi}'_{00}+\overline{\psi}_{00}+\varphi_{00} &= \pm\int_0^s (lX_n - Y_n)\,ds + c_{00},\\
z\overline{\varphi}'_{10}+\overline{\psi}_{10}+\varphi_{10} &= \frac{1}{4}\left(z^2\overline{\varphi}''_{00}+2\overline{z}\varphi'_{00}\right)+c_{10},\\
z\overline{\varphi}'_{20}+\overline{\psi}_{20}+\varphi_{20} &= \frac{1}{4}\left(z^2\overline{\varphi}''_{10}+2\overline{z}\varphi'_{10}\right)-\\
&\quad -\frac{1}{8}\left(\frac{z^3}{3}\overline{\varphi}'''_{00}+z^2\overline{\varphi}''_{00}+\overline{z}^2\varphi''_{00}+2\overline{z}\varphi'_{00}\right)+c_{20},\\
z\overline{\varphi}'_{11}+\overline{\psi}_{11}+\varphi_{11} &= \frac{1}{2}\left(z^2\overline{\varphi}''_{10}+2\overline{z}\varphi'_{10}\right)-\\
&\quad -\frac{1}{8}\left(\frac{z^3}{3}\overline{\varphi}'''_{00}-z^2\overline{\psi}''_{00}+\overline{z}^2\varphi''_{00}-2\overline{z}\psi'_{00}\right)+c_{11}.
\end{aligned}\right\} \qquad (47.12)
$$

As we see from Eqs. (47.9), (47.10) and Conditions (47.12), the functions φ_{00} and ψ_{00} give the stress distribution in an isotropic plate of the same form as the slightly isotropic one and loaded by the same forces X_n, Y_n. Having obtained φ_{00} and ψ_{00} on the basis of the second condition of (47.12), we determine φ_{10}, ψ_{10}; these functions give the stress distribution in an isotropic plate loaded on the contour by forces whose distribution law depends on the form of the functions φ_{00} and ψ_{00}. Having obtained φ_{10} and ψ_{10} on the basis of the third and fourth conditions of (47.12) we determine φ_{20}, ψ_{20} and φ_{11}, ψ_{11}. In this way, in order to determine the stresses in a slightly anisotropic plate in a second approximation we must solve four one-dimensional problems for an isotropic plate of the same form. The theory of the one-dimensional problem of the isotropic body is available in a very good elaboration (thanks to the papers by G.V. Kolosov, N.I. Muskhelishvili and their pupils); we have effective methods of solving this problem at our disposal: the method of series expansions, the methods based on the application of conformal mapping and Cauchy integrals, and others.* These methods enable us to obtain approximate solutions to a very broad class of problems of slightly anisotropic plates and to derive approximate formulas in order to determine the stresses in such plates.

§48. DETERMINATION OF THE STRESSES IN A SLIGHTLY ANISOTROPIC PLATE WITH AN APERTURE

Let us consider a slightly anisotropic plate with an arbitrary aperture, which is exposed to the action of forces applied to the outer edge and the edge of the aperture. The aperture is assumed small compared to the plate dimensions and remote from its edge.

An approximate solution of the problem can be obtained by superimposing the stresses in an infinitely large plate with an aperture to the edge of which certain forces are applied; these forces are chosen such that the necessary boundary conditions are satisfied at the contour of the aperture. In this way we obtain stresses which satisfy precisely the conditions of the aperture's contour, and with increasing distance to them they tend to the stresses existing in a plate without aperture, on the outer edge of which the given forces are distributed.

Assume we have obtained the stresses $\overset{0}{\sigma}_x$, $\overset{0}{\sigma}_y$, $\overset{0}{\tau}_{xy}$, which would be

generated under the action of the given forces in such a plate which, however, in massive, without aperture. The well-known functions $\varphi^0_{ij}(z)$ and $\psi^0_{ij}(z)$ of the complex variable correspond to these stresses. Let us represent the functions φ_{ij}, ψ_{ij} engering Eqs. (47.7)-(47.10) in the form of sums

$$\left.\begin{aligned}\varphi_{ij}(z)&=\varphi^0_{ij}(z)+f_{ij}(z),\\ \psi_{ij}(z)&=\psi^0_{ij}(z)+F_{ij}(z),\end{aligned}\right\}\qquad(48.1)$$

where f_{ij} and F_{ij} are unknown functions to which stresses correspond which tend to zero as the distance to the aperture increases.

Let us map conformally an infinite plane with a cutout in the form of the aperture to an infinite plane with a cutout of the form of the unit circle (i.e., to the exterior of the unit circle). Let the function performing this conformal mapping have the form

$$z=\omega(\zeta).\qquad(48.2)$$

We then replace in all formulas z by Expression (48.2) and introduce the denotations

$$\left.\begin{aligned}&f_{ij}(z)=f_{ij}[\omega(\zeta)]=\Phi_{ij}(\zeta),\\ &f'_{ij}(z)=\frac{\Phi'_{ij}(\zeta)}{\omega'(\zeta)}=u_{ij}(\zeta),\\ &f''_{ij}(z)=v_{ij}(\zeta),\quad f'''_{ij}(z)=w_{ij}(\zeta);\\ &F_{ij}(z)=F_{ij}[\omega(\zeta)]=\Psi_{ij}(\zeta),\\ &F'_{ij}(z)=\frac{\Psi'_{ij}(\zeta)}{\omega'(\zeta)}=U_{ij}(\zeta),\\ &F''_{ij}(z)=V_{ij}(\zeta).\end{aligned}\right\}\qquad(48.3)$$

The problem is reduced to the determination of the functions $\Phi_{ij}(\zeta)$ and $\Psi_{ij}(\zeta)$, which satisfy the definite conditions of regularity outside the unit circle and the boundary conditions which are obtained from (47.12) after having replaced ζ for z.

In order to determine the unknown functions various methods can be used. When $\omega(\zeta)$ is an algebraic function Φ_{ij} and Ψ_{ij} can generally be represented in the form of series*

$$\left.\begin{aligned}\Phi_{ij}(\zeta)&=A_{ij}\ln\zeta+A^0_{ij}+\sum_{m=1}^{\infty}A^m_{ij}\zeta^{-m},\\ \Psi_{ij}(\zeta)&=B_{ij}\ln\zeta+B^0_{ij}+\sum_{m=1}^{\infty}B^m_{ij}\zeta^{-m}.\end{aligned}\right\}\qquad(48.4)$$

The coefficients of the logarithmic functions can be given in terms of the projection of the vector sum of the forces acting on the contour of the aperture. The coefficients A^m_{ij} and B^m_{ij} are determined from the boundary conditions in which the given quantities must be represented in the form of series expanded in powers of $\sigma=e^{\theta i}$; this is the boundary value of the variable ζ (in the general case the expressions for the given quantities will

also contain ln σ besides σ). In certain cases it is more favorable to use a method which is based on the application of Cauchy integrals.

When all functions have been determined we obtain the stress components from Eqs. (47.9) and (47.10). The stresses σ_ϑ in surfaces normal to the contour of the aperture, at the contour itself, are in the general case determined according to the formula

$$\begin{aligned} \sigma_\vartheta = & -\sigma_n + \sigma_x^0 + \sigma_y^0 + 4\,\mathrm{Re}\,[u_{00}(\sigma)] + \\ & + (\alpha_1 + \alpha_2)\,\mathrm{Re}\left[4u_{10}(\sigma) - 2\,\overline{\omega}\left(\frac{1}{\sigma}\right) v_{00}(\sigma)\right] + \\ & + (\alpha_1^2 + \alpha_2^2)\,\mathrm{Re}\left\{4u_{20}(\sigma) + \overline{\omega}\left(\frac{1}{\sigma}\right)\left[-2v_{10}(\sigma) + v_{00}(\sigma) + \right.\right. \\ & \left.\left. + \frac{1}{2}\,\overline{\omega}\left(\frac{1}{\sigma}\right) w_{00}(\sigma)\right]\right\} + \alpha_1\alpha_2\,\mathrm{Re}\left\{4u_{11}(\sigma) + \overline{\omega}\left(\frac{1}{\sigma}\right) \times \right. \\ & \left. \times \left[-4v_{10}(\sigma) - V_{00}(\sigma) + \frac{1}{2}\,\overline{\omega}\left(\frac{1}{\sigma}\right) w_{00}(\sigma)\right]\right\}. \end{aligned} \tag{48.5}$$

Here σ_n is the normal stress acting on the edge of the aperture (a given quantity), $\sigma = e^{\vartheta i}$, ϑ is the polar angle determining the positions of the points on the contour of the unit circle and, at the same time, in the basis to Eq. (48.2), also the positions of the points on the contour of the aperture; $\overline{\omega}\left(\frac{1}{\sigma}\right)$ is the quantity conjugate to ω(σ).

In the following section we consider by way of example the stress distribution in a plate with an almost quadratic aperture in greater detail.

§49. EXTENSION AND PURE BENDING OF A SLIGHTLY ANISOTROPIC PLATE WITH AN ALMOST QUADRATIC APERTURE

We consider a plate weakened by a small aperture whose contour is given by the equations

$$x = a(\cos\vartheta + \varepsilon\cos 3\vartheta), \qquad y = a(\sin\vartheta - \varepsilon\sin 3\vartheta), \tag{49.1}$$

where ϑ is a parameter varying from zero to 2π; a and ε are constant coefficients where $a > 0$, $|\varepsilon| < 1/3$. A suitable choice of the coefficient ε enables us to obtain a shape which differs but slightly from a square with rounded corners.

The problem of the stress distribution in an isotropic plate with an almost quadratic aperture, which is deformed by forces applied to the outer edge, has been considered by several authors who solved it with the methods by N.I. Muskhelishvili. P.A. Sokolov considered extension and shear,* G.N. Savin extension and bending** and M.I. Nayman bending alone.*** Using the equation of the aperture's contour in the form of (49.1), M.I. Nayman assumed ε = 1/9 and G.N. Savin supposed ε = ±1/6; Savin also considered contours determined by more complex equations which give a better approach to the square.

A function mapping an infinite plane with a cutout (49.1) on the exterior of a unit circle has the form

$$\omega(\zeta) = a\left(\zeta + \frac{\varepsilon}{\zeta^3}\right). \tag{49.2}$$

1. Extension. A slightly anisotropic but orthotropic rectangular plate is weakened by a small aperture in its center and deformed by normal forces p distributed uniformly on two sides. It is supposed that the principal directions of elasticity are parallel to the sides so that the tension direction coincides with the principal directions.

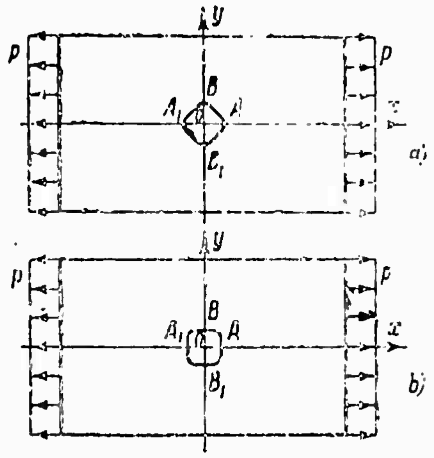

Fig. 109

Placing the origin of coordinates in the center of the aperture and direct the x, y-axes of symmetry of the plate, we may consider the contour of the aperture given by Eq. (49.1). With positive ε the "square" has a position as shown in Fig. 109a and the tension acts in a diagonal direction; with negative ε the position of the aperture corresponds to Fig. 109b.

In the case of extension

$$\sigma_x^0 = p, \qquad \sigma_y^0 = \tau_{xy}^0 = 0; \tag{49.3}$$

$$\left.\begin{array}{ll}
\varphi_{00}^0(z) = \dfrac{p}{4}z, & \psi_{00}^0(z) = -\dfrac{p}{2}z,\\
\varphi_{10}^0(z) = 0, & \psi_{10}^0(z) = \dfrac{p}{8}z,\\
\varphi_{20}^0(z) = 0, & \psi_{20}^0(z) = -\dfrac{p}{16}z,\\
\varphi_{11}^0(z) = 0, & \psi_{11}^0(z) = -\dfrac{p}{8}z,
\end{array}\right\} \tag{49.4}$$

which is easy to prove when these values and $\sigma_x = p$, $\sigma_y = \tau_{xy} = 0$ are substituted in Eqs. (47.9) and (47.10).

The functions Φ_{ij} and Ψ_{00} which must be known in order to determine the stress σ_ϑ around the aperture read*

$$
\left.
\begin{aligned}
&\Phi_{00}(\zeta)=\frac{pa}{2}\left(\frac{1}{1-\varepsilon}\cdot\frac{1}{\zeta}-\frac{\varepsilon}{\zeta^3}\right),\\
&\Psi_{00}(\zeta)=-\frac{pa\varepsilon}{2(1-\varepsilon)}\zeta+\frac{pa}{2}\left(\frac{\varepsilon}{\zeta^3}-\frac{1}{\zeta}\right)-\bar{\omega}\left(\frac{1}{\zeta}\right)u_{00}(\zeta),\\
&\Phi_{10}(\zeta)=\frac{pa\varepsilon}{4(1-\varepsilon)}\left(\zeta-\frac{3\varepsilon^2}{\zeta}+\frac{\varepsilon}{\zeta^3}\right)+\frac{1}{2}\bar{\omega}\left(\frac{1}{\zeta}\right)u_{00}(\zeta),\\
&\Phi_{20}(\zeta)=-\frac{p\varepsilon(1+3\varepsilon^3)}{8(1-\varepsilon)}\omega(\zeta)+\\
&\quad+\frac{pa\varepsilon^3}{8(1-\varepsilon)}\left[\zeta^3+\frac{10\varepsilon^3+3\varepsilon-2}{1-\varepsilon}\cdot\frac{1}{\zeta}-\frac{4\varepsilon^3(1-\varepsilon)}{\zeta^3}+\frac{\varepsilon}{\zeta^5}\right]+\\
&\quad+\frac{1}{8}\bar{\omega}\left(\frac{1}{\zeta}\right)\left[4u_{10}(\zeta)-2u_{00}(\zeta)-\bar{\omega}\left(\frac{1}{\zeta}\right)v_{00}(\zeta)\right],\\
&\Phi_{11}(\zeta)=-\frac{p\varepsilon}{8(1-\varepsilon)}[3\varepsilon^2(1+\varepsilon)+1-\varepsilon]\,\omega(\zeta)+\\
&\quad+\frac{pa\varepsilon^3}{8(1-\varepsilon)}\left[\zeta^3+\frac{13\varepsilon^2+3\varepsilon-2}{1-\varepsilon}\cdot\frac{1}{\zeta}-\frac{4\varepsilon^3(1-\varepsilon)}{\zeta^3}+\frac{\varepsilon}{\zeta^5}\right]+\\
&\quad+\frac{1}{8}\bar{\omega}\left(\frac{1}{\zeta}\right)\left[8u_{10}(\zeta)+2U_{00}(\zeta)-\bar{\omega}\left(\frac{1}{\zeta}\right)v_{00}(\zeta)\right].
\end{aligned}
\right\}
\tag{49.5}
$$

Supposing that $\zeta=\sigma=e^{\theta i}$ in these expressions and substituting it in (48.5) we obtain the stress distribution along the contour of the aperture. The formulas for σ_ϑ are very cumbersome and we do not give them here; we only give the values of the stresses in the two most important points.

At the points B and B_1 at the ends of the diameter which is perpendicular to the tensile forces (Fig. 109 a and b) we obtain:

$$
\begin{aligned}
\sigma_\theta=&\frac{p}{(1-\varepsilon)(1-3\varepsilon)}\Big[3+2\varepsilon-3\varepsilon^2+(\alpha_1+\alpha_2)(1+4\varepsilon-6\varepsilon^2-3\varepsilon^3)+\\
&+(\alpha_1^2+\alpha_2^2)\frac{\varepsilon^2}{2(1-\varepsilon)}(-5+5\varepsilon+32\varepsilon^2-33\varepsilon^3+12\varepsilon^4)+\\
&+\alpha_1\alpha_2\frac{\varepsilon}{2(1-\varepsilon)}(1-8\varepsilon+\varepsilon^2+38\varepsilon^3-21\varepsilon^4+3\varepsilon^5)\Big].
\end{aligned}
\tag{49.6}
$$

In the case of extension in the direction of the diagonal (Fig. 109a) we obtain for ε = 1/6

$$\sigma_\theta=p[7{,}80+(\alpha_1+\alpha_2)\,3{,}57-(\alpha_1^2+\alpha_2^2)\,0{,}15-\alpha_1\alpha_2\cdot 0{,}03]; \tag{49.7}$$

and for ε = 1/9

$$\sigma_\theta=p[5{,}37+(\alpha_1+\alpha_2)\,2{,}31-(\alpha_1^2+\alpha_2^2)\,0{,}05+\alpha_1\alpha_2\cdot 0{,}02]. \tag{49.8}$$

In the case of extension in the direction of a side of the "square" (Fig. 109b) for ε = -1/6

$$\sigma_\theta=p[1{,}47+(\alpha_1+\alpha_2)\,0{,}10-(\alpha_1^2+\alpha_2^2)\,0{,}03-\alpha_1\alpha_2\cdot 0{,}09]; \tag{49.9}$$

for ε = -1/9

$$\sigma_\theta=p[1{,}85+(\alpha_1+\alpha_2)\,0{,}32-(\alpha_1^2+\alpha_2^2)\,0{,}02+\alpha_1\alpha_2\cdot 0{,}06]. \tag{49.10}$$

With ε = 0 we obtain a round aperture of radius a. At the points at the end of a diameter perpendicular to the forces we obtain according to Eq. (49.6):

$$\sigma_\theta=p(3+\alpha_1+\alpha_2). \tag{49.11}$$

This result coincides (obviously by chance) with the accurate result [see Eq. (39.14)].

2. Pure bending. A rectangular slightly anisotropic orthotropic plate which is weakened by a small central aperture of the form (49.1) is deformed by forces distributed on two sides, which can be reduced to the moments M (Fig. 110 a and b).

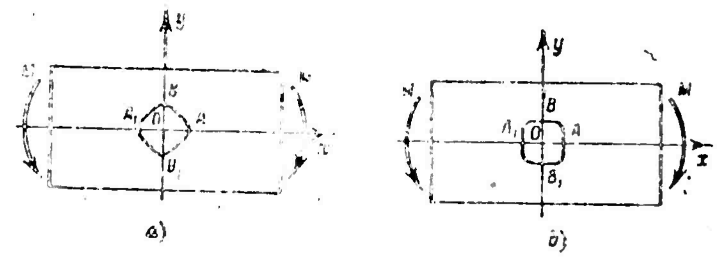

Fig. 110

In this case*

$$\sigma_x^0 = \frac{M}{J} y, \quad \sigma_y^0 = \tau_{xy}^0 = 0 \tag{49.12}$$

(J is the moment of inertia of a cross section),

$$\left.\begin{aligned}
\varphi_{00}^0(z) &= -\frac{Mi}{8J} z^2, & \psi_{00}^0(z) &= \frac{Mi}{8J} z^2, \\
\varphi_{10}^0(z) &= \frac{Mi}{16J} z^2, & \psi_{10}^0(z) &= 0, \\
\varphi_{20}^0(z) &= -\frac{Mi}{16J} z^2, & \psi_{20}^0(z) &= -\frac{Mi}{32J} z^2, \\
\varphi_{11}^0(z) &= -\frac{3Mi}{32J} z^2, & \psi_{11}^0(z) &= -\frac{Mi}{32J} z^2;
\end{aligned}\right\} \tag{49.13}$$

$$\left.\begin{aligned}
\Phi_{00}(\zeta) &= \frac{Ma^3 i}{8J}\left(\frac{1+2\varepsilon}{\zeta^2} - \frac{2\varepsilon}{\zeta^4} + \frac{\varepsilon^3}{\zeta^8}\right), \\
\Psi_{00}(\zeta) &= -\Phi_{00}(\zeta) - \overline{\omega}\left(\frac{1}{\zeta}\right) u_{00}(\zeta), \\
\Phi_{10}(\zeta) &= -\frac{3Ma^2 i}{16J}(1+2\varepsilon)\frac{\varepsilon^2}{\zeta^2} + \frac{1}{2}\overline{\omega}\left(\frac{1}{\zeta}\right) u_{00}(\zeta), \\
\Phi_{20}(\zeta) &= \frac{Ma^2 i}{32J}\varepsilon^2\Big[3(1+2\varepsilon)\zeta^2 + \\
&\quad + (3+6\varepsilon+\varepsilon^2+18\varepsilon^3)\frac{1}{\zeta^2} - \frac{4\varepsilon(1+2\varepsilon)}{\zeta^4}\Big] + \\
&\quad + \frac{1}{4}\overline{\omega}\left(\frac{1}{\zeta}\right)\Big[2u_{10}(\zeta) - u_{00}(\zeta) - \frac{1}{2}\overline{\omega}\left(\frac{1}{\zeta}\right) v_{00}(\zeta)\Big], \\
\Phi_{11}(\zeta) &= \frac{Ma^2 i}{32J}\varepsilon^2\Big[3(1+2\varepsilon)\zeta^2 + \\
&\quad + (3+6\varepsilon-38\varepsilon^2+36\varepsilon^3)\frac{1}{\zeta^2} - \frac{4\varepsilon(1+2\varepsilon)}{\zeta^4}\Big] + \\
&\quad + \frac{1}{4}\overline{\omega}\left(\frac{1}{\zeta}\right)\Big[4u_{10}(\zeta) + U_{00}(\zeta) - \frac{1}{2}\overline{\omega}\left(\frac{1}{\zeta}\right) v_{00}(\zeta)\Big].
\end{aligned}\right\} \tag{49.14}$$

At the points A and A_1 (Fig. 110 a and b) $\sigma_\theta = 0$. At the points B and B_1

$$\sigma_\theta = \pm \frac{Ma}{J(1-3\varepsilon)}[2(1+2\varepsilon)+0,5(\alpha_1+\alpha_2)(1+6\varepsilon-6\varepsilon^3)- -0,25(\alpha_1^2+\alpha_2^2)\varepsilon^2(3-2\varepsilon-17\varepsilon^2-18\varepsilon^3)- -0,5\alpha_1\alpha_2\varepsilon^2(3+2\varepsilon+11\varepsilon^2-18\varepsilon^3)]. \quad (49.15)$$

When the aperture has been cut out as shown in Fig. 110a, the results of the calculation according to Eq. (49.15) lead us to the following:

for ε = 1/6

$$\sigma_\theta = \pm \frac{Ma}{J}[5,33+(\alpha_1+\alpha_2)1,97-(\alpha_1^2+\alpha_2^2)0,002-\alpha_1\alpha_2\cdot 0,09]; \quad (49.16)$$

for ε = 1/9

$$\sigma_\theta = \pm \frac{Ma}{J}[3,67+(\alpha_1+\alpha_2)1,24-(\alpha_1^2+\alpha_2^2)0,01-\alpha_1\alpha_2\cdot 0,03]. \quad (49.17)$$

When the aperture has a position as shown in Fig. 110b we have for ε = -1/6

$$\sigma_\theta = \pm \frac{Ma}{J}[0,89+(\alpha_1+\alpha_2)0,09-(\alpha_1^2+\alpha_2^2)0,004-\alpha_1\alpha_2\cdot 0,03]; \quad (49.18)$$

for ε = -1/9

$$\sigma_\theta = \pm \frac{Ma}{J}[1,17+(\alpha_1+\alpha_2)0,13-(\alpha_1^2+\alpha_2^2)0,01-\alpha_1\alpha_2\cdot 0,01]. \quad (49.19)$$

Assuming in Eq. (49.15) ε = 0 we obtain the stress at the edge of a round aperture at the points $\theta = \pm\frac{\pi}{2}$:

$$\sigma_\theta = \pm \frac{Ma}{J}\left(2+\frac{\alpha_1+\alpha_2}{2}\right). \quad (49.20)$$

It is interesting to note that also this result agrees with the accurate result [see Eq. (40.10)].

§50. DISTRIBUTION OF STRESSES IN A HOMOGENEOUS RING COMPRESSED BY UNIFORM PRESSURE

In §26 we considered the solution of the problem on the stress distribution in a ring with cylindrical anisotropy compressed by normal forces distributed uniformly on the inner and outer contours. The solution has a very simple form: the radial directions are equivalent with respect to the elastic properties so that the stress only depends of the single coordinate r (distance from the center). The same problem for a homogeneous ring with rectilinear anisotropy is much more complex owing to the fact that the radial stresses are in this case not equivalent and it is therefore not justified to expect the stresses to be independent of the polar angle θ. An exact solution to this problem has not yet been obtained. We shall here give the solution for a slightly anisotropic orthotropic ring obtained in a second approximation.

Placing the origin of coordinates at the center of the ring, we assume the principal directions of elasticity in the directions of the axes x and y. We denote by a and b the inner and outer radii of the ring ($c = a/b$) and by p and q the quantities of the inner and outer pressures per unit surface area (Fig. 44 in §26).

Let us return to Eqs. (47.8)-(47.10). The functions φ_{00} and ψ_{00} determining the stresses in an isotropic ring are well known, namely*

$$\varphi_{00}(z) = \frac{pc^2 - q}{2(1 - c^2)} z, \quad \psi_{00}(z) = \frac{(q - p) b^2 c^2}{1 - c^2} \cdot \frac{1}{z}. \tag{50.1}$$

The other functions φ_{10}, ψ_{10}, φ_{20}, ψ_{20}, φ_{11} and ψ_{11} are obtained in an elementary manner from the given functions φ_{00} and ψ_{00} in the form of polynomials with positive and negative powers of z; the coefficients of these polynomials are determined from the boundary conditions (47.12).

We arrive at the result that $\varphi_{10} = \psi_{10} = \varphi_{20} = \psi_{20} = 0$ and only φ_{11} and ψ_{11} are nonzero. Omitting all the intermediary calculations which are elementary, anyway, we only give the final formulas for the stresses:

$$\left.\begin{aligned}
\sigma_r &= \frac{pc^2 - q}{1 - c^2} - \frac{(p - q) c^2}{1 - c^2}\left(\frac{b}{r}\right)^2 + \alpha_1\alpha_2\left[A\left(\frac{r}{b}\right)^4 + B\left(\frac{r}{b}\right)^2 + \right.\\
&\quad \left. + 3C\left(\frac{b}{r}\right)^4 + D\left(\frac{b}{r}\right)^6 + 2\left(\frac{b}{r}\right)^2\right]\frac{(p - q) c^2}{1 - c^2}\cos 4\theta,\\
\sigma_\theta &= \frac{pc^2 - q}{1 - c^2} + \frac{(p - q) c^2}{1 - c^2}\left(\frac{b}{r}\right)^2 + \\
&\quad + \alpha_1\alpha_2\left[-3A\left(\frac{r}{b}\right)^4 - B\left(\frac{r}{b}\right)^2 - C\left(\frac{b}{r}\right)^4 - D\left(\frac{b}{r}\right)^6\right]\frac{(p - q) c^2}{1 - c^2}\cos 4\theta,\\
\tau_{r\theta} &= \alpha_1\alpha_2\left[-2A\left(\frac{r}{b}\right)^4 - B\left(\frac{r}{b}\right)^2 + 2C\left(\frac{b}{r}\right)^4 + \right.\\
&\quad \left. + D\left(\frac{b}{r}\right)^6 + 0{,}5\left(\frac{b}{r}\right)^2\right]\frac{(p - q) c^2}{1 - c^2}\sin 4\theta.
\end{aligned}\right\} \tag{50.2}$$

Here

$$\left.\begin{aligned}
A &= \frac{5}{2\delta}(1 + c^2), \qquad B = -\frac{3}{2\delta}(3 + 4c^2 + 3c^4),\\
C &= -\frac{3c^6}{2\delta}(1 + c^2)(1 + 5c^2 + c^4), \quad D = \frac{5c^4}{2\delta}(1 + 4c^2 + c^4),\\
\delta &= 1 + 4c^2 + 10c^4 + 4c^6 + c^8.
\end{aligned}\right\} \tag{50.3}$$

Formulas (50.2) which give the stress distribution in a homogeneous, slightly anisotropic ring in a second approximation, only contain products $\alpha_1\alpha_2$ and do not contain first or second powers of α_1 and α_2. Hence it follows that the stress distribution in such a ring agrees in a first approximation with the distribution in a ring of isotropic material; the corrections for anisotropy are quantities which are small of second order.

When for a given material one of the complex parameters is equal to i, i.e., $\alpha_2 = 0$, the stress distribution in an anisotropic ring will be precisely the same as in an isotropic one. This holds true not only for a slightly anisotropic material and in a

second approximation, but also for a material for which the second complex parameter is an arbitrary number, even when its absolute value is very high. In fact, with $\alpha_2 = 0$ the equation for the stress function (47.4) assumes the form

$$\left[(1+\alpha_1)^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] \nabla^2 F = 0. \tag{50.4}$$

The stress function for an isotropic ring has the form*

$$F = Ar^2 + B \ln r. \tag{50.5}$$

It is easy to verify the simple statement that it will also satisfy Eq. (50.4) for a ring with an anisotropy of a given type, from which we can deduce the validity of what has been said above.

It follows from Eq. (50.2) that in a ring compressed by pressures of equal intensities ($p = q$) and also in a disk without aperture compressed by the pressure q, which is distributed uniformly along the edge ($c = 0$) a uniform field of stresses will establish:

$$\sigma_r = \sigma_\theta = -q, \quad \tau_{r\theta} = 0. \tag{50.6}$$

These formulas prove to be exact for a homogeneous ring and a disk with rectilinear anisotropy with arbitrary α_1 and α_2.

Applying the method of small parameters described in this chapter it is easy to obtain, in a second approximation, formulas for the stresses in a homogeneous ring which is deformed by forces distributed arbitrarily along the contours.

Manuscript Page No.	[Footnotes]
213*	In the first edition of our book we used the symbols λ_1 and λ_2 instead of α_1 and α_2.
213**	A brief review of papers of this kind published until 1948 and references are to be found in the collection entitled "Mekhanika v SSSR za 30 let" [30 Years Mechanics in USSR], Gostekhizdat, Moscow-Leningrad, 1950.
213***	Lekhnitskiy, S.G., Ploskaya zadacha teorii uprugosti dlya sredy so slabo vyrazhennoy anizotropiyey [The Plane Problem in the Theory of Elasticity for Media Which are Slightly Anisotropic] I and II, DAN SSSR, Vol. XXXI, No. 5 and No. 9, 1941.
214	The function F_{00} is biharmonic; we do not know a general expression for it in terms of a function of a complex variable. Knowing F_{00} we obtain for F_{10} a nonharmo-

geneous equation which is easy to integrate and, having obtained F_{10} we use nonhomogeneous equations to determine F_{20} and F_{11} and so on.

215 $$c_{00} + (\alpha_1 + \alpha_2) c_{10} + (\alpha_1^2 + \alpha_2^2) c_{20} + \alpha_1 \alpha_2 c_{11} = c_1$$

216 These methods and their applications to concrete problems are discussed in detail in the book by N.I. Muskhelishvili "Nekotoryye osnovnyye zadachi matematicheskoy teorii uprugosti" [Some Fundamental Problems of the Mathematical Theory of Elasticity], Izd. AN SSSR, 1933, 1935, 1949 and 1954.

217 Muskhelishvili, N.I., Nekotoryye osnovnyye zadachi matematicheskoy teorii uprugosti [Some Fundamental Problems of the Mathematical Theory of Elasticity], Izd. AN SSSR, Moscow, 1954, page 185.

218* Sokolov, P.A., Raspredeleniye napryazheniy v ploskom pole, oslablennom otverstiyami [Stress Distribution in a Plane Field Weakened by an Aperture], Buylleten' nauchno-tekhn. komiteta [Bulletin of the Scientifical-Technical Committee] UVMS RKKA, No. 1, IV, 1930.

218** Savin, G.N., 1) Raspredeleniye napryazheniy v ploskom pole, oslablennom kakim-lobo otverstiyem [Stress Distribution in a Plane Field Weakened by an Arbitrary Aperture] Trudy Dnepropetrovskogo inzhenerno-stroytel'nogo in-ta [Transactions of the Dnepropetrovsk Institute of Engineering and Construction] Contribution 10, 1936; 2) Kontsentratsiya napryazheniy vozle malykh otverstiy v neodnorodno napryazhennom pole [Stress Concentration Around Small Apertures in a Nonuniformly Stressed Field] Ibid. No. 20, 1937; see also: G.N. Savin, Kontsentratsiya napryazheniy okolo otverstiy [Stress Concentration Around an Aperture], Gostekhizdat, Moscow-Leningrad, 1951, Ch. II.

218*** Nayman, M.I., Napryazheniya v balke s krivolineynym otverstiyem [Stresses in a Beam with Curvilinear Aperture] Transactions of the TsAGI, No. 313, 1937.

219 These functions were obtained by N.I. Muskhelishvili's method with the help of Cauchy integrals; for details see our paper mentioned in §47, Part II, pages 844-847.

221 An approximate solution of this problem was obtained by I.S. Malyutin in his Diploma Thesis "Izgib slabo anizotropnoy plastinki s otverstiyem, blizkim k kvadratnomu" [The Bending of a Slightly Anisotropic Plate with an Almost Quadratic Aperture] (Saratov State University, Saratov, 1951).

223 Muskhelishvili, N.I., Nekotoryye osnovnyye zadachi matematicheskoy teorii uprugosti [Some Fundamental Problems of the Mathematical Theory of Elasticity] Izd. AN SSSR, Moscow, 1954, page 219.

224 Timoshenko, S.P., Teoriya uprugosti [Theory of Elasticity] ONTI 1937, page 68.

Chapter 8

APPROXIMATION METHOD FOR THE DETERMINATION OF THE STRESSES IN AN ANISOTROPIC PLATE WITH AN APERTURE

§51. GENERAL FORMULAS FOR A PLATE WITH AN APERTURE WHICH IS ALMOST ELLIPTIC

An exact solution of the problem of the stress distribution in an anisotropic plate with an aperture is only known for the case where the aperture is elliptic or circular; it has been dealt with in Chapter 6. For no other form of aperture an exact solution has been achieved so far owing to the great mathematical difficulties. When, however, the form of the aperture in an isotropic plate is such that it may be considered to be only slightly different from elliptic or circular form, it is not difficult to obtain an approximate solution to it by introducing a small parameter characterizing the deviation of the aperture from the elliptic or circular form and neglecting powers of this parameter, beginning with the, say, third or fourth power. The present chapter is devoted to this approximation method.

Let us consider an anisotropic plate with an aperture whose contour is given by the equations

$$\left.\begin{aligned} x &= a\left[\cos\vartheta + \varepsilon\sum_{n=1}^{N}(a_n\cos n\vartheta + b_n\sin n\vartheta)\right],\\ y &= a\left[c\sin\vartheta + \varepsilon\sum_{n=1}^{N}(-a_n\sin n\vartheta + b_n\cos n\vartheta)\right].\end{aligned}\right\} \tag{51.1}$$

With $\varepsilon = 0$ we obtain an ellipse with the semiaxes a and ac, and with small ε, a form which differs only slightly from an ellipse. Assume given forces X_n, Y_n (per unit area) distributed along the edge of the aperture. For simplicity we shall assume that the vector sum of these forces is equal to zero. It is required to determine the stress at an arbitrary point of the plate, first of all the stress σ_ϑ in surfaces normal to the edge of the aperture, at the aperture itself (Fig. 111).

The course of the problem's solution is the following: we map conformally an infinite plane with a cutout given by (51.1) on the ζ plane with a cutout in the form of the unit circle $|\zeta| = 1$; the mapping function has the form:

$$z = a\left[\frac{1+c}{2}\zeta + \frac{1-c}{2}\cdot\frac{1}{\zeta} + \varepsilon\psi(\zeta)\right]. \tag{51.2}$$

where

$$\psi(\zeta)=\sum_{n=1}^{N}(a_n+ib_n)\zeta^{-n}. \qquad (51.3)$$

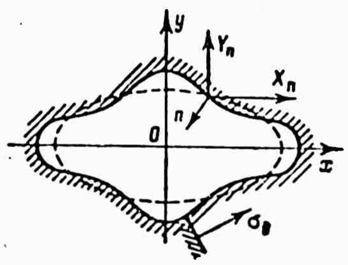

Fig. 111

In order to obtain a mapping which is in one-to-one correspondence and conformal it is necessary that all solutions to the equation

$$\frac{1+c}{2}-\frac{1-c}{2}\cdot\frac{1}{\zeta^2}+\varepsilon\psi'(\zeta)=0 \qquad (51.4)$$

are mapped in the ζ plane as points inside the unit circle $|\zeta| = 1$. The coefficients a_n, b_n and the parameter ε are always chosen so that this condition is satisfied.

In the given case it is more convenient instead of Φ_1 and Φ_2 to introduce the functions $\varphi_1(z_1')$ and $\varphi_2(z_2')$ of the variables $z_1'=z+\lambda_1\bar{z}$ and $z_2'=z+\lambda_2\bar{z}$, where λ_1 and λ_2 are complex parameters of second kind (see §7 and §8).

When the external forces are given, the boundary conditions for these functions have the form

$$\left.\begin{aligned}2\,\mathrm{Re}\,[\varphi_1(z_1')+\varphi_2(z_2')]&=\int_0^s Y_n\,ds+c_1,\\ 2\,\mathrm{Re}\,[\mu_1\varphi_1(z_1')+\mu_2\varphi_2(z_2')]&=-\int_0^s X_n\,ds+c_2,\end{aligned}\right\} \qquad (51.5)$$

where c_1, c_2 are arbitrary constants and the integral is taken along the arc of the contour for which the anticlockwise circumvention is considered to be positive.

Replacing the variable z by its expression (51.2) we represent the functions φ_1 and φ_2 in the form of series expanded in powers of the parameter ε:

$$\left.\begin{aligned}\varphi_1&=\varphi_{10}+\varepsilon\varphi_{11}+\varepsilon^2\varphi_{12}+\ldots,\\ \varphi_2&=\varphi_{20}+\varepsilon\varphi_{21}+\varepsilon^2\varphi_{22}+\ldots\end{aligned}\right\} \qquad (51.6)$$

Here φ_{1k} and φ_{2k} are functions which do not depend on ε and are expressed in the following way*

$$\left.\begin{aligned}
&\varphi_{10}=f_{10}(\zeta_1'), \qquad \varphi_{20}=f_{20}(\zeta_2'),\\
&\varphi_{11}=f_{11}(\zeta_1')+[\psi(\zeta)+\lambda_1\overline{\psi}(\overline{\zeta})]f_{10}'(\zeta_1'),\\
&\varphi_{21}=f_{21}(\zeta_2')+[\psi(\zeta)+\lambda_2\overline{\psi}(\overline{\zeta})]f_{20}'(\zeta_2'),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&\varphi_{1k}=f_{1k}(\zeta_1')+[\psi(\zeta)+\lambda_1\overline{\psi}(\overline{\zeta})]f_{1,k-1}'(\zeta_1')+\dots\\
&\qquad\dots+\frac{1}{k!}[\psi(\zeta)+\lambda_1\overline{\psi}(\overline{\zeta})]^k f_{10}^{(k)}(\zeta_1'),\\
&\varphi_{2k}=f_{2k}(\zeta_2')+[\psi(\zeta)+\lambda_2\overline{\psi}(\overline{\zeta})]f_{2,k-1}'(\zeta_2')+\dots\\
&\qquad\dots+\frac{1}{k!}[\psi(\zeta)+\lambda_2\overline{\psi}(\overline{\zeta})]^k f_{20}^{(k)}(\zeta_2'),\\
&\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}\right\} \tag{51.7}$$

where $f_{1k}(\zeta_1')$, $f_{2k}(\zeta_2')$ are unknown functions of complex variables

$$\zeta_i'=\frac{1+c}{2}\zeta+\frac{1-c}{2}\cdot\frac{1}{\zeta}+\lambda_i\left(\frac{1+c}{2}\overline{\zeta}+\frac{1-c}{2}\cdot\frac{1}{\overline{\zeta}}\right) \tag{51.8}$$

($i=1, 2$; $\overline{\zeta}$ is the variable conjugate to ζ).

Assuming the forces X_n, Y_n depending on ε, we expand then in a power series of this parameter and then obtain in the general case

$$\left.\begin{aligned}
&\int_0^s Y_n\,ds+c_1=\sum_{k=0}\varepsilon^k\left[\overline{\alpha}_{k0}+\sum_{m=1}^{\infty}(\alpha_{km}\sigma^m+\overline{\alpha}_{km}\sigma^{-m})\right],\\
&-\int_0^s X_n\,ds+c_2=\sum_{k=0}\varepsilon^k\left[\overline{\beta}_{k0}+\sum_{m=1}^{\infty}(\beta_{km}\sigma^m+\overline{\beta}_{km}\sigma^{-m})\right].
\end{aligned}\right\} \tag{51.9}$$

Here α_{km}, β_{km} are given coefficients depending on the law of distribution of the forces on the edge of the aperture, $\overline{\alpha}_{km}$, $\overline{\beta}_{km}$ are the conjugate quantities $\overline{\alpha}_{k0}$ and $\overline{\beta}_{k0}$ are arbitrary constants; for brevity of formulation we introduced the denotation $\sigma=e^{\theta i}$, which will be used in the following throughout this chapter.

Substituting the boundary values φ_1 and φ_2 in the boundary conditions (51.5) and comparing the coefficients of equal powers of ε, we obtain a series of pairs of boundary conditions for the functions f_{1k} and f_{2k} (the arguments of these functions, their derivatives, and also the conjugate quantities and the functions ψ and $\overline{\psi}$ are not given):

$$
\left.
\begin{aligned}
& f_{10}+f_{20}+\bar{f}_{10}+\bar{f}_{20}= \\
& \qquad =\bar{\alpha}_{00}+\sum_{m=1}^{\infty}(\alpha_{0m}\sigma^m+\bar{\alpha}_{0m}\sigma^{-m}), \\
& \mu_1 f_{10}+\mu_2 f_{20}+\bar{\mu}_1\bar{f}_{10}+\bar{\mu}_2\bar{f}_{20}= \\
& \qquad =\bar{\beta}_{00}+\sum_{m=1}^{\infty}(\beta_{0m}\sigma^m+\bar{\beta}_{0m}\sigma^{-m});
\end{aligned}
\right\} \tag{51.10.0}
$$

$$
\left.
\begin{aligned}
& f_{1k}+f_{2k}+(\psi+\lambda_1\bar{\psi})f'_{1,k-1}+(\psi+\lambda_2\bar{\psi})f'_{2,k-1}+\ldots \\
& \ldots+\frac{1}{k!}(\psi+\lambda_1\bar{\psi})^k f_{10}^{(k)}+\frac{1}{k!}(\psi+\lambda_2\bar{\psi})^k f_{20}^{(k)}+ \\
& +\text{conjugate quantities}= \\
& =\bar{\alpha}_{k0}+\sum_{m=1}^{\infty}(\alpha_{km}\sigma^m+\bar{\alpha}_{km}\sigma^{-m}), \\
& \mu_1 f_{1k}+\mu_2 f_{2k}+\mu_1(\psi+\lambda_1\bar{\psi})f'_{1,k-1}+\mu_2(\psi+\lambda_2\bar{\psi})f'_{2,k-1}+\ldots \\
& \ldots+\frac{\mu_1}{k!}(\psi+\lambda_1\bar{\psi})^k f_{10}^{(k)}+\frac{\mu_2}{k!}(\psi+\lambda_2\bar{\psi})^k f_{20}^{(k)}+ \\
& +\text{conjugate quantities}= \\
& =\bar{\beta}_{k0}+\sum_{m=1}^{\infty}(\beta_{km}\sigma^m+\bar{\beta}_{km}\sigma^{-m}) \\
& \qquad\qquad (k=1,\ 2,\ 3,\ \ldots).
\end{aligned}
\right\} \tag{51.10.k}
$$

When the variable ζ runs along the contour of the unit circle, the variable

$$\zeta'=\frac{1+c}{2}\zeta+\frac{1-c}{2}\cdot\frac{1}{\zeta} \tag{51.11}$$

runs along the contour of an ellipse with poles equal to 1 and c. Owing to this, the functions f_{1k}, f_{2k} of the variables $\zeta_1'=\zeta'+\lambda_1\bar{\zeta}'$ and $\zeta_2'=\zeta'+\lambda_2\bar{\zeta}'$ determine the stresses in an infinite anisotropic plate with an aperture in the form of an ellipse with the poles 1 and c, somehow loaded on the edge of the aperture. The form of these functions is well known [see §37, Eqs. (37.14)-(37.15)]:

$$
\left.
\begin{aligned}
f_{1k} &= A_{k0}+\sum_{m=1}^{\infty}A_{km}t_1^{-m}, \\
f_{2k} &= B_{k0}+\sum_{m=1}^{\infty}B_{km}t_2^{-m},
\end{aligned}
\right\} \tag{51.12}
$$

where

$$
\left.
\begin{aligned}
t_1 &= \frac{\zeta_1'+\sqrt{\zeta_1'^2-2(1+c^2)\lambda_1-(1-c^2)(1+\lambda_1^2)}}{1+c+(1-c)\lambda_1}, \\
t_2 &= \frac{\zeta_2'+\sqrt{\zeta_2'^2-2(1+c^2)\lambda_2-(1-c^2)(1+\lambda_2^2)}}{1+c+(1-c)\lambda_2}.
\end{aligned}
\right\} \tag{51.13}
$$

On the contour of the aperture

$$\left.\begin{aligned} f_{1k} &= A_{k0} + \sum_{m=1}^{\infty} A_{km}\sigma^{-m}, \\ f_{2k} &= B_{k0} + \sum_{m=1}^{\infty} B_{km}\sigma^{-m}. \end{aligned}\right\} \qquad (51.14)$$

The constants A_{k0}, B_{k0} have no influence on the stress distribution and can be arbitrary while A_{km}, B_{km} are determined from the boundary conditions. From Conditions (51.10.0) we obtain

$$A_{0m} = \frac{\bar{\beta}_{0m} - \mu_2\bar{\alpha}_{0m}}{\mu_1 - \mu_2}, \quad B_{0m} = -\frac{\bar{\beta}_{0m} - \mu_1\bar{\alpha}_{0m}}{\mu_1 - \mu_2} \qquad (51.15)$$

and so we determine the functions f_{10} and f_{20} yielding a solution to the problem "in a zero approximation."

In the next boundary condition [(51.10.k) for $k = 1$] the values of the unknown functions f_{11} and f_{21} and the derivatives of the known functions f_{10} and f_{20} are contained. From these conditions we determine the coefficients of the functions f_{11}, f_{21} which, instead of f_{10}, f_{20}, determine the solution to the problem in a first approximation. Knowing now f_{10}, f_{20}, f_{11}, f_{21} we find from Conditions (51.10.k) with $k = 2$ the coefficients of the functions f_{12} and f_{22} which, together with those obtained previously, determine the solution to our problem in a second approximation, and so on.

For an actual determination of the coefficients A_{km}, B_{km} from Conditions (51.10.k) we must expand in power series of σ all products of the form

$$\frac{1}{n!}(\psi + \lambda_1\bar{\psi})^n f_{1k}^{(n)}, \quad \frac{1}{n!}(\psi + \lambda_2\bar{\psi})^n f_{2k}^{(n)}. \qquad (51.16)$$

The coefficients of these series will determine on the coef icients A_{km}, B_{km} of the functions f_{1k} and f_{2k} themselves. In order to establish all these functions it is, obviously, necessary that the form of the function $\psi(\zeta)$ is given, i.e., we must know which terms in Eq. (51.3) are nonvanishing.

§52. FORMULAS FOR A PLATE WITH AN APERTURE OF PARTICULAR FORM

In this chapter we consider various cases of a plate with apertures whose forms are given by equations of the type

$$\left.\begin{aligned} x &= a(\cos\theta + \varepsilon\cos N\theta), \\ y &= a(c\sin\theta - \varepsilon\sin N\theta), \end{aligned}\right\} \qquad (52.1)$$

where $0 < c \leqslant 1$, and N is an integral number. With $c = 1$ and $N = 2$ the aperture has three axes of symmetry and with a due choice of the parameter ε it will differ only slightly from an equilateral triangle with rounded corners. With $c = 1$ and $N = 3$ we have four axes of symmetry and with certain values of ε the aperture resembles a square with rounded corners. With $c < 1$ and $N = 3$ an oval hole of special form.

The mapping function has the form

$$z = a\left(\frac{1+c}{2}\zeta + \frac{1-c}{2}\cdot\frac{1}{\zeta} + \frac{\epsilon}{\zeta^N}\right),$$
$$\psi(\zeta) = \zeta^{-N}. \tag{52.2}$$

Let us give general formulas according to which we can determine the coefficients of the products (51.16) with arbitrary values of n and N.*

We introduce the notation

$$\left.\begin{gathered} \omega_i = \frac{2}{1+c+\lambda_i(1-c)}, \\ \alpha_i = \frac{1-c+\lambda_i(1+c)}{1+c+\lambda_i(1-c)}, \quad \beta_i = \omega_i\lambda_i \\ (i = 1,\ 2). \end{gathered}\right\} \tag{52.3}$$

Each of the products of the type (51.16) is represented in the form of a sum of a finite number of terms with positive powers of σ and infinite series expanded with respect to negative powers of σ. The first of the products of (51.16) with arbitrary (integral) k, n and N is written as follows

$$\begin{aligned} \frac{1}{n!}(\psi + \lambda_1\bar{\psi})^n f_{1k}^{(n)} &= \frac{1}{n!}\left(\frac{1}{\sigma^N} + \lambda_1\sigma^N\right)^n f_{1k}^{(n)} = \\ &= \sum_{m=0}^{Nn-n-1} A_{k,-m}^n\sigma^m + \sum_{m=1}^{\infty} A_{km}^n\sigma^{-m}. \end{aligned} \tag{52.4}$$

All coefficients A_{km}^n are determined by means of the following general formula:

$$A_{km}^n = \sum_{i=1} A_{k,\,m+Nn-n+2-2i}\, g_{mi}^n(\lambda_1,\ \alpha_1). \tag{52.5}$$

The g_{mi}^n are here integral polynomials with respect to λ_1 and α_1

$$\left.\begin{aligned} g_{mi}^n(\lambda_1,\ \alpha_1) = {} & \frac{(-1)^n\omega_1^n}{(n-1)!\,n!}(m+Nn-n+2-2i)\,\alpha_1^{i-Nn-1}\times \\ & \times\{(m+Nn-n-i+2)(m+Nn-n-i+3)\ldots \\ & \ldots(m+Nn-i)\,i\,(i+1)\ldots(i+n-2)(\lambda_1\alpha_1^N)^n + \\ & + n\,(m+Nn-n-i+2-N)(m+Nn-n-i+3-N)\ldots \\ & \ldots(m+Nn-i-N)(i-N)(i+1-N)\ldots \\ & \ldots(i+n-2-N)(\lambda_1\alpha_1^N)^{n-1} + \binom{n}{2}\times \\ & \times(m+Nn-n-i+2-2N)(m+Nn-n-i+3-2N)\ldots \\ & \ldots(m+Nn-i-2N)(i-2N)(i+1-2N)\ldots \\ & \ldots(i+n-2-2N)(\lambda_1\alpha_1^N)^{n-2} + \ldots \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & \ldots + \binom{n}{2}(m-n-i+2+2N)(m-n-i+3+2N)\ldots \\ & \ldots(m-i+2N)(i-Nn+2N)(i-Nn+2N+1)\ldots \\ & \ldots(i-Nn+n+2N-2)(\lambda_1\alpha_1^N)^2 + n\,(m-n-i+2+N)\times \\ & \times(m-n-i+3+N)\ldots(m-i+N)(i-Nn+N)\times \\ & \times(i-Nn+N+1)\ldots(i-Nn+n+N-2)(\lambda_1\alpha_1^N) + \\ & + (m-n-i+2)(m-n-i+3)\ldots(m-i)(i-Nn)\times \\ & \times(i-Nn+1)\ldots(i-Nn+n-2)\} \end{aligned}\right\} \tag{52.6}$$

$\left[\binom{n}{2}, \binom{n}{3}\right.$ etc., are the coefficients of Newton's binomial$\left.\right]$.

The use of Formulas (52.5) and (52.6) must be done in the following way: carrying out the summation in Expressions (52.5), beginning with $i = 1$, we must eliminate all terms containing the coefficient A with negative second subscript, since the function f_{1k} on the contour is represented by a series expanded with respect to negative powers of σ such that all $A_{k,-m}=0$. Further, determining the g^n_{mi} from Formula (52.6), we must eliminate all terms with negative powers of α_1 when they appear in the summation. With this reservation we obtain from Eqs. (52.5) and (52.6) the coefficients of the product (52.4) for arbitrary integers k, n, m, among them also for negative m. The formula for the second product (51.16) has the same form, $A, \lambda_1, \alpha_1, \omega_1$ must everywhere be replaced by B, λ_2, α_2 and ω_2.

If $c = 1$ (aperture slightly differing from circular form), $\omega_i = 1$, $\alpha_i=\beta_i=\lambda_i$ $(i=1, 2)$. For this case Eq. (52.6) which in general, is very complex, can be simplified a little.

When we solve the problem in a first approximation, retaining in the expressions of φ_1 and φ_2 only the first powers of the parameter ε and neglecting the higher ones, it is reduced to two problems for an infinitely large plate with elliptic, or, in particular, round aperture. Retaining in (51.6) also the second powers of ε, we obtain finally a solution in a second approximation; for this we have to solve three problems for a plate with an elliptic aperture, and so on. Considering concrete problems, we restrict ourselves to the third approximation and in a series of cases even to the second one.

In every particular case we are first of all interested in the stress $\sigma\vartheta$ in surfaces normal to the contour of the aperture, at the edge itself. From the magnitude of this stress we may conclude as to the stress concentration near the aperture. It is determined according to the formula

$$\sigma\vartheta = \frac{2}{aC^4}\,\mathrm{Re}\left[\frac{d\varphi_1}{d\vartheta}\cdot\frac{(A+\mu_1 B)^3}{\mu_1 A - B} + \frac{d\varphi_2}{d\vartheta}\cdot\frac{(A+\mu_2 B)^3}{\mu_2 A - B}\right]. \qquad (52.7)$$

Here we have introduced new denotations.

$$\left.\begin{aligned} A &= c\cos\vartheta - \varepsilon N\cos N\vartheta,\\ B &= \sin\vartheta + \varepsilon N\sin N\vartheta,\\ C^2 &= A^2 + B^2. \end{aligned}\right\} \qquad (52.8)$$

φ_1 and φ_2 in Eq. (52.7) denote the values of the functions of complex variables on the contour of the aperture, determined in the one or other degree of accuracy.

In the following we shall consider plates with three types of apertures: similar to an equilateral triangle, an oval form of special contour, and similar to a square with rounded corners. We give the boundary values of the functions φ_{1k} for a nonortho-

tropic plate with an aperture, on the edge of which the forces X_n, Y_n are distributed such that their vector sum is vanishing. Extension and bending by moments of orthotropic plates is investigated in greater detail.

For such plates we give the formulas for the stress σ on the contour of the aperture and at individual points of it.

In the case of equal complex parameters the limiting transition must be carried out in the stress formulas, assuming $\mu_2 = \mu_1$.

Considering extension and bending of a plate with an aperture, we also give the results of calculations for a veneer plate for which, as already mentioned, $\mu_1 = 4{,}11i$, $\mu_2 = 0{,}343i$, when the x-axis is directed along the fivers of the sheet and $\mu_1 = 0{,}243i$, $\mu_2 = 2{,}91i$, when the axis is perpendicular to the fibers.

At the end of this section we give a compilation of the denotations used when studying the extension and bending of an orthotropic plate with an aperture:

$$k = -\mu_1\mu_2 = \sqrt{\frac{E_1}{E_2}}, \quad n = -i(\mu_1 + \mu_2); \tag{52.9}$$

$$\left.\begin{aligned} L &= (B^2 - \mu_1^2A^2)(B^2 - \mu_2^2A^2), \\ D^4 &= -A^4k + A^2B^2(1 - 2k - k^2) + B^4(2 + k - n^2), \\ E^4 &= A^4(2k^2 - n^2 + k) + A^2B^2(k^2 - 2k - 1) - B^4k; \end{aligned}\right\} \tag{52.10}$$

$$\left.\begin{aligned} g &= \frac{8(1-k)}{(1+k+n)^3}, \quad h = \frac{2}{(1+k+n)^2}[(1-n)^2 + k(k+2n-6)], \\ d &= \frac{4}{1+k+n}, \quad l = \frac{2}{1+k+n}(1-k-n), \\ r &= \frac{8}{(1+k+n)^3}[10k - 3(1+k^2) + n(1+k)], \\ s &= \frac{2}{(1+k+n)^3}[k^3 + 3k^2(n-11) + k(3n^2 - 22n + 27) + \\ &+ (n-1)^2(n-3)]; \end{aligned}\right\} \tag{52.11}$$

$$\left.\begin{aligned} a_{11} &= \frac{\beta_1\mu_1 - \beta_2\mu_2}{\mu_1 - \mu_2}, \quad b_{11} = i\frac{\beta_1 - \beta_2}{\mu_1 - \mu_2}, \quad c_{11} = \frac{\beta_1\mu_2 - \beta_2\mu_1}{\mu_1 - \mu_2}; \\ a_{21} &= a_{11}^2 + kb_{11}^2 + 2\frac{\alpha_1\beta_1^2\mu_1 - \alpha_2\beta_2^2\mu_2}{\mu_1 - \mu_2}, \\ b_{21} &= nb_{11}^2 + 2i\frac{\alpha_1\beta_1^2 - \alpha_2\beta_2^2}{\mu_1 - \mu_2}, \\ c_{21} &= c_{11}^2 + kb_{11}^2 + 2\frac{\alpha_1\beta_1^2\mu_2 - \alpha_2\beta_2^2\mu_1}{\mu_1 - \mu_2}; \\ a_{23} &= \frac{\beta_1^2\mu_1 - \beta_2^2\mu_2}{\mu_1 - \mu_2}, \quad b_{23} = i\frac{\beta_1^2 - \beta_2^2}{\mu_1 - \mu_2}, \quad c_{23} = \frac{\beta_1^2\mu_2 - \beta_2^2\mu_1}{\mu_1 - \mu_2}. \end{aligned}\right\} \tag{52.12}$$

For an isotropic plate

$$
\left.
\begin{aligned}
&\mu_1 = \mu_2 = i, \quad \lambda_1 = \lambda_2 = 0, \quad k = 1, \quad n = 2, \\
&g = h = 0, \quad d = r = 1, \quad l = s = -1, \\
&a_{11} = b_{11} = c_{11} = \frac{1}{1+c}, \\
&a_{21} = b_{21} = c_{21} = \frac{2}{(1+c)^3}, \\
&a_{23} = b_{23} = c_{23} = 0, \\
&L = C^4, \quad D^4 = E^4 = C^4.
\end{aligned}
\right\} \qquad (52.13)
$$

§53. DETERMINATION OF THE STRESSES IN A PLATE WITH TRIANGULAR APERTURE

Let us consider an anisotropic plate with an aperture whose contour is given by the equations

$$
\left.
\begin{aligned}
x &= a(\cos\theta + \varepsilon\cos 2\theta), \\
y &= a(\sin\theta - \varepsilon\sin 2\theta) \\
&(c = 1, \quad N = 2).
\end{aligned}
\right\} \qquad (53.1)
$$

With an appropriate choice of the parameter ε (whose absolute value must be smaller than 0.5) this aperture will differ only slightly from an equilateral triangle with rounded corners. With ε = 0.25 the curvature of the middle sections of the sides is equal to zero. For simplicity, let us agree upon an aperture of the form (53.1) which in the following will be called "triangular."

Some problems on the stress distribution in an isotropic plate with triangular aperture were considered in papers by G.N. Savin* and M.I. Nayman.**

Let us assume the forces X_n, Y_n distributed on the edge of an aperture, the vector sum of them being equal to zero; in other respects, the distribution may be arbitrary.

In order to obtain a solution to this problem in a third approximation, in the expressions of the functions φ_1 and φ_2 the third powers of ε must be retained and the higher ones are omitted. We have

$$
\left.
\begin{aligned}
\varphi_1 &= \varphi_{10} + \varepsilon\varphi_{11} + \varepsilon^2\varphi_{12} + \varepsilon^3\varphi_{13}, \\
\varphi_2 &= \varphi_{20} + \varepsilon\varphi_{21} + \varepsilon^2\varphi_{22} + \varepsilon^3\varphi_{23}.
\end{aligned}
\right\} \qquad (53.2)
$$

Since $c = 1$ the form of the functions f_{1k} and f_{2k} [Eqs. (51.12) and (51.13)] is simplified a little; in the case given

$$
\left.\begin{aligned}
f_{1k}(\zeta_1') &= A_{k0} + \sum_{m=1}^{\infty} A_{km}\left(\frac{2}{\zeta_1' + \sqrt{\zeta_1'^2 - 4\lambda_1}}\right)^m, \\
f_{2k}(\zeta_2') &= B_{k0} + \sum_{m=1}^{\infty} B_{km}\left(\frac{2}{\zeta_2' + \sqrt{\zeta_2'^2 - 4\lambda_2}}\right)^m, \\
\zeta_1' &= \zeta + \lambda_1\bar{\zeta}, \quad \zeta_2' = \zeta + \lambda_2\bar{\zeta}.
\end{aligned}\right\} \qquad (53.3)
$$

On the contour of the aperture Conditions (51.10.0) and (50.10.k) must be satisfied, where k = 1, 2, 3.

The final expressions for the functions φ_{1k} on the contour of the aperture have the form;*

$$
\left.\begin{aligned}
\varphi_{10} &= A_0 + \sum_{m=1}^{\infty} \frac{\bar{\beta}_{0m} - \mu_2\bar{\alpha}_{0m}}{\mu_1 - \mu_2}\sigma^{-m}, \\
\varphi_{11} &= A_1 + \sum_{m=1}^{\infty} \frac{\bar{\beta}_{1m} - \mu_2\bar{\alpha}_{1m}}{\mu_1 - \mu_2}\sigma^{-m}, \\
\varphi_{12} &= A_2 + \sum_{m=1}^{\infty} \frac{\bar{\beta}_{2m} - \mu_2\bar{\alpha}_{2m}}{\mu_1 - \mu_2}\sigma^{-m} + \\
&\quad + A_{01}\lambda_1^2\sigma + \frac{\bar{A}_{01}\bar{\lambda}_1^2(\mu_2 - \bar{\mu}_1) + \bar{B}_{01}\bar{\lambda}_2^2(\mu_2 - \bar{\mu}_2)}{\mu_1 - \mu_2}\cdot\frac{1}{\sigma}, \\
\varphi_{13} &= A_3 + \sum_{m=1}^{\infty} \frac{\bar{\beta}_{3m} - \mu_2\bar{\alpha}_{3m}}{\mu_1 - \mu_2}\sigma^{-m} + \\
&\quad + (A_{11}\lambda_1^2 - 4A_{02}\lambda_1^3)\sigma - A_{01}\lambda_1^3\sigma^2 + \\
&\quad + \frac{(\bar{A}_{11}\bar{\lambda}_1^2 - 4\bar{A}_{02}\bar{\lambda}_1^3)(\mu_2 - \bar{\mu}_1) + (\bar{B}_{11}\bar{\lambda}_2^2 - 4\bar{B}_{02}\bar{\lambda}_2^3)(\mu_2 - \bar{\mu}_2)}{\mu_1 - \mu_2}\cdot\frac{1}{\sigma} - \\
&\quad - \frac{\bar{A}_{01}\bar{\lambda}_1^3(\mu_2 - \bar{\mu}_1) + \bar{B}_{01}\bar{\lambda}_2^3(\mu_2 - \bar{\mu}_2)}{\mu_1 - \mu_2}\cdot\frac{1}{\sigma^2}.
\end{aligned}\right\} \qquad (53.4)
$$

Here $\sigma = e^{i\theta}$; A_0, A_1, A_2 and A_3 are arbitrary constants, $\bar{\alpha}_{km}$, $\bar{\beta}_{km}$ are coefficients entering the integrals of (51.9) and which depend on the distribution of the external forces.

Expressions for the functions φ_{2k} will not be given, neither here nor in the following sections of this chapter; they are obtained from the expressions for φ_{1k} if everywhere in the latter $A, B, \mu_1, \mu_2, \lambda_1, \lambda_2$ is replaced by $B, A, \mu_2, \mu_1, \lambda_2, \lambda_1$, respectively (or, briefly, when the letters B and A and the subscripts of μ and λ are exchanged). The constants A_{11} and B_{11} are determined from the formulas

$$
\left.\begin{aligned}
A_{11} &= \frac{\bar{\beta}_{11} - \mu_2\bar{\alpha}_{11}}{\mu_1 - \mu_2} + 2A_{02}\lambda_1, \\
B_{11} &= -\frac{\bar{\beta}_{11} - \mu_1\bar{\alpha}_{11}}{\mu_1 - \mu_2} + 2B_{02}\lambda_2
\end{aligned}\right\} \qquad (53.5)
$$

(A_{02}, B_{02} are the coefficients at σ^{-2} in the functions φ_{10} and φ_{20}).

In order to find the stresses σ_ϑ at the edge of the aperture, we must use Formula (52.7) substituting in it

$$\left.\begin{aligned} A &= \cos\vartheta - 2\varepsilon\cos 2\vartheta, \\ B &= \sin\vartheta + 2\varepsilon\sin 2\vartheta, \\ C^2 &= 1 + 4\varepsilon^2 - 4\varepsilon\cos 3\vartheta. \end{aligned}\right\} \qquad (53.6)$$

§54. EXTENSION OF AN ORTHOTROPIC PLATE WITH TRIANGULAR APERTURE*

An orthotropic rectangular plate with small triangular aperture in the center is extended by normal forces p which are distributed uniformly along two sides. The principal directions of elasticity are assumed parallel to the sides of the plate and the aperture has been cut out such that the direction of one of the three sides is parallel to a principal direction.

Just as in the case of an elliptic aperture the stress distribution in this plate is obtained by superposing the stresses in a massive plate, without aperture, which is extended by the forces p, and the stresses in an infinitely large plate with an aperture loaded by forces arranged on the edge of the aperture. The additional stresses are determined by the functions $\varphi_1(z_1')$ and $\varphi_2(z_2')$, which are chosen in such a way that the conditions $X_n = 0$, $Y_n = 0$ are satisfied at the contour of the aperture. This application of the solution to an infinite plate will be justified if the dimensions of the aperture are small compared to the dimensions of the plate sides and if it is located in the center of it. For each of these cases we give the values of the coefficients $\bar{\beta}_{km}$, $\bar{\alpha}_{km}$, A_{0m}, B_{0m} in the formula for the stress σ_ϑ on the contour of the aperture and at individual points of it obtained in a third approximation.

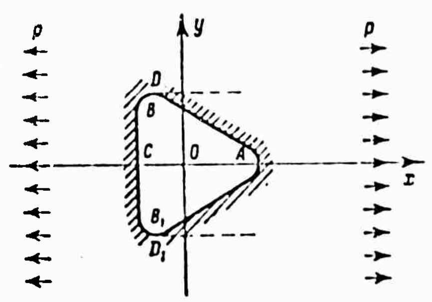

Fig. 112

Case I. The forces are perpendicular to one of the sides of the aperture (Fig. 112).

$$\bar{\beta}_{01} = -\frac{pal}{2}, \quad \bar{\beta}_{12} = \frac{pal}{2}; \qquad (54.1)$$

the other $\bar{\beta}_{km}$ and $\bar{\alpha}_{km}$ are vanishing.

$$\left.\begin{aligned} A_{01} = -\frac{pal}{2(\mu_1 - \mu_2)}, \quad B_{01} = \frac{pal}{2(\mu_1 - \mu_2)}, \\ A_{0m} = B_{0m} = 0 \;\; (m \geqslant 2); \end{aligned}\right\} \qquad (54.2)$$

$$\begin{aligned} \sigma_\vartheta = p\frac{B^2}{C^2} + \frac{p}{LC^2}[AD^4\cos\vartheta + BC^4 n\sin\vartheta - \\ - 2\varepsilon(AD^4\cos 2\vartheta + BC^4 n\sin 2\vartheta) + \varepsilon^2 C^4 n(Agk\cos\vartheta + Bh\sin\vartheta) - \\ - \varepsilon^3 AC^4 kn(dh + gl + dgn)\cos 2\vartheta + \varepsilon^3 BC^4 n(dgk - hl)\sin 2\vartheta]. \end{aligned} \qquad (54.3)$$

Remember that C^2, D^4 and L are determined according to Eqs. (52.8) and (52.10) where we must put

$$\left.\begin{array}{l} A=\cos\vartheta-2\varepsilon\cos 2\vartheta, \\ B=\sin\vartheta+2\varepsilon\sin 2\vartheta. \end{array}\right\} \tag{54.4}$$

At point A (ϑ = 0, Fig. 112)

$$(\sigma_\vartheta)_A=\frac{p}{k}\left[-1+\frac{\varepsilon^2}{1-2\varepsilon}gn-\frac{\varepsilon^3}{1-2\varepsilon}n(dh+gl+dgn)\right]. \tag{54.5}$$

At point C ($\vartheta=\pi$)

$$(\sigma_\vartheta)_C=\frac{p}{k}\left[-1+\frac{\varepsilon^2}{1+2\varepsilon}gn+\frac{\varepsilon^3}{1+2\varepsilon}n(dh+gl+dgn)\right]. \tag{54.6}$$

The investigations show that the highest value of σ_ϑ is at the points D and D_1 where the tangent to the contour is parallel to the extending forces (Fig. 112). The position of these points is determined by the values $\vartheta=\pm\vartheta_0$, where ϑ_0 is the smallest solution of the equation

$$\cos\vartheta_0-2\varepsilon\cos 2\vartheta_0=0. \tag{54.7}$$

In particular, for ε = 0.25 we obtain $\vartheta_0=111°30'$. At the points D and D_1 the stress is determined by the formula

$$(\sigma_\vartheta)_D=p\left\{1+\frac{n}{B}[\sin\vartheta_0-2\varepsilon\sin 2\vartheta_0+\varepsilon^2 h\sin\vartheta_0+\varepsilon^3(dgk-hl)\sin 2\vartheta_0]\right\}. \tag{54.8}$$

For an isotropic plate $\mu_1=\mu_2=i$, $g=h=0$, $d=1$, $l=-1$

$$\sigma_\vartheta=\frac{p}{C^2}(1-2\cos 2\vartheta+4\varepsilon\cos\vartheta-4\varepsilon^2) \tag{54.9}$$

and Eq. (54.3) obtained in a third approximation proves to be identical with the exact formula for an isotropic plate.*

Case 2. The forces are parallel to one side of the aperture (Fig. 113).

$$\bar{\alpha}_{01}=\bar{\alpha}_{12}=-\frac{pa}{2}; \tag{54.10}$$

the other $\bar{\alpha}_{km}$ and all $\bar{\beta}_{km}$ are vanishing;

$$\left.\begin{array}{l} A_{01}=\dfrac{pa\mu_2}{2(\mu_1-\mu_2)}, \quad B_{01}=-\dfrac{pa\mu_1}{2(\mu_1-\mu_2)}, \\ A_{0m}=B_{0m}=0 \quad (m\geqslant 2); \end{array}\right\} \tag{54.11}$$

$$\begin{aligned} \sigma_\vartheta=p\frac{A^2}{C^2}+\frac{p}{LC^2}\{&AC^4kn\cos\vartheta+BE^4\sin\vartheta+ \\ &+2\varepsilon(AC^4kn\cos 2\vartheta+BE^4\sin 2\vartheta)+ \\ &+\varepsilon^2C^4kn[-A(gn+h)\cos\vartheta+Bg\sin\vartheta]+ \\ &+\varepsilon^3AC^4kn[dg(n^2-k)+(dh+gl)n+hl]\cos 2\vartheta- \\ &-\varepsilon^3BC^4kn(dh+gl+dgn)\sin 2\vartheta\}. \end{aligned} \tag{54.12}$$

The stress will be maximum at point A (Fig. 113):

$$(\sigma_\theta)_A = p + \frac{p}{1-2\varepsilon}\cdot\frac{n}{k}[1 + 2\varepsilon - \varepsilon^2(gn + h) - \\ - \varepsilon^3 dgk + \varepsilon^3(gn + h)(dn + l)]. \tag{54.13}$$

By means of a limiting transition from the approximate equation (54.12) we obtain the stress formula for the isotropic plate extended as shown in Fig. 113:

$$\sigma_\vartheta = \frac{p}{C^2}(1 + 2\cos 2\vartheta - 4\varepsilon\cos\vartheta - 4\varepsilon^2). \tag{54.14}$$

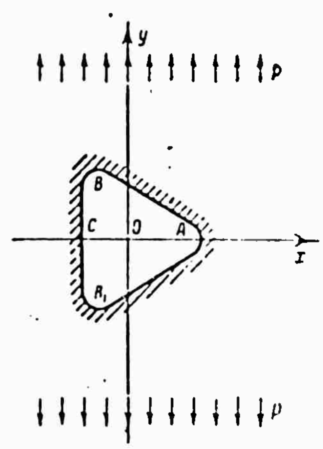

Fig. 113

This formula is also identical with the exact one.

Let us give some results of calculations for a veneer plate with an aperture characterized by the parameter $\varepsilon = 0.25$.

In Table 6 the numerical values of the stress σ_ϑ (in fractions of p) are compiled for the most important points of the contour as obtained in first, second and third approximations. In the case of tension in the direction of which is perpendicular to the side of the aperture (Case 1) the numerical results obtained are different according to whether the direction of x is directed parallel to the fibers of the sheet or perpendicular to the fibers. The same holds true for the extension in the principal direction of y (Case 2). For each of the cases 1 and 2 the values given refer to both extension along the fibers of the sheet ($E_x = E_{max}$ and $E_y = E_{min}$) and across the fibers ($E_x = E_{min}$ and $E_y = E_{min}$). Everywhere two significant ciphers have been retained after the comma.

The results of a calculation of the stresses in a fourth approximation (which are not given here) differ only very little from the results of the third approximation; in any case, in the fourth approximation the first cipher after the comma remains one and the same as in the third approximation. It is therefore unnecessary to calculate higher approximations, it is sufficient to use

TABLE 6

The Stresses σ_ϑ at Points on the Contour of a Triangular Aperture

	1 Случай 1						2 Случай 2			
	$E_x = E_{max}$			$E_x = E_{min}$			$E_y = E_{max}$		$E_y = E_{min}$	
3 Точки / Прибл. 4	A	D	C	A	D	C	A	C	A	C
1-е	−0,71	10,61	−0,71	−1,41	7,80	−1,11	14,38	2,49	10,45	2,05
2-е	−0,74	10,94	−0,72	−1,36	7,93	−1,40	13,95	2,31	10,27	1,99
3-е	−0,77	10,86	−0,70	−1,43	7,90	−1,37	14,09	2,30	10,33	1,97

1) Case 1; 2) case 2; 3) points; 4) approximation.

the third or even the second approximation in order to estimate the stress concentration.

For an isotropic plate with an aperture at which $\varepsilon = 0.25$ we obtain the following results.

Case 1. At the points A and C

$$(\sigma_\theta)_A = -p; \tag{54.15}$$

at point D

$$(\sigma_\theta)_D = 5{,}32p. \tag{54.16}$$

Case 2. At point A

$$(\sigma_\theta)_A = 7p; \tag{54.17}$$

at point C

$$(\sigma_\theta)_C = 1{,}67p. \tag{54.18}$$

From the table given we see that the stress concentration in both cases (1 and 2) is higher if the plate is extended in the direction of the fibers of the sheet (in the direction of higher E). With an extension in a direction perpendicular to the fibers of the sheet (i.e., in the direction of smaller E), the stress σ_ϑ is distributed over the contour more uniformly and the difference between the maximum and minimum stresses on the contour is considerably smaller than in the case of extension in the direction of large E.

Comparing the results obtained for a veneer plate and an isotropic plate we see that in all cases the maximum stresses in a veneer plate are considerably higher than in an isotropic plate of the same type.

All these conclusions are valid also for a series of other anisotropic materials applied in practice (for example, for various forms of wood).

The solution of the problem on the extension of an anisotropic plate with a triangular aperture by means of the method described was obtained for the first time by V.P. Krasyukov.* Another approximation method of determining the stresses in an extended orthotropic plate with triangular aperture has been suggested in a paper by K. Stephens.**

§55. PURE BENDING OF AN ORTHOTROPIC PLATE WITH TRIANGULAR APERTURE***

An orthotropic rectangular plate with a triangular aperture in the center is deformed by forces distributed on two sides, which can be attributed to the action of the moments M in the mid-plane. It is suggested that the directions of the sides of the plate and one of the sides of the aperture are parallel to the principal direction of elasticity of the material.

Approaching the problem in the same way as the problem of the extension considered in &54 we distinguish also here between two cases.

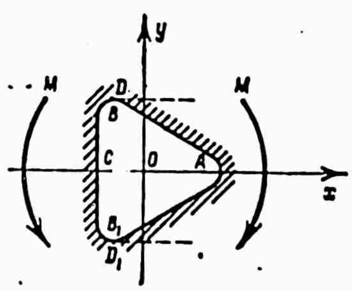

Fig. 114

Case I. The sides of the plate parallel to one of the sides of the aperture are loaded (Fig. 114).

$$\left.\begin{array}{ll} \bar{\beta}_{02}=\frac{Ma^2}{8J}, & \bar{\beta}_{11}=\frac{Ma^2}{4J}, \\ \bar{\beta}_{13}=-\frac{Ma^2}{4J}, & \bar{\beta}_{24}=\frac{Ma^2}{8J} \end{array}\right\} \qquad (55.1)$$

(J — moment of inertia of a plate cross section perpendicular to the x-axis, the other $\bar{\beta}_{km}$ and all $\bar{\alpha}_{km}$ are vanishing;

$$\left.\begin{array}{c} A_{01}=B_{01}=0; \\ A_{02}=\frac{Ma^2}{8J(\mu_1-\mu_2)}, \quad B_{02}=-\frac{Ma^2}{8J(\mu_1-\mu_2)}, \\ A_{0m}=B_{0m}=0 \quad (m\geqslant 3); \end{array}\right\} \qquad (55.2)$$

$$\sigma_\theta = \frac{Ma}{J}(\sin\theta - \varepsilon\sin 2\theta)\frac{B^2}{C^2} + \\ + \frac{Ma}{2JLC^2}\{-BC^4 n\cos 2\theta + AD^4\sin 2\theta + \\ + \varepsilon[-BC^4 n(\cos\theta - 3\cos 3\theta) + AD^4(\sin\theta - 3\sin 3\theta)] + \\ + 2\varepsilon^2[-BC^4 n\cos 4\theta + AD^4\sin 4\theta] + \\ + \varepsilon^3 BC^4 n[h + 0{,}5(dgk - lh)]\cos\theta - \varepsilon^3 AC^4 gkn\sin\theta + \\ + 0{,}5\varepsilon^3 AC^4 kn(dh + gl + dgn)\sin\theta\}, \qquad (55.3)$$

where A and B are determined according to Eqs. (54.4)

At the points A and C (Fig. 114) $\sigma_\theta = 0$. The stress reaches its highest absolute value at the points D and D_1 where the tangent to the aperture's contour is parallel to the x-axis (i.e., the non-loaded side of the plate). At point D

$$(\sigma_\theta)_D = \frac{Ma}{J}(\sin\vartheta_0 - \varepsilon\sin 2\vartheta_0) + \frac{Ma}{2J}\cdot\frac{n}{B}\{-\cos 2\vartheta_0 + \\ + \varepsilon(3\cos 3\vartheta_0 - \cos\vartheta_0) - 2\varepsilon^2\cos 4\vartheta_0 + \\ + \varepsilon^3[h + 0{,}5(dgk - hl)]\cos\vartheta_0\} \qquad (55.4)$$

[the angle ϑ_0 is determined from Eq. (54.7)].

Carrying out the limiting transition, we obtain from (55.3) a formula for an isotropic plate which coincides with the exact expression:*

$$\sigma_\theta = \frac{Ma}{JC^2}(\sin\theta - \sin 3\theta + \varepsilon\sin 4\theta - 6\varepsilon^2\sin\theta). \qquad (55.5)$$

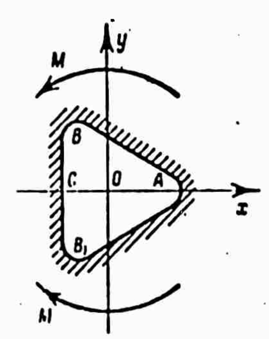

Fig. 115

Case 2. The sides perpendicular to one of the sides of the aperture are loaded (Fig. 115).

$$\bar{\alpha}_{02} = -\frac{Ma^2}{8J}, \quad \bar{\alpha}_{11} = \bar{\alpha}_{13} = -\frac{Ma^2}{4J}, \\ \bar{\alpha}_{24} = -\frac{Ma^2}{8J} \qquad (55.6)$$

(J is the moment of inertia of a plate cross section perpendicular to the x-axis); $\bar{\alpha}_{km}$ with other subscripts, and all β_{km} are

equal to zero;

$$\left.\begin{array}{c} A_{01} = B_{01} = 0, \\ A_{02} = \frac{Ma^2\mu_2}{8J(\mu_1 - \mu_2)}, \quad B_{02} = -\frac{Ma^2\mu_1}{8J(\mu_1 - \mu_2)}, \\ A_{0m} = B_{0m} = 0 \quad (m \geq 3); \end{array}\right\} \tag{55.7}$$

$$\begin{aligned} \sigma_\vartheta = {} & \frac{Ma}{J}(\cos\vartheta + \varepsilon\cos 2\vartheta)\frac{A^2}{C^2} + \\ & + \frac{Ma}{2JLC^4}\{AC^4kn\cos 2\vartheta + BE^4\sin 2\vartheta + \\ & + \varepsilon[AC^4kn(\cos\vartheta + 3\cos 3\vartheta) + BE^4(\sin\vartheta + 3\sin 3\vartheta)] + \\ & + 2\varepsilon^2[AC^4kn\cos 4\vartheta + BE^4\sin 4\vartheta] + \\ & + 0{,}5\varepsilon^3AC^4kn[h(l-2) + dg(n^2-k) + (dh + gl - 2g)n]\cos\vartheta + \\ & + \varepsilon^3BC^4gkn\sin\vartheta - 0{,}5\varepsilon^3BC^4kn(dh + gl + dgn)\sin\vartheta\}. \end{aligned} \tag{55.8}$$

At point A (Fig. 115)

$$\begin{aligned} (\sigma_\vartheta)_A = {} & \frac{Ma}{J}(1+\varepsilon) + \frac{Ma}{J(1-2\varepsilon)}\cdot\frac{n}{k}\{0{,}5 + 2\varepsilon + \varepsilon^2 + \\ & + 0{,}25\varepsilon^3[h(l-2) + dg(n^2-k) + (dh + gl - 2g)n]\}. \end{aligned} \tag{55.9}$$

From (55.8) we obtain the formula for the isotropic plate; it also coincides with the accurate expression:*

$$\sigma_\vartheta = \frac{Ma}{JC^2}[\cos\vartheta + \cos 3\vartheta + \varepsilon(\cos 4\vartheta - 2) - 6\varepsilon^2\cos\vartheta]. \tag{55.10}$$

In Table 7 the values of the stresses σ_ϑ (in fractions of Ma/J) are given for a veneer plate calculated in first, second and third approximations (for Case 1 at point D and in Case 2 at points A and C). The parameter ε is taken to be equal to 0.25.

In each of the two cases numerical values are given for a plate in which the fibers of the sheet are perpendicular to the external sides ($(E_x = E_{max}$ and $E_y = E_{max})$, and for a plate where the fibers are parallel to the external sides $(E_x = E_{min}$ and $E_y = E_{min})$.

TABLE 7

The Stresses σ_θ at Points on the Contour of a Triangular Aperture

Прибл. 4 \ 3 Точки	1 Случай 1		2 Случай 2			
	$E_x = E_{max}$	$E_x = E_{min}$	$E_y = E_{max}$		$E_y = E_{min}$	
	)	D	A	C	A	C
1-е	6,77	5,11	10,16	−0,75	7,56	−0,75
2-е	6,73	5,09	10,71	−0,91	7,95	−0,88
3-е	6,71	5,08	10,70	−0,91	7,94	−0,88

1) Case 1; 2) case 2; 3) points; 4) approximation.

From the table we see that the third approximation, within the limits of accuracy accepted (two significant ciphers behind the comma) differs very little from the second or is even the same. Therefore already the second approximation renders it possible to estimate the magnitude of the coefficient of concentration with an accuracy sufficient for practice.

For an isotropic plate with the same triangular opening we obtain:

Case 1. At point D

$$(\sigma_\theta)_D = 3{,}63 \frac{Ma}{J}. \tag{55.11}$$

Case 2. At point A

$$(\sigma_\theta)_A = 5{,}50 \frac{Ma}{J}; \tag{55.12}$$

at point C

$$(\sigma_\theta)_C = -0{,}83 \frac{Ma}{J}. \tag{55.13}$$

Table 7 shows that the coefficient of stress concentration in a veneer plate in all cases reaches much higher values than the coefficient of concentration in an isotropic plate of the same form. Of the two basic cases of orientation of the principal directions with respect to the loaded sides of the plate the first one is less favorable as the forces are there acting on the sides perpendicular to the fibers of the sheet and in this case the stress concentration around the hole is higher.

§56. DETERMINATION OF THE STRESSES IN A PLATE WITH AN OVAL APERTURE

Let us consider an infinite anisotropic plate with an aperture whose contour equation reads

$$\left.\begin{aligned} x &= a(\cos\theta + \varepsilon\cos 3\theta),\\ y &= a(c\sin\theta - \varepsilon\sin 3\theta). \end{aligned}\right\} \tag{56.1}$$

When c is positive and smaller than unity the aperture has a lengthy form and the x and y-axes are the axes of symmetry. With c = 0.36 and ε = -0.04 we obtain a figure which differs only slightly from a rectangle whose short sides have been replaced by half circles; with such a form the length-to-width ratio is equal to 3 and the curvature in the middle of the long sides is equal to zero. With c = 0.537 and ε = -0.038 the hole has an oval shape; the length-to-width ratio is equal to 1.93. An aperture of the type (56.1) with c < 1 will be called oval in the following. The problem on the extension of an isotropic plate with such an aperture (and, in particular, with an aperture for which c = 0.537 and ε = -0.038) has been considered by Greenspan* and the problem on bending caused by moments by Joseph and Brock.** Solutions to a series of other problems for an isotropic plate with oval aperture were obtained by Ye.F. Burmistrov*** (using the method by

N.I. Muskhelishvili).

Assuming in Eq. (56.1) $c = 1$, we arrive, as already indicated, at a figure with four axes of symmetry which, with certain values of ε will differ only little from a square with rounded corners.

Assume given forces X_n, Y_n attacking at the edge of the oval aperture (56.1), whose vector sum is equal to zero. In this case the functions $\varphi_{1k}(z_1')$ and $\varphi_{2k}(z_2')$ have the form (51.7) and $f_1(\zeta_1')$ and $f_2(\zeta_2')$ are represented by the series (51.12)-(51.13)

Solving the problem in a second approximation, we omit in the expressions for φ_1 and φ_2 powers of ε higher than the second, i.e., we have

$$\left.\begin{aligned} \varphi_1 &= \varphi_{10} + \varepsilon\varphi_{11} + \varepsilon^2\varphi_{12}, \\ \varphi_2 &= \varphi_{20} + \varepsilon\varphi_{21} + \varepsilon^2\varphi_{22}. \end{aligned}\right\} \tag{56.2}$$

In the right-hand sides of the boundary conditions (51.5) we also neglect all powers of ε higher than the second.

After some transformations we obtain the following expressions for the boundary conditions of the functions φ_{1k}:

$$\begin{aligned}
\varphi_{10} &= A_0 + \sum_{m=1}^{\infty} \frac{\bar{\beta}_{0m} - \mu_2\bar{\alpha}_{0m}}{\mu_1 - \mu_2}\,\sigma^{-m}, \\
\varphi_{11} &= A_1 + \sum_{m=1}^{\infty} \frac{\bar{\beta}_{1m} - \mu_2\bar{\alpha}_{1m}}{\mu_1 - \mu_2}\,\sigma^{-m} - A_{01}\beta_1\sigma - \\
&\qquad - \frac{\bar{A}_{01}\bar{\beta}_1(\mu_2 - \bar{\mu}_1) + \bar{B}_{01}\bar{\beta}_2(\mu_2 - \bar{\mu}_2)}{\mu_1 - \mu_2}\cdot\frac{1}{\sigma}, \\
\varphi_{12} &= A_2 + \sum_{m=1}^{\infty} \frac{\bar{\beta}_{2m} - \mu_2\bar{\alpha}_{2m}}{\mu_1 - \mu_2}\,\sigma^{-m} + \\
&\quad + [(3A_{01}\alpha_1 + 6A_{03})\beta_1^2 - A_{11}\beta_1]\,\sigma + 3A_{02}\beta_1^2\sigma^2 + A_{01}\beta_1^2\sigma^3 + \\
&\quad + \frac{[(3\bar{A}_{01}\bar{\alpha}_1 + 6\bar{A}_{03})\bar{\beta}_1^2 - \bar{A}_{11}\bar{\beta}_1](\mu_2 - \bar{\mu}_1) + [(3\bar{B}_{01}\bar{\alpha}_2 + 6\bar{B}_{03})\bar{\beta}_2^2 - \bar{B}_{11}\bar{\beta}_2](\mu_2 - \bar{\mu}_2)}{\mu_1 - \mu_2} \times \\
&\quad \times \frac{1}{\sigma} + 3\,\frac{\bar{A}_{02}\bar{\beta}_1^2(\mu_2 - \bar{\mu}_1) + \bar{B}_{02}\bar{\beta}_2^2(\mu_2 - \bar{\mu}_2)}{\mu_1 - \mu_2}\cdot\frac{1}{\sigma^2} + \\
&\qquad + \frac{\bar{A}_{01}\bar{\beta}_1^2(\mu_2 - \bar{\mu}_1) + \bar{B}_{01}\bar{\beta}_2^2(\mu_2 - \bar{\mu}_2)}{\mu_1 - \mu_2}\cdot\frac{1}{\sigma^3}.
\end{aligned} \tag{56.3}$$

Here A_0, A_1, A_2 are arbitrary constants,

$$\left.\begin{aligned}
A_{11} &= \frac{\bar{\beta}_{11} - \mu_2\bar{\alpha}_{11}}{\mu_1 - \mu_2} - \frac{\bar{A}_{01}\bar{\beta}_1(\mu_2 - \bar{\mu}_1) + \bar{B}_{01}\bar{\beta}_2(\mu_2 - \bar{\mu}_2)}{\mu_1 - \mu_2} + \\
&\qquad + (A_{01}\alpha_1 + 3A_{03})\beta_1, \\
B_{11} &= -\frac{\bar{\beta}_{11} - \mu_1\bar{\alpha}_{11}}{\mu_1 - \mu_2} - \frac{\bar{A}_{01}\bar{\beta}_1(\mu_1 - \bar{\mu}_1) + \bar{B}_{01}\bar{\beta}_2(\mu_1 - \bar{\mu}_2)}{\mu_1 - \mu_2} + \\
&\qquad + (B_{01}\alpha_2 + 3B_{03})\beta_2;
\end{aligned}\right\} \tag{56.4}$$

A_{0m} and B_{0m} are obtained from Eqs. (15.15).

The expressions for the boundary values of the functions φ_{2k} are obtained from (56.3) by an exchange of the letters A and B and the subscripts of μ, α and β.

For the determination of the stresses σ_ϑ at the edge of the aperture Eq. (52.7) is used in which we must substitute

$$\left.\begin{aligned} A &= c\cos\vartheta - 3\varepsilon\cos 3\vartheta, \\ B &= \sin\vartheta + 3\varepsilon\sin 3\vartheta. \end{aligned}\right\} \tag{56.5}$$

The formulas given here can also be used in order to determine the stresses at the edge of the aperture for which $c = 1$, $\alpha_1 = \beta_1 = \lambda_1$, $\alpha_2 = \beta_2 = \lambda_2$.

§57. EXTENSION OF AN ORTHOTROPIC PLATE WITH OVAL HOLE*

A rectangular orthotropic plate with small oval aperture is deformed by tensile forces p distributed uniformly on two sides. The principal directions of elasticity and the axes of symmetry of the aperture are parallel to the sides.

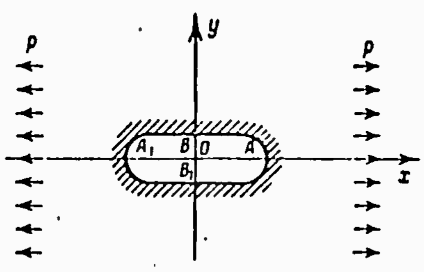

Fig. 116

Solving the problem in a second approximation (by means of superposition), we consider two cases (denotations see §52).

Case I. The plate is extended in the direction of the major axis of the aperture (Fig. 116),

$$\bar\beta_{01} = -\frac{1}{2}pacl, \quad \bar\beta_{13} = \frac{1}{2}pal; \tag{57.1}$$

$\bar\beta_{km} = 0$ for the other k, m; $\bar\alpha_{km} = 0$ for all k, m;

$$A_{01} = -\frac{pacl}{2(\mu_1 - \mu_2)}, \quad B_{01} = \frac{pacl}{2(\mu_1 - \mu_2)}, \tag{57.2}$$

$$A_{0m} = B_{0m} = 0 \quad (m \geqslant 2);$$

$$\begin{aligned} \sigma_\vartheta = p\frac{B^2}{C^4} + \frac{p}{LC^4}\{&c(AD^4\cos\vartheta + BC^4n\sin\vartheta) + \\ &+ \varepsilon[2AC^4b_{11}ckn\cos\vartheta - 3AD^4\cos 3\vartheta - \\ &- BC^4n(2a_{11}c\sin\vartheta + 3\sin 3\vartheta)] + \\ &+ 2\varepsilon^2C^4cn[-Ak(b_{21}\cos\vartheta + 3b_{23}\cos 3\vartheta) + \\ &\qquad + B(a_{21}\sin\vartheta + 3a_{23}\sin 3\vartheta)]\}. \end{aligned} \tag{57.3}$$

The stress distribution on the contour is symmetrical relative to the axes of the aperture. It reaches its highest (absolute) value at the points A and A_1 or at the points B and B_1 at the ends of the axes (Fig. 116).

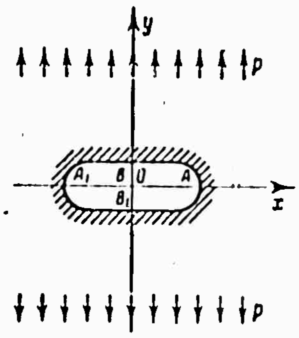

Fig. 117

At the points A and A_1

$$(\sigma_\theta)_A = \frac{p}{c-3\varepsilon}\cdot\frac{1}{k}[-c+ \\ +\varepsilon(3+2b_{11}cn) - 2\varepsilon^2 cn(b_{21}+3b_{23})]; \tag{57.4}$$

at the points B and B_1

$$(\sigma_0)_B = p + \frac{p}{1-3\varepsilon} n[c+\varepsilon(3-2a_{11}c)+ \\ +2\varepsilon^2 c(a_{21}-3a_{23})]. \tag{57.5}$$

In an isotropic plate at the same points A, A_1 and B, B_1, by virtue of (57.4) and (57.5)

$$(\sigma_\theta)_A = -\frac{p}{c-3\varepsilon}\left[c-\varepsilon\frac{3-c}{1+c}+\varepsilon^2\frac{8c}{(1+c)^2}\right]; \tag{57.6}$$

$$(\sigma_0)_B = p + \frac{2p}{1-3\varepsilon}\left[c+\varepsilon\frac{3+5c}{1+c}+\varepsilon^2\frac{4c}{(1+c)^3}\right]. \tag{57.7}$$

In contrast to the plates with triangular apertures, Eqs. (57.6) and (57.7) for an isotropic material do not agree with the exact formulas but are approximations [to the same degree as Eqs. (57.4) and (57.5)].

Case 2. The plate is extended in the direction of the minor axis of the aperture (Fig. 117),

$$\bar{\alpha}_{01} = \bar{\alpha}_{13} = -\frac{1}{2}pa; \tag{57.8}$$

the other $\bar{\alpha}_{km}$ and all $\bar{\beta}_{km}$ are vanishing;

$$A_{01} = \frac{pa\mu_2}{2(\mu_1-\mu_2)}, \quad B_{01} = -\frac{pa\mu_1}{2(\mu_1-\mu_2)}, \\ A_{0m} = B_{0m} = 0 \quad (m \geqslant 2); \tag{57.9}$$

$$\sigma_\vartheta = p\frac{A^2}{C^2} + \frac{p}{LC^4}\{AC^4kn\cos\vartheta + BE^4\sin\vartheta + \\ + \varepsilon[AC^4kn(-2c_{11}\cos\vartheta + 3\cos 3\vartheta) + \\ + 2BC^4b_{11}kn\sin\vartheta + 3BE^4\sin 3\vartheta] + \\ + 2\varepsilon^2C^4kn[A(c_{21}\cos\vartheta + 3c_{23}\cos 3\vartheta) - \\ - B(b_{21}\sin\vartheta + 3b_{23}\sin 3\vartheta)]\}. \tag{57.10}$$

At the points A and A_1 (Fig. 117)

$$(\sigma_\vartheta)_A = p + \frac{p}{c-3\varepsilon}\cdot\frac{n}{k}[1 + \varepsilon(3-2c_{11}) + 2\varepsilon^2(c_{21}+3c_{23})]; \tag{57.11}$$

at points B and B_1

$$(\sigma_\vartheta)_B = \frac{p}{1-3\varepsilon}k[-1 + \varepsilon(3+2b_{11}n) + 2\varepsilon^2 n(-b_{21}+3b_{23})]. \tag{57.12}$$

At the same points of an isotropic plate

$$(\sigma_\vartheta)_A = p + \frac{2p}{c-3\varepsilon}\left[1 + \varepsilon\frac{5+3c}{1+c} + \varepsilon^2\frac{4}{(1+c)^2}\right]; \tag{57.13}$$

$$(\sigma_\vartheta)_B = -\frac{p}{1-3\varepsilon}\left[1 + \varepsilon\frac{1-3c}{1+c} + \varepsilon^2\frac{8}{(1+c)^2}\right]. \tag{57.14}$$

In Table 8 the results of calculating the stresses σ_ϑ at the points A and B of a veneer plate with an aperture for which c = 0.36 and ε = -0.04 are compiled; we consider the cases where the major axis of the aperture is parallel to the sheet fibers and where it is perpendicular to the fibers. The values of σ_ϑ are given in fractions of p.

For an isotropic plate with such an aperture we obtain the following values for the stress σ_ϑ at the points A and B:

Case 1.

$$(\sigma_\vartheta)_A = -0{,}92p, \quad (\sigma_\vartheta)_B = 1{,}39p. \tag{57.15}$$

Case 2.

$$(\sigma_\vartheta)_A = 4{,}44p, \quad (\sigma_\vartheta)_B = -0{,}90p. \tag{57.16}$$

In Table 9 we give the results of calculations for a veneer plate with an aperture characterized by the parameters $c = 0{,}537,\ \varepsilon = -0{,}038$. For such an isotropic plate we have at the points A and B:

Case 1.

$$(\sigma_\vartheta)_A = -0{,}92p, \quad (\sigma_\vartheta)_B = 1{,}71p. \tag{57.17}$$

Case 2.

$$(\sigma_\vartheta)_A = 3{,}58p, \quad (\sigma_\vartheta)_B = -0{,}92p. \tag{57.18}$$

In Tables 8 and 9 only the first and second approximations have been given. The calculations show that with given values of the parameters of the aperture and other constants the third and higher approximations do not alter the first two ciphers behind the comma.

Comparing the ciphers of these tables for different cases and juxtaposing them to the values obtained for the isotropic

TABLE 8

Stresses σ_{ϑ} at Points of the Aperture's Contour, $c = 0{,}36,\ \varepsilon = -0{,}04$

3 Точки / Прибл. 4	1 Случай 1				2 Случай 2			
	$E_x = E_{max}$		$E_x = E_{min}$		$E_y = E_{max}$		$E_y = E_{min}$	
	A	B	A	B	A	B	A	B
1-е	−0,62	1,79	−1,25	1,59	8,54	−0,61	6,39	−1,20
2-е	−0,63	1,78	−1,24	1,58	8,50	−0,61	6,38	−1,19

1) Case 1; 2) case 2; 3) points; 4) approximations.

TABLE 9

Stresses σ_{θ} at Points of the Aperture's Contour, $c = 0{,}537,\ \varepsilon = -0{,}038$

3 Точки / Прибл. 4	1 Случай 1				2 Случай 2			
	$E_x = E_{max}$		$E_x = E_{min}$		$E_y = E_{max}$		$E_y = E_{min}$	
	A	B	A	B	A	B	A	B
1-е	−0,63	2,51	−1,27	2,10	6,65	−0,63	5,04	−1,24
2-е	−0,63	2,50	−1,27	2,09	6,63	−0,63	5,03	−1,24

1) Case 1; 2) case 2; 3) points; 4) approximations.

plate, we arrive at the same conclusions as in the case of the extended plate with triangular aperture, namely: 1) the stress concentration is highest in the case where the plate is extended in the direction of the higher Young's modulus (i.e., along the sheet fibers); 2) the coefficients of concentration for a veneer plate are in both cases 1 and 2 higher than the coefficients of concentration for the same isotropic plate.

§58. PURE BENDING OF AN ORTHOTROPIC PLATE WITH OVAL HOLE*

On a rectangular orthotropic plate with a small oval hole, as considered in the preceding section, forces are assumed to act which are distributed on two sides and which are due to bending moments M.

Case 1. The forces are applied to the sides parallel to the minor axis of the aperture (Fig. 118).

$$\bar{\beta}_{02}=\frac{Ma^2c^2}{8J},\quad \bar{\beta}_{12}=\frac{Ma^2c}{4J},\quad \bar{\beta}_{14}=-\frac{Ma^2c}{4J},\quad \bar{\beta}_{26}=\frac{Ma^2}{8J}, \tag{58.1}$$

the other $\bar{\beta}_{km}$ and all $\bar{\alpha}_{km}$ are equal to zero;

$$\left.\begin{array}{c} A_{01}=B_{01}=0, \\ A_{02}=\dfrac{Ma^2c^2}{8J(\mu_1-\mu_2)},\quad B_{02}=-\dfrac{Ma^2c^2}{8J(\mu_1-\mu_2)}, \\ A_{0m}=B_{0m}=0 \quad (m\geqslant 3) \end{array}\right\} \tag{58.2}$$

(J is the moment of inertia of a cross section of the plate parallel to the minor axis of the aperture);

$$\begin{aligned}\sigma_\vartheta &= \frac{Ma}{J}(c\sin\vartheta-\varepsilon\sin 3\vartheta)\frac{B^2}{C^2}+ \\ &+\frac{Ma}{2JLC^2}\{c^2(-BC^4n\cos 2\vartheta+AD^4\sin 2\vartheta)+ \\ &+2\varepsilon c[-BC^4n(\cos 2\vartheta-2\cos 4\vartheta)+ \\ &+AD^4(\sin 2\vartheta-2\cos 4\vartheta)]+ \\ &+3\varepsilon^2[BC^4n(2a_{23}c^2\cos 2\vartheta-\sin 6\vartheta)+ \\ &+2AC^4c^2b_{23}kn\sin 2\vartheta+AD^4\sin 6\vartheta]\}.\end{aligned} \tag{58.3}$$

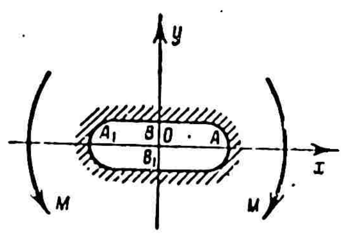

Fig. 118

At the points A and A_1 at the ends of the major axis (Fig. 118) $\sigma_\vartheta=0$.

At the points B and B_1, at the ends of the minor axis, we obtain the highest (absolute) values of the stress; at point B

$$(\sigma_\vartheta)_B=\frac{Ma}{J}\left\{c+\varepsilon+\frac{n}{1-3\varepsilon}[0{,}5c^2+3\varepsilon c+1{,}5\varepsilon^2(1-2a_{23}c^2)]\right\}. \tag{58.4}$$

The stress at the same point of an isotropic plate is equal to

$$(\sigma_\vartheta)_B=\frac{Ma}{J(1-3\varepsilon)}[c(1+c)+\varepsilon(1+3c)]. \tag{58.5}$$

It is interesting to note that this formula, obtained from the approximate relation (58.4), agrees with the exact expression,* just as in the case of the triangular aperture.

Case 2. Forces are applied to the sides parallel to the major axis of the aperture (Fig. 119),

$$\bar{\alpha}_{02} = -\frac{Ma^2}{8J}, \quad \bar{\alpha}_{12} = \bar{\alpha}_{14} = -\frac{Ma^3}{4J}, \quad \bar{\alpha}_{20} = -\frac{Ma^3}{8J}; \tag{58.6}$$

$\bar{\alpha}_{km} = 0$ for other k, m; $\bar{\beta}_{km} = 0$;

$$\left.\begin{array}{c} A_{01} = B_{01} = 0, \\ A_{02} = \dfrac{Ma^2\mu_2}{8J(\mu_1 - \mu_2)}, \quad B_{02} = -\dfrac{Ma^2\mu_1}{8J(\mu_1 - \mu_2)}, \\ A_{0m} = B_{0m} = 0 \quad (m \geqslant 3) \end{array}\right\} \tag{58.7}$$

(J is the moment of inertia in a cross section of the plate parallel to the major axis of the aperture);

$$\begin{aligned} \sigma_\theta = \frac{Ma}{J}(\cos\theta + \varepsilon\cos 3\theta)\frac{A^2}{C^2} + \frac{Ma}{2JLC^4}\{AC^4\,kn\cos 2\theta + BE^4\sin 2\theta + \\ + 2\varepsilon[AC^4kn(\cos 2\theta + 2\cos 4\theta) + BE^4(\sin 2\theta + 2\sin 4\theta)] + \\ + 3\varepsilon^2[AC^4kn(2c_{23}\cos 2\theta + \cos 6\theta) - \\ - 2BC^4b_{23}kn\sin 2\theta + BE^4\sin 6\theta]\}. \end{aligned} \tag{58.8}$$

At the points B and B_1 (Fig. 119) $\sigma_\theta = 0$.

At point A, at the end of the major axis,

$$(\sigma_0)_A = \frac{Ma}{J}\left\{1 + \varepsilon + \frac{1}{c - 3\varepsilon}\cdot\frac{n}{k}[0,5 + 3\varepsilon + 1,5\varepsilon^2(1 + 2c_{23})]\right\}. \tag{58.9}$$

At the same point of an isotropic plate

$$(\sigma_0)_A = \frac{Ma}{J(c - 3\varepsilon)}[1 + c + \varepsilon(3 + c)], \tag{58.10}$$

and this formula agrees with the exact one.

The numerical values of the stress σ_ϑ (in fractions of Ma/J) at point B in Case 1 and at point A in Case 2 are compiled in tables for a veneer plate.

Table 10 applies to a plate with an aperture whose parameters are: c = 0.36, ε = -0.04. For an isotropic plate of this form we obtain:

Case I. At point B

$$(\sigma_\theta)_B = 0,36\frac{Ma}{J}. \tag{58.11}$$

Case 2. At point A

$$(\sigma_\theta)_A = 2,55\frac{Ma}{J}. \tag{58.12}$$

In Table 11 we give the values for a plate whose aperture is characterized by the parameters c = 0.537 and ε = -0.038. For an isotropic plate with such an opening we obtain the following values:

Case I. At point B

$$(\sigma_\vartheta)_B = 0{,}65\frac{Ma}{J}. \qquad (58.13)$$

Case 2. At point *B*

$$(\sigma_\vartheta)_A = 2{,}16\frac{Ma}{J}. \qquad (58.14)$$

Two significant ciphers are everywhere retained after the comma.

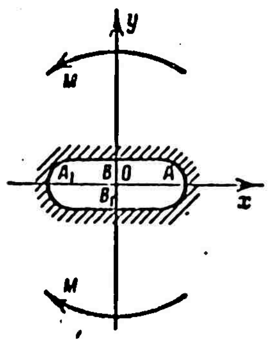

Fig. 119

TABLE 10

Stresses σ_ϑ at the Points of the Aperture's Contour, $c = 0{,}36,\ \varepsilon = -0{,}04$

Прибл. [3]	Случай 1, точка *B* [1]		Случай 2, точка *A* [2]	
	$E_x = E_{max}$	$E_x = E_{min}$	$E_y = E_{max}$	$E_y = E_{min}$
1-е	0,41	0,38	4,50	3,47
2-е	0,41	0,39	4,49	3,46

1) Case 1, point *B*; 2) case 2, point *A*; 3) approximation.

TABLE 11

Stresses σ_ϑ at the Points of the Aperture's Contour, $c = 0{,}537,\ \varepsilon = -0{,}038$

Прибл. [3]	Случай 1, точка *B* [1]		Случай 1, точка *A* [2]	
	$E_x = E_{max}$	$E_x = E_{min}$	$E_y = E_{max}$	$E_y = E_{min}$
1-е	0,84	0,74	3,61	2,84
2-е	0,84	0,74	3,60	2,84

1) Case 1, point *B*; 2) case 2, point *A*; 3) approximation.

As we see from these tables, the coefficient of stress concentration is higher in the case where the sides of the plate perpendicular to the fibers of the sheet are loaded.

§59. EXTENSION OF AN ORTHOTROPIC PLATE WITH A SQUARE HOLE*

Consider an orthotropic plate with the principal directions of elasticity parallel to the directions of the sides, which is weakened by a small central hole whose contour is given by the equation

$$\left.\begin{aligned} x &= a(\cos\vartheta + \varepsilon\cos 3\vartheta), \\ y &= a(\sin\vartheta - \varepsilon\sin 3\vartheta) \end{aligned}\right\} \qquad (59.1)$$

(the axes x and y are parallel to the sides).

With certain values of the parameter ε (e.g., with $\varepsilon = \pm 1/9$), the aperture differs only little from a square with rounded corners; in the following, for the sake of simplicity, we shall call it square.

With positive ε the (rounded) corners of the square lie on the axes x and y, i.e., these axes coincide with the diagonals; with negative ε the sides are parallel to the coordinate axes.

Let us assume the plate extended by the forces p which are distributed uniformly on two sides (Figs. 120 and 121).

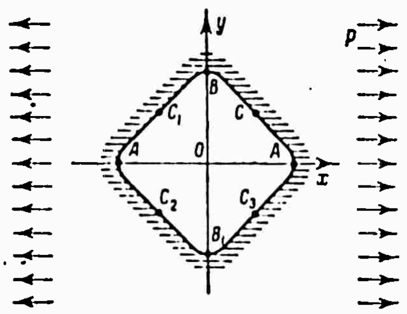

Fig. 120

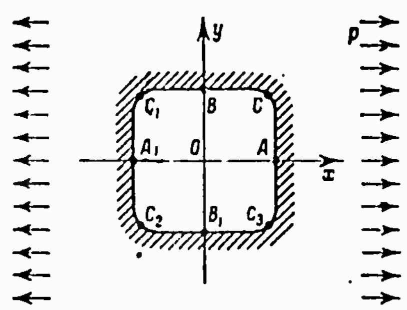

Fig. 121

The formulas in order to calculate the stresses σ_ϑ on the contour of the hole and at individual points of it are obtained in a second approximation from Eqs. (57.3)=(57.7), assuming $c = 1$.

$$\begin{aligned} \sigma_\vartheta = p\frac{B^2}{C^2} + \frac{p}{LC^4}\{ & AD^4\cos\vartheta + BC^4 n\sin\vartheta - \\ & -\varepsilon[AC^4 dkn\cos\vartheta + 3AD^4\cos 3\vartheta + BC^4 n(l\sin\vartheta + 3\sin 3\vartheta)] - \\ & -\varepsilon^2 C^4 n[Ak(r\cos\vartheta - 3g\cos 3\vartheta) + B(s\sin\vartheta - 3h\sin 3\vartheta)]\}. \end{aligned} \qquad (59.2)$$

At the points A and A_1 at the ends of the diameter parallel to the forces the stress is given by

$$(\sigma_0)_A = \frac{p}{1-3\varepsilon}\cdot\frac{1}{k}[-1+\varepsilon(3-nd)+\varepsilon^2 n(3g-r)]. \qquad (59.3)$$

At the points B and B_1, at the end of the diameter perpendicular to the tensile forces,

$$(\sigma_0)_B = \frac{p}{1-3\varepsilon}[1+n-\varepsilon(3+nd-5n)-\varepsilon^2 n(s+3h)]. \qquad (59.4)$$

At the points C, C_1, C_2 and C_3 at the ends of the diameters making angles of 45° with the forces, we obtain

$$(\sigma_\theta)_C = \frac{p}{1+3\varepsilon}\cdot\frac{2}{n^2+(k-1)^2}\{1-k+n+\varepsilon[3(1-k)-n(3+kd+l)]+ \\ +\varepsilon^2 n[3(h-kg)-kr-s]\}. \qquad (59.5)$$

For an isotropic plate Formulas (59.3), (59.4) and (59.5) assume the form

$$(\sigma_\theta)_A = \frac{p}{1-3\varepsilon}(-1+\varepsilon-2\varepsilon^2-2\varepsilon^3); \qquad (59.6)$$

$$(\sigma_\theta)_B = \frac{p}{1-3\varepsilon}(3+5\varepsilon+2\varepsilon^2+2\varepsilon^3); \qquad (59.7)$$

$$(\sigma_\theta)_C = p\,\frac{1-3\varepsilon}{1+3\varepsilon}. \qquad (59.8)$$

The first two of them are approximative, the third agrees with the exact formula.*

From Eqs. (59.2)-(59.8) one can calculate the stresses in a plate with aperture, both in the case of extension along the diagonals (Case 1, $\varepsilon > 0$, Fig. 120), and in the case of extension in the directions of the sides of the square (Case 2, $\varepsilon < 0$, Fig. 121). In order to pass over from one case to the other, we must only change the sign of ε wihout changing the value of this parameter.

The values of σ at the points A, B and C (in fractions of p) for a veneer plate with aperture, for which $|\varepsilon| = 1/9$, are given in Table 12. In both cases we considered the extension in the direction of the higher young's modulus ($E_x = E_{max}$) and of the smaller one ($E_x = E_{min}$). The results given have been calculated not only in first and second approximations but also in the third approximation [according to a formula which is not given here);** we retained everywhere two ciphers after the comma].

For an isotropic plate with the same aperture we obtain the following values for the stresses σ_ϑ at the points A, B and C:

Case 1, $\varepsilon = 1/9$

$$\left.\begin{aligned}(\sigma_\theta)_A &= -1{,}38\,p,\\ (\sigma_\theta)_B &= 5{,}38\,p,\\ (\sigma_\theta)_C &= 0{,}5\,p.\end{aligned}\right\} \qquad (59.9)$$

Case 2, $\varepsilon = -1/9$

$$\left.\begin{aligned}(\sigma_\theta)_A &= -0{,}85p,\\(\sigma_0)_B &= 1{,}85p,\\(\sigma_0)_C &= 2p.\end{aligned}\right\} \qquad (59.10)$$

TABLE 12

Stresses σ_ϑ at Points of the Contour of a Square Hole

Точки [3] / Приближ [4]	Случай 1, $\varepsilon = \frac{1}{9}$ [1]						Случай 2, $\varepsilon = -\frac{1}{9}$ [2]					
	$E_x = E_{max}$			$E_x = E_{min}$			$E_x = E_{max}$			$E_x = E_{min}$		
	A	B	C	A	B	C	A	B	C	A	B	C
1-е	−1,01	10,96	0,20	−2,03	7,92	0,10	−0,55	2,70	0,81	−1,11	2,27	1,26
2-е	−1,05	10,79	0,22	−2,05	7,85	0,11	−0,57	2,61	0,82	−1,12	2,23	1,27
3-е	−1,05	10,82	0,21	−2,06	7,86	0,11	−0,57	2,60	0,83	−1,11	2,22	1,28

1) Case 1; 2) case 2; 3) points; 4) approximation.

Within the limits of accuracy accepted, the third approximation for the plate given differs only little from the second or is even equal to it.

The conclusions which can be drawn when comparing the results for a veneer plate and an isotropic plate are quite analogous to the conclusions drawn previously with respect to the plates with other apertures. The stress σ_ϑ in an isotropic plate varies along the contour more rapidly than the stress in an analogous isotropic plate, forming "peaks" at individual points. This also explains the fact that the maximum stress in a veneer plate is in all cases higher than in an isotropic plate. The stress concentration coefficient in a veneer plate is higher in the case of tension in the direction of the high Young's modulus (along the fibers of the sheet) and smaller when tension is exerted in the direction of the small E (across the fibers); in the latter case the stress σ_ϑ is distributed more uniformly along the contour. It should also be mentioned that the maximum stress in a plate stressed in the direction of the diagonal of the aperture is much higher than the maximum stress when the plate is extended in the direction of the side of the aperture (at least for the plates considered here).

§60. PURE BENDING OF AN ORTHOTROPIC PLATE WITH SQUARE APERTURE*

Consider a rectangular plate with a small quadratic hole in its center, as considered in the preceding section, which is subject to the action of forces applied to two sides, which are due to the moments M. With $\varepsilon > 0$ the moments act in such a way as shown in Fig. 122, with $c < 0$ such as in Fig. 123.

From the formula for a plate with an oval opening (58.3) with $c = 1$ we obtain

$$\sigma_\theta = \frac{Ma}{J}(\sin\theta - \varepsilon\sin 3\theta)\frac{B^2}{C^2} + \frac{Ma}{2JLC^2}\{-BC^4n\cos 2\theta + AD^4\sin 2\theta + \\ + 2\varepsilon[-BC^4n(\cos 2\theta - 2\cos 4\theta) + AD^4(\sin 2\theta - 2\sin 4\theta)] + \\ + 3\varepsilon^2[BC^4n(h\cos 2\theta - \cos 6\theta) - \\ - AC^4gkn\sin 2\theta + AD^4\sin 6\theta]\}. \qquad (60.1)$$

At the points A and A_1 (Figs. 122 and 123) $\sigma_\theta = 0$. At point B

$$(\sigma_\theta)_B = \frac{Ma}{J(1-3\varepsilon)}[1 + 0{,}5n - \varepsilon(2-3n) - 3\varepsilon^2(1 + 0{,}5hn - 0{,}5n)]. \qquad (60.2)$$

At the opposite point B_1 σ_θ will have the same absolute value but the opposite sign.

At the points C and C_1

$$(\sigma_\theta)_C = \frac{Ma}{J(1+3\varepsilon)}\cdot\frac{\sqrt{2}}{n^2+(k-1)^2}[1 - k + 2\varepsilon(1 - k - 2n) + \\ + 3\varepsilon^2(k - gkn - 1)]. \qquad (60.3)$$

The stress at the symmetrical points C_2 and C_3 differs from the stress at the points C and C_1 only by the sign.

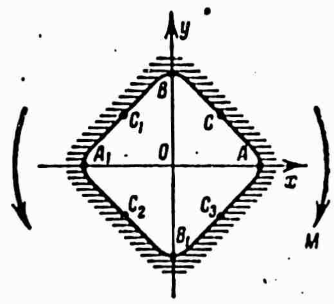

Fig. 122

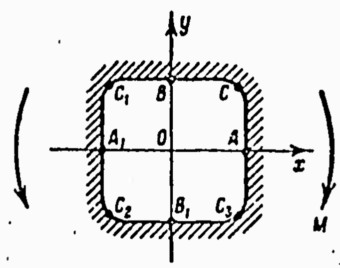

Fig. 123

For an isotropic plate we obtain for the points B and C:

$$(\sigma_\theta)_B = \frac{2Ma}{J}\cdot\frac{1+2\varepsilon}{1-3\varepsilon}; \qquad (60.4)$$

$$(\sigma_\theta)_C = -\frac{2\sqrt{2}Ma}{J}\cdot\frac{\varepsilon}{1+3\varepsilon}. \qquad (60.5)$$

These formulas, which have been obtained from the approximate formulas by means of a limiting transition, are also identical with the exact ones.*

In Table 13 the numerical values of σ_ϑ are compiled in fractions of Ma/J for the points B and C of a veneer plate obtained in first and second approximations; the aperture parameter is taken equal to $\varepsilon = \pm 1/_9$. The third approximation is not given; in the worst case it differs from the second by at most 0.01.

In an isotropic plate the stresses at the points B and C are

Case 1, ε = 1/9

$$(\sigma_\theta)_B = 1{,}17\frac{Ma}{J};\quad (\sigma_\theta)_C = 0{,}47\frac{Ma}{J}. \tag{60.6}$$

Case 2, ε = -1/9,

$$(\sigma_\theta)_B = 3{,}67\frac{Ma}{J};\quad (\sigma_\theta)_C = -0{,}24\frac{Ma}{J}. \tag{60.7}$$

TABLE 13

Values of σ_ϑ at Points on the Contour of a Square Aperture

	Случай 1, $\epsilon = \frac{1}{9}$ (1)				Случай 2, $\epsilon = -\frac{1}{9}$ (2)			
	$E_x = E_{max}$		$E_x = E_{min}$		$E_x = E_{max}$		$E_x = E_{min}$	
(3) Точки / Прибл. (4)	B	C	B	C	B	C	B	C
1-е	6,73	−0,13	5,11	−0,11	1,47	0,18	1,31	0,34
2-е	6,71	−0,13	5,10	−0,11	1,46	0,18	1,30	0,34

1) Case 1; 2) case 2; 3) points; 4) approximation.

A comparison of the values for a veneer plate and an isotropic plate leads us to conclusions which are quite analogous to those drawn for plates with triangular or oval apertures.

Manuscript Page No. — [Footnotes]

229 Lekhnitskiy, S.G., Nekotoryye sluchai uprugogo ravnovesiya anizotropnoy plastinki s nekruglym otverstiyem [Certain Cases of Elastic Equilibrium of an Anisotropic Plate with Noncircular Aperture], Inzhenernyy sbornik, Vol. XXII, 1955.

232 See our paper mentioned in the preceding section.

235* Savin, G.N., Kontsentratsiya napryazheniy okolo otverstiy [Stress Concentration Around an Aperture], Gostekhizdat, Moscow, Leningrad, 1951, Ch. II.

235** Nayman, M.I., Napryazheniya v balke s krivolineynym otverstiyem [Stresses in a Beam with Curvilinear Aperture] Transactions of the TsAGI, No. 313, 1937.

236 See our paper mentioned in §51.

237 See our paper mentioned in §51.

238 Cf. M.I. Nayman, Napryazheniya v balke s krivolineynym otverstiyem [Stresses in a Beam with Curvilinear Aperture] Transactions of the TsAGI, No. 313, 1937 (Eqs. (6.7) on page 41 where we must put $R=a=0,\ h=-p/A$).

241* Krasyukov, V.P., Rastyazheniye ortotropnoy plastinki s otverstiyem, imeyushchim tri osi simmetrii i malo otlichayushchimsya ot krugovogo [Extension of an Orthotropic Plate With an Aperture Having Three Axes of Symmetry and Being Almost Circular], Diploma thesis, Saratov State University, Saratov 1952.

241** Stephens Katheleen, M.A., Boundary Problem in Orthotropic Generalized Plane Stress, Quart. Journal Mech. and Appl. Math., Vol. 5, Part 2, 1952.

241*** See our paper mentioned in §51.

242 See M.I. Nayman's paper, page 44, referred to in §54.

243 See paper by M.I. Nayman, page 42.

244* Greenspan, M., Effect of a Small Hole on the Stress in a Uniformly Loaded Plate, Quart. Appl. Math., Vol. 2, 1944, pages 69-71.

244** Joseph, J.A., Brock, J.S., The Stress Around a Small Opening in a Beam Subjected to Pure Bending, Journ. of Appl. Mech., Vol. 17, No. 4, 1950.

244*** Burmistrov, Ye.F., O kontsentratsii napryazheniy okolo oval'nykh otverstiy nekotorogo vida [On the Stress Concentration About an Oval Aperture of Certain Form] Inzhenernyy sbornik, Vol. XVII, 1953.

246 See our paper mentioned in §51.

249 See our paper mentioned in §51.

250 It is obtained from a formula given in the paper by Joseph and Brock using our notation.

253 See the paper by S.G. Lekhnitskiy, Priblizhenyy metod opredeleniya napryazheniy v uprugoy anizotropnoy plastinke vblizi otverstiya, malo otlichayushchegosya ot krugovogo [Approximation Method of Determining the Stresses in an Elastic Anisotropic Plate Near an Aperture Which is Almost Circular], Inzhenernyy sbornik, Vol. XVII, 1953.

254* See paper by M.I. Nayman, page 54, mentioned in §53.

254** See our paper, reference on page 253.

255 See our paper which has been referred to in §59.

256 See the paper by M.I. Nayman, page 54, Eq. (80), mentioned repeatedly.

Chapter 9

THE THEORY OF BENDING OF ANISOTROPIC PLATES (THIN SHEETS)

§61. APPROXIMATE THEORY OF BENDING OF ANISOTROPIC PLATES (THIN SHEETS)

The cases considered in Chapters 2-13, dealing with the deformation of anisotropic plates are characterized by the fact that the median surface of the plates remains plane. In the present chapter we shall consider the general theory of deformations of anisotropic plates with which the median surface becomes curved, that is, the theory of bending. Here we shall only treat this category of plates which has become known as the "thin sheets" and we shall deal with the approximate theory of thin sheets. The basic conceptions of the theory of bending of anisotropic plates may be found in the papers by Gehring* and Boussinesque.** The approximate theory of bending of anisotropic plates was developed mainly in the papers by Huber.***

In the theory of elasticity the term thin sheet is applied to a plate whose thickness is small compared with the other dimensions, carrying out studies on bending and investigating deflections which are small relative to the thickness (in any case they must not exceed the thickness).

Since within the framework of the present book we are only concerned with thin sheets, we shall call them in the following simply "plates."

Let us consider the elastic equilibrium of a plane homogeneous anisotropic plate of constant thickness, which is fixed on its whole edge or partly, and which is deformed by a bending load.

In the general case the bending load consists of the load q in kg/cm^2, distributed on plane surfaces and normal to the median surface in its nonstrained state, and of loads which are distributed on the edges, in the form of bending moments m and forces p normal to the nonstrained median surface; the latter may be given or may be reactive moments and strains arising at the fastened places.

In the most general case the plate is assumed to be nonorthotropic but possessing at every point a plane of elastic symmetry which is parallel to the median surface.

We assume the mid-plane of a nondeformed plate in the xy-plane; the origin of coordinates is placed at an arbitrary point 0, the z-axis is directed to the nonloaded outer surface (Fig.

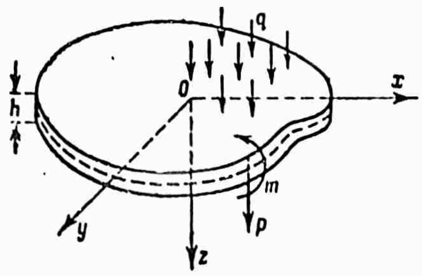

Fig. 124

124). Volume forces will be neglected. On the basis of the suppositions made as to the elastic properties we shall assume the equations of Hooke's generalized law in the form (2.5) applicable to the plate.

The approximate theory of bending of plates (thin sheets) is developed on the following two suppositions:

1) rectilinear sections which in the nonstrained state of the plate were normal to the plane of its median surface remain rectilinear and normal to the curved median surface after bending (hypothesis of the straight normals");

2) the normal stress σ_z in sections parallel to the median surface is a small quantity compared with the stresses in the cross sections σ_x, σ_y and τ_{xy}.

Let us denote by h the thickness of the plate, by u and v displacements of arbitrary points in the directions of the axes x and y and by $w(x, y)$ the deflection of the median surface, i.e., the displacement in the direction of the z-axis of particles of the plate lying in the median surface; the form of the function w depends on the form of the curved median surface.

From the first supposition it follows that

$$u = -z\frac{\partial w}{\partial x}, \quad v = -z\frac{\partial w}{\partial y}; \tag{61.1}$$

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_y = -z\frac{\partial^2 w}{\partial y^2}, \quad \gamma_{xy} = -2z\frac{\partial^2 w}{\partial x\,\partial y}. \tag{61.2}$$

Assuming the equations of Hooke's generalized law (2.5) valid for the plate, we use three of them (neglecting σ_z):

$$\left.\begin{aligned} \varepsilon_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{16}\tau_{xy}, \\ \varepsilon_y &= a_{12}\sigma_x + a_{22}\sigma_y + a_{26}\tau_{xy}, \\ \gamma_{xy} &= a_{16}\sigma_x + a_{26}\sigma_y + a_{66}\tau_{xy}. \end{aligned}\right\} \tag{61.3}$$

Solving these equations with respect to the stress components we obtain

$$
\left.\begin{aligned}
\sigma_x &= -z\left(B_{11}\frac{\partial^2 w}{\partial x^2} + B_{12}\frac{\partial^2 w}{\partial y^2} + 2B_{16}\frac{\partial^2 w}{\partial x\,\partial y}\right),\\
\sigma_y &= -z\left(B_{12}\frac{\partial^2 w}{\partial x^2} + B_{22}\frac{\partial^2 w}{\partial y^2} + 2B_{26}\frac{\partial^2 w}{\partial x\,\partial y}\right),\\
\tau_{xy} &= -z\left(B_{16}\frac{\partial^2 w}{\partial x^2} + B_{26}\frac{\partial^2 w}{\partial y^2} + 2B_{66}\frac{\partial^2 w}{\partial x\,\partial y}\right).
\end{aligned}\right\} \qquad (61.4)
$$

The constants B_{ij} are expressed in terms of a_{ij}, namely:

$$
\left.\begin{aligned}
&B_{11} = \frac{1}{\Delta}\left(a_{22}a_{66} - a_{26}^2\right), \qquad B_{22} = \frac{1}{\Delta}\left(a_{11}a_{66} - a_{16}^2\right),\\
&B_{12} = \frac{1}{\Delta}\left(a_{16}a_{26} - a_{12}a_{66}\right), \qquad B_{66} = \frac{1}{\Delta}\left(a_{11}a_{22} - a_{12}^2\right),\\
&B_{16} = \frac{1}{\Delta}\left(a_{12}a_{26} - a_{22}a_{16}\right), \qquad B_{26} = \frac{1}{\Delta}\left(a_{12}a_{16} - a_{11}a_{26}\right),\\
&\Delta = \begin{vmatrix} a_{11}, & a_{12}, & a_{16}\\ a_{12}, & a_{22}, & a_{26}\\ a_{16}, & a_{26}, & a_{66} \end{vmatrix}.
\end{aligned}\right\} \qquad (61.5)
$$

The components of the strains τ_{zx} and τ_{zy} are determined from the equilibrium equations

$$
\left.\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0,\\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0.
\end{aligned}\right\} \qquad (61.6)
$$

Taking into account that on the outer surfaces $z = \pm h/2$ $\tau_{zx} = \tau_{zy} = 0$, we obtain

$$
\left.\begin{aligned}
\tau_{zx} &= \frac{1}{2}\left(z^2 - \frac{h^2}{4}\right)\left[B_{11}\frac{\partial^3 w}{\partial x^3} + 3B_{16}\frac{\partial^3 w}{\partial x^2\,\partial y} + \right.\\
&\qquad \left. + (B_{12} + 2B_{66})\frac{\partial^3 w}{\partial x\,\partial y^2} + B_{26}\frac{\partial^3 w}{\partial y^3}\right],\\
\tau_{zy} &= \frac{1}{2}\left(z^2 - \frac{h^2}{4}\right)\left[B_{16}\frac{\partial^3 w}{\partial x^3} + (B_{12} + 2B_{66})\frac{\partial^3 w}{\partial x^2\,\partial y} + \right.\\
&\qquad \left. + 3B_{26}\frac{\partial^3 w}{\partial x\,\partial y^2} + B_{22}\frac{\partial^3 w}{\partial y^3}\right].
\end{aligned}\right\} \qquad (61.7)
$$

Let us consider separate areas in the plate which are normal to the initial mid-plane xy, with a height equal to the thickness of the plate and the base sides dx and dy; the stress components σ_x, σ_y, τ_{xy} can then be reduced to the moments $M_y\,dx$, $H_{xy}\,dx$ and $M_x\,dy$, $H_{yx}\,dy$, and τ_{zx}, τ_{zy} to the forces $N_x\,dy$, $N_y\,dx$. The quantities M_x, M_y are called the bending moments, H_{xy}, H_{yx} the torsion moments and N_x, N_y are the crosscut forces (all referred to a unit length of the median surface). Obviously,

$$
\left.\begin{aligned}
M_x = \int_{-h/2}^{h/2}\sigma_x z\,dz, \quad M_y = \int_{-h/2}^{h/2}\sigma_y z\,dz, \quad H_{xy} = H_{yx} = \int_{-h/2}^{h/2}\tau_{xy} z\,dz,\\
N_x = \int_{-h/2}^{h/2}\tau_{zx}\,dz, \quad N_y = \int_{-h/2}^{h/2}\tau_{zy}\,dz.
\end{aligned}\right\} \qquad (61.8)
$$

From this and from (61.4) and (61.7) we can derive the functions

$$\left.\begin{aligned} M_x &= -\left(D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2} + 2D_{16}\frac{\partial^2 w}{\partial x\,\partial y}\right),\\ M_y &= -\left(D_{12}\frac{\partial^2 w}{\partial x^2} + D_{22}\frac{\partial^2 w}{\partial y^2} + 2D_{26}\frac{\partial^2 w}{\partial x\,\partial y}\right),\\ H_{xy} &= -\left(D_{16}\frac{\partial^2 w}{\partial x^2} + D_{26}\frac{\partial^2 w}{\partial y^2} + 2D_{66}\frac{\partial^2 w}{\partial x\,\partial y}\right); \end{aligned}\right\} \tag{61.9}$$

$$\left.\begin{aligned} N_x &= -\left[D_{11}\frac{\partial^3 w}{\partial x^3} + 3D_{16}\frac{\partial^3 w}{\partial x^2\,\partial y} + \right.\\ &\qquad \left. + (D_{12} + 2D_{66})\frac{\partial^3 w}{\partial x\,\partial y^2} + D_{26}\frac{\partial^3 w}{\partial y^3}\right],\\ N_y &= -\left[D_{16}\frac{\partial^3 w}{\partial x^3} + (D_{12} + 2D_{66})\frac{\partial^3 w}{\partial x^2\,\partial y} + \right.\\ &\qquad \left. + 3D_{26}\frac{\partial^3 w}{\partial x\,\partial y^2} + D_{22}\frac{\partial^3 w}{\partial y^3}\right]. \end{aligned}\right\} \tag{61.10}$$

The constants D_{ij} are connected with the B_{ij}:

$$D_{ij} = B_{ij}\frac{h^3}{12}. \tag{61.11}$$

In analogy to the isotropic plate for which we introduced the conception of the "rigidity" we call the constants D_{ij} the rigidities of the anisotropic plate, namely: D_{11}, D_{22} — the rigidity of bending about the axes y, x; D_{66} the rigidities of torsion; D_{16}, D_{26} the secondary rigidities; taking the formulas for the orthotropic plate into account (which will be given below) the ratios $D_{12}/D_{22} = \nu_1, D_{12}/D_{11} = \nu_2$ can be called the reduced Poisson coefficients.

The stress components, moments and crosscut forces are linked through simple relations which follow from the formulas given above:

$$\left.\begin{aligned} &\sigma_x = \frac{12M_x}{h^3}z, \quad \sigma_y = \frac{12M_y}{h^3}z, \quad \tau_{xy} = \tau_{yx} = \frac{12H_{xy}}{h^3}z,\\ &\tau_{xz} = \tau_{zx} = \frac{6N_x}{h^3}\left(\frac{h^2}{4} - z^2\right), \quad \tau_{yz} = \tau_{zy} = \frac{6N_y}{h^3}\left(\frac{h^2}{4} - z^2\right). \end{aligned}\right\} \tag{61.12}$$

In Fig. 125a the stress components are shown for the areas normal to the axes x and y, and in Fig. 125b the moments and the crosscut forces inducing the stresses.

If the plate is orthotropic and the directions of the axes x and y coincide with the principal directions of elasticity, we shall have instead of Eqs. (61.3), (61.9) and (61.10)

$$e_x = \frac{1}{E_1}(\sigma_x - \nu_1\sigma_y), \quad e_y = \frac{1}{E_2}(\sigma_y - \nu_2\sigma_x), \quad \gamma_{xy} = \frac{1}{G}\tau_{xy}; \tag{61.13}$$

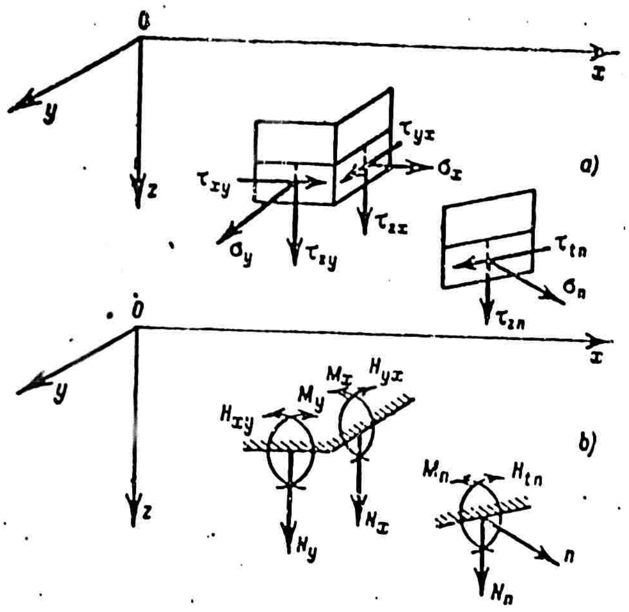

Fig. 125

$$\left.\begin{aligned} M_x &= -D_1\left(\frac{\partial^2 w}{\partial x^2} + \nu_2 \frac{\partial^2 w}{\partial y^2}\right), \\ M_y &= -D_2\left(\frac{\partial^2 w}{\partial y^2} + \nu_1 \frac{\partial^2 w}{\partial x^2}\right), \\ H_{xy} &= -2D_k \frac{\partial^2 w}{\partial x \partial y}; \end{aligned}\right\} \qquad (61.14)$$

$$\left.\begin{aligned} N_x &= -\frac{\partial}{\partial x}\left(D_1 \frac{\partial^2 w}{\partial x^2} + D_3 \frac{\partial^2 w}{\partial y^2}\right), \\ N_y &= -\frac{\partial}{\partial y}\left(D_3 \frac{\partial^2 w}{\partial x^2} + D_2 \frac{\partial^2 w}{\partial y^2}\right). \end{aligned}\right\} \qquad (61.15)$$

Here $E_1, E_2, \nu_1, \nu_2, G$ are Young's moduli, Poisson's coefficients and the modulus of shear for the principal directions;

$$D_1 = \frac{E_1 h^3}{12(1-\nu_1\nu_2)}, \quad D_2 = \frac{E_2 h^3}{12(1-\nu_1\nu_2)}, \quad D_k = \frac{Gh^3}{12} \qquad (61.16)$$

the rigidity of bending and the rigidity of torsion for the principal directions of elasticity, or the principal rigidities

$$D_3 = D_1\nu_2 + 2D_k; \qquad (61.17)$$

the secondary rigidities for the principal directions are equal to zero.

In the case of an isotropic plate $E_1 = E_2 = E$, $\nu_1 = \nu_2 = \nu$, $G = \frac{E}{2(1+\nu)}$ and all rigidities can be reduced to a single one:

$$D = \frac{Eh^3}{12(1-\nu^2)}. \qquad (61.18)$$

Formulas (61.14) and (61.15) assume the form

$$\left.\begin{aligned} M_x &= -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right), \\ M_y &= -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right), \\ H_{xy} &= -D(1-\nu)\frac{\partial^2 w}{\partial x\,\partial y}; \end{aligned}\right\} \qquad (61.19)$$

$$\left.\begin{aligned} N_x &= -D\frac{\partial}{\partial x}(\nabla^2 w), \\ N_y &= -D\frac{\partial}{\partial y}(\nabla^2 w) \end{aligned}\right\} \qquad (61.20)$$

$$\left(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

We shall here give those expressions for the potential energy of bending for nonorthotropic and orthotropic plates which result from Eqs. (2.2)-(2.4) when we neglect τ_{xz}, τ_{yz} and σ_z:

$$V = \frac{1}{2}\int\int\left[D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{12}\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} + D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \right. \\ \left. + 4D_{66}\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 + 4\left(D_{16}\frac{\partial^2 w}{\partial x^2} + D_{26}\frac{\partial^2 w}{\partial y^2}\right)\frac{\partial^2 w}{\partial x\,\partial y}\right]dx\,dy; \qquad (61.21)$$

$$V = \frac{1}{2}\int\int\left[D_1\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_1\nu_2\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} + D_2\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \right. \\ \left. + 4D_k\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]dx\,dy \qquad (61.22)$$

(the integrals are taken over the areas occupied by the plate).

Apart from the components of the stresses, moments and crosscut forces acting on the elements of the plate perpendicular to the axes x and y, we may sometimes encounter the stress components $\sigma_n, \tau_{tn}, \tau_{zn}$ in a surface making an arbitrary angle with the direction of the normal n, which correspond to the moments M_n, H_{tn} of bending and torsion and the crosscut force N_n (Fig. 125). The latter are determined from formulas resulting from (8.5):

$$\left.\begin{aligned} M_n &= M_x\cos^2(n,x) + M_y\cos^2(n,y) + 2H_{xy}\cos(n,x)\cos(n,y), \\ H_{tn} &= (M_y - M_x)\cos(n,x)\cos(n,y) + H_{xy}[\cos^2(n,x) - \cos^2(n,y)], \\ N_n &= N_x\cos(n,x) + N_y\cos(n,y); \end{aligned}\right\} \qquad (61.23)$$

$$\sigma_n = \frac{12M_n}{h^3}z, \quad \tau_{tn} = -\frac{12H_{tn}}{h^3}z, \quad \tau_{zn} = \frac{6N_n}{h^3}\left(\frac{h^2}{4} - z^2\right). \qquad (61.24)$$

The signs of the moments and crosscut forces will be determined according to Eqs. (61.12) and (61.24), respectively; the moments and crosscut forces are considered to be positive when they are produced in positive directions with $z > 0$.

§62. EQUATION OF A BENT SURFACE AND THE BOUNDARY CONDITIONS

As we see from the formulas of the preceding section, the moments and crosscut forces (and thus also the stresses) can be expressed in terms of the deflection w of the median surface. This

function satisfies a differential equation of fourth order which we obtain when we consider the equilibrium of a plate element in the form of a rectangular parallelepiped with the sides *dx*, *dy*, *h*. A cross section of this element with the area *xy* is shown in Fig. 126, together with the forces and moments acting on it.

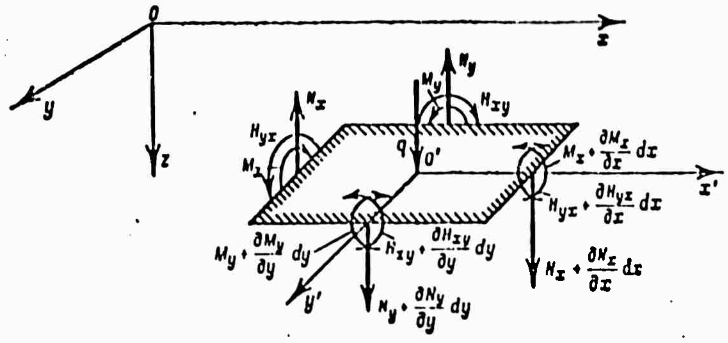

Fig. 126

The conditions of equilibrium of the element (the sum of the projections of the forces on the *x*-axis is vanishing and the sum of the moments relative to the axes *x'*, *y'* parallel to the axes *x* and *y* are vanishing) have the form

$$\left.\begin{aligned} &\frac{\partial N_x}{\partial y}+\frac{\partial N_y}{\partial y}+q=0,\\ &N_x=\frac{\partial M_x}{\partial x}+\frac{\partial H_{xy}}{\partial y},\\ &N_y=\frac{\partial M_y}{\partial y}+\frac{\partial H_{xy}}{\partial x}.\end{aligned}\right\} \qquad (62.1)$$

q is here the load distributed on the outer surface, per unit area.

Substituting in the first equation expressions for the cross-cut forces (61.10) we obtain the equation of deflection of a non-orthotropic plate

$$D_{11}\frac{\partial^4 w}{\partial x^4}+4D_{16}\frac{\partial^4 w}{\partial x^3\partial y}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+ \\ +4D_{26}\frac{\partial^4 w}{\partial x\partial y^3}+D_{22}\frac{\partial^4 w}{\partial y^4}=q. \qquad (62.2)$$

In particular, for an orthotropic plate where the directions of the axes x and y coincide with the principal directions, we obtain

$$D_1\frac{\partial^4 w}{\partial x^4}+2D_3\frac{\partial^4 w}{\partial x^2\partial y^2}+D_2\frac{\partial^4 w}{\partial y^4}=q. \qquad (62.3)$$

For an isotropic plate

$$D_1=D_2=D_3=D=\frac{Eh^3}{12(1-\nu^2)}$$

and Eq. (62.3) assumes the form*

$$D\nabla^2\nabla^2 w = q. \qquad (62.4)$$

The problem of the elastic equilibrium of a plate bent by some arbitrary forces is thus reduced to determining the function $w(x, y)$ in the zone occupied by the plate. This function satisfies a differential equation of fourth order, Eq. (62.2) [or (62.3) and (62.4), respectively] and the boundary conditions on the edge of the plate which depend on the way of fastening or loading. Let us give the boundary conditions for five fundamental cases. In the general case the edge is considered to be curvilinear, with the normal n whose direction relative to the axes x and y is arbitrary:*

1) The edge is rigidly fixed (pinched):

$$w = 0, \ \frac{dw}{dn} = 0. \qquad (62.5)$$

2) The edge is resting (hinged):

$$w = 0, \ M_n = 0; \qquad (62.6)$$

3) The edge is free:

$$M_n = 0, \ N_n + \frac{\partial H_{tn}}{\partial s} = 0 \qquad (62.7)$$

($\frac{\partial}{\partial s}$ is the derivative with respect to the arc s of the contour).

4) The edge is loaded by given bending moments and forces whose magnitudes per unit length are equal to m and p, respectively:

$$M_n = m, \ N_n + \frac{\partial H_{tn}}{\partial s} = p. \qquad (62.8)$$

5) The edge has been deformed and we know the deflection w^* and the angle α^* of slope of the curved surface with respect to the xy-plane:

$$w = w^*, \ \frac{dw}{dn} = \alpha^*. \qquad (62.9)$$

In Eqs. (62.8) and (62.9) the known right-hand sides of the equations are assumed given either in the form of functions of the arc s of the contour, or as functions of any other variable determining the position of a point on the contour.

§63. CONNECTION BETWEEN THE THEORY OF BENDING OF A PLATE AND THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

The equation of the deflections of an anisotropic plate, §62.2), belongs to the same type as the equation for the stress function (5.9) in the theory of the plane state of tension. We know a general expression for the function w; it depends on the roots $\mu_1, \mu_2, \bar{\mu}_1, \bar{\mu}_2$ of the characteristic equation

$$D_{22}\mu^4 + 4D_{26}\mu^3 + 2(D_{12} + 2D_{66})\mu^2 + 4D_{16}\mu + D_{11} = 0. \qquad (63.1)$$

As we have shown, for an arbitrary elastic homogeneous material, this equation cannot possess any real solutions.* The complex of purely imaginary quantities $\mu_1 = \alpha + \beta i$ and $\mu_2 = \gamma + \delta$ are called the complex parameters of bending (of first kind); in general they differ from the complex parameters of the plane state of stress for this pla-e and only in the case of an isotropic material they and others are equal to i.

The general expressions for the deflections read:

1) in the case of different complex parameters ($\mu_2 \neq \mu_1$):

$$w = w_0 + 2\mathrm{Re}\,[w_1(z_1) + w_2(z_2)]; \qquad (63.2)$$

2) in the case of equal complex parameters ($\mu_2 = \mu_1$):

$$w = w_0 + 2\mathrm{Re}\,[w_1(z_1) + \bar{z}_1 w_2(z_1)]. \qquad (63.3)$$

Here w_0 is a particular solution of the nonhomogeneous equation (62.2) whose form depends on the distribution of the loads q over the surface, $w_1(z_1)$, $w_2(z_2)$ are arbitrary analytical functions of the complex variables $z_1 = x + \mu_1 y$ and $z_2 = x + \mu_2 y$.

On the basis of Eqs. (61.9) and (61.10) we obtain general expressions for the moments and the crosscut forces (for the case $\mu_2 \neq \mu_1$):

$$\left.\begin{aligned} M_x &= M_x^0 - 2\mathrm{Re}\,[p_1 w_1''(z_1) + p_2 w_2''(z_2)],\\ M_y &= M_y^0 - 2\mathrm{Re}\,[q_1 w_1''(z_1) + q_2 w_2''(z_2)],\\ H_{xy} &= H_{xy}^0 - 2\mathrm{Re}\,[r_1 w_1''(z_1) + r_2 w_2''(z_2)]; \end{aligned}\right\} \qquad (63.4)$$

$$\left.\begin{aligned} N_x &= N_x^0 - 2\mathrm{Re}\,[\mu_1 s_1 w_1'''(z_1) + \mu_2 s_2 w_2'''(z_2)],\\ N_y &= N_y^0 + 2\mathrm{Re}\,[s_1 w_1'''(z_1) + s_2 w_2'''(z_2)]. \end{aligned}\right\} \qquad (63.5)$$

Here M_x^0, M_y^0, ..., N_y^0 are the moments and crosscut forces corresponding to the functions w_0 [they are determined from Eqs. (61.9) and (61.10)];

$$\left.\begin{aligned} &p_1 = D_{11} + D_{12}\mu_1^2 + 2D_{16}\mu_1, \quad p_2 = D_{11} + D_{12}\mu_2^2 + 2D_{16}\mu_2,\\ &q_1 = D_{12} + D_{22}\mu_1^2 + 2D_{26}\mu_1, \quad q_2 = D_{12} + D_{22}\mu_2^2 + 2D_{26}\mu_2,\\ &r_1 = D_{16} + D_{26}\mu_1^2 + 2D_{66}\mu_1, \quad r_2 = D_{16} + D_{26}\mu_2^2 + 2D_{66}\mu_2,\\ &s_1 = \frac{D_{11}}{\mu_1} + 3D_{16} + (D_{12} + 2D_{66})\mu_1 + D_{26}\mu_1^2,\\ &s_2 = \frac{D_{11}}{\mu_2} + 3D_{16} + (D_{12} + 2D_{66})\mu_2 + D_{26}\mu_2^2,\\ &s_1 - r_1 = \frac{p_1}{\mu_1}, \qquad s_2 - r_2 = \frac{p_2}{\mu_2},\\ &s_1 + r_1 = -q_1\mu_1 \quad s_2 + r_2 = -q_2\mu_2. \end{aligned}\right\} \qquad (63.6)$$

Let us consider the case where the plate is bent only by forces and moments distributed along the edge ($q = 0$). Equations (62.2)-(62.4) become homogeneous and in Eqs. (63.2)-§63.5) we must put $w_0 = M_x^0 = M_y^0 = H_{xy}^0 = N_x^0 = N_y^0 = 0$.

If at the edge of the plate the bending moments $m(s)$ and the forces $p(s)$ (per unit length) are given as functions of s, the boundary conditions (62.8) are transformed in conditions for the functions w_1' and w_2':*

$$\left.\begin{aligned} 2\,\mathrm{Re}\left[\frac{p_1}{\mu_1} w_1'(z_1) + \frac{p_2}{\mu_2} w_2'(z_2)\right] &= -\int_0^s (m\,dy + f\,dx) - Cx + C_1, \\ 2\,\mathrm{Re}\,[q_1 w_1'(z_1) + q_2 w_2'(z_2)] &= \int_0^s (-m\,dx + f\,dy) + Cy + C_2. \end{aligned}\right\} \quad (63.7)$$

C, C_1, C_2 are here unknown constants, $f = \int_0^s p\,ds$, the integrals are taken along the arc of the contour, from the initial to the variable point.

With given strain of the edge of the aperture caused by unknown forces [$w^*(s)$ and $\alpha^*(s)$ given] the boundary conditions (62.9) assume the form:

$$\left.\begin{aligned} 2\,\mathrm{Re}\,[w_1'(z_1) + w_2'(z_2)] &= -\frac{dw^*}{ds}\cos(n,\ y) + \alpha^*\cos(n,\ x), \\ 2\,\mathrm{Re}\,[\mu_1 w_1'(z_1) + \mu_2 w_2'(z_2)] &= \frac{dw^*}{ds}\cos(n,\ x) + \alpha^*\cos(n,\ y) \end{aligned}\right\} \quad (63.8)$$

(n is the direction of the outer normal to the plate's contour).

Inside the region of the plate bent by forces distributed on the edge, the functions w_1' and w_2' must satisfy a series of conditions derived in our paper and given in §62, namely:

1) if the region S of the plate is simply connected (the plate has no aperture), the functions $w_1'(z_1)$ and $w_2'(z_2)$ must be holomorph and unambiguous in their regions S_1 and S_2 (see §8, Fig. 9);

2) if the region of the plate has holes but the forces distributed on the edges of the holes are in equilibrium on each of them (vector sum and resulting moment being equal to zero), the functions $w_1'(z_1)$ and $w_2'(z_2)$ are holomorph and unambiguous in their domains S_1 and S_2;

3) when the zone of the plate is limited by certain contours (and the plate as a hole) where for one of the contours the vector sum and the resulting moment of forces are not vanishing, the functions $w_1'(z_1)$ and $w_2'(z_2)$ will be multiple-valued.

Consider, for example, a plate with one hole and forces and moments act on the edge of this hole which can be reduced to the resultant (vector sum) P_z and a moment with the components m_x and m_y. On a circumvention on an arbitrary closed contour surrounding the aperture, the functions w_1' and w_2' grow by the increments Δ_1' and Δ_2' and their derivatives w_1'' and w_2'' (i.e., the second derivatives of the functions w_1 and w_2) by the increments Δ_1'' and Δ_2'' which are determined from the equations:**

$$\begin{aligned}
&\Delta_1'' + \Delta_2'' + \bar{\Delta}_1'' + \bar{\Delta}_2'' = 0,\\
&\mu_1\Delta_1'' + \mu_2\Delta_2'' + \bar{\mu}_1\bar{\Delta}_1'' + \bar{\mu}_2\bar{\Delta}_2'' = 0,\\
&\mu_1^2\Delta_1'' + \mu_2^2\Delta_2'' + \bar{\mu}_1^2\bar{\Delta}_1'' + \bar{\mu}_2^2\bar{\Delta}_2'' = 0,\\
&\frac{1}{\mu_1}\Delta_1'' + \frac{1}{\mu_2}\Delta_2'' + \frac{1}{\bar{\mu}_1}\bar{\Delta}_1'' + \frac{1}{\bar{\mu}_2}\bar{\Delta}_2'' = \frac{P_s}{D_{11}};
\end{aligned}\quad (63.9)$$

$$\begin{aligned}
&\Delta_1' + \Delta_2' + \bar{\Delta}_1' + \bar{\Delta}_2' = 0,\\
&\mu_1\Delta_1' + \mu_2\Delta_2' + \bar{\mu}_1\bar{\Delta}_1' + \bar{\mu}_2\bar{\Delta}_2' = 0,\\
&\mu_1^2\Delta_1' + \mu_2^2\Delta_2' + \bar{\mu}_1^2\bar{\Delta}_1' + \bar{\mu}_2^2\bar{\Delta}_2' = -\frac{m_x}{D_{22}} - \frac{yP_s}{D_{22}},\\
&\frac{1}{\mu_1}\Delta_1' + \frac{1}{\mu_2}\Delta_2' + \frac{1}{\bar{\mu}_1}\bar{\Delta}_1' + \frac{1}{\bar{\mu}_2}\bar{\Delta}_2' = -\frac{m_y}{D_{11}} + \frac{xP_s}{D_{11}}.
\end{aligned}\quad (63.10)$$

On the basis of this the nature of the functions $w_1(z_1)$ and $w_2(z_2)$ becomes clear. We see that the problem on the bending of a plate by a load distributed on the edge has much in common with the plane problem and at the same time it presents the same inherent difficulties, which are connected with the determination of the functions of the different complex variables z_1 and z_2, just as the plane problem. With simple contours where the plate has the form of a rectangle, strip or circle, in a series of cases the problem can be solved simply, without complex representations; in more complicate cases, however, the complex representation may prove to be very valuable.

The deflection, the moments and crosscut forces may also be expressed in terms of functions of the generalized complex variables $z_1' = z + \lambda_1\bar{z}$, and $z_2' = z + \lambda_2\bar{z}$ where

$$\lambda_1 = \frac{1 + i\mu_1}{1 - i\mu_1}, \quad \lambda_2 = \frac{1 + i\mu_2}{1 - i\mu_2} \quad (63.11)$$

are complex parameters of second kind for the case of bending. The deflection can then be represented in the form

$$w = w_0 + 2\,\mathrm{Re}\,[\Theta_1(z_1') + \Theta_2(z_2')], \quad (63.12)$$

where Θ_1 and Θ_2 are arbitrary analytical functions of the variables z_1' and z_2'. In the following this representation of deflections will not be used so that we restrict ourselves to what has been said above.

Let us also give the formulas for the isotropic plate which are analogous to the formulas of the plane problem:*

$$w = w_0 + \mathrm{Re}\,[\bar{z}\varphi(z) + \chi(z)]; \quad (63.13)$$

$$\begin{aligned}
M_y - M_x + 2iH_{xy} &= M_y^0 - M_x^0 + 2iH_{xy}^0 +\\
&+ 2D(1-\nu)[\bar{z}\varphi''(z) + \psi'(z)],\\
M_x + M_y &= M_x^0 + M_y^0 - 4D(1+\nu)\,\mathrm{Re}\,[\varphi'(z)];
\end{aligned}\quad (63.14)$$

$$N_x - iN_y = N_x^0 - iN_y^0 - 4D\varphi''(z). \quad (63.15)$$

Here w_0 is an arbitrary particular solution of the nonhomoge-

neous equation (62.4) depending on the distribution of load q; $M_x^0, M_y^0, H_{xy}^0, N_x^0, N_y^0$ are the moments and crosscut forces corresponding to the deflection w_0 [they are determined from Eqs. (61.19) and (61.20)]; $\varphi(z), \chi(z)$ are analytic functions of the variable $z = x + iy$, $\psi(z) = \chi'(z)$; D is the rigidity of the isotropic plate (61.18).

Owing the complex representations of the deflections, moments and crosscut forces, the methods by G.V. Kolosov and N.I. Muskhelishvili developed for a plane problem can also be applied to find a solution of the problems on the bending of isotropic plates. The first complex representations of bending was used by A.I. Lur'ye who considered the bending of an arbitrarily loaded round plate.* M.M. Fridman obtained solutions of a series of problems on the bending of isotropic plates; in particular, he studied the bending of plates weakened by various forms of holes.**

§64. THE EQUATION OF A CURVED SURFACE WHERE THE LONGITUDINAL FORCES ARE TAKEN INTO ACCOUNT. PLATE WITH LARGE DEFLECTIONS AND PLATE ON AN ELASTIC BASE

Assume a homogeneous anisotropic plate loaded by forces and moments which cause a bending (as considered in §§61-62) and, moreover, by forces which act in the median surface (longitudinal forces). When only the first forces and moments are acting the stress distribution which develops in the plate corresponds to bending; when only the longitudinal forces are acting, the state of stress in the plate will be generally plane.

When, however, these forces act simultaneously, it would be incorrect to determine the stress by a simple summation of the stresses due to bending and the stresses corresponding to the plane state of stress; the longitudinal forces exert an influence on the bending and Eqs. (62.2)-(62.4) are no longer applicable.

When we investigate (approximately) the simultaneous action of the bending loads and the longitudinal forces we assume that the stress components in the plate consist of two parts:

$$\left.\begin{array}{c}\sigma_x = \bar{\sigma}_x + \sigma'_x, \quad \sigma_y = \bar{\sigma}_y + \sigma'_y, \quad \tau_{xy} = \bar{\tau}_{xy} + \tau'_{xy}, \\ \tau_{xz} = \tau'_{xz}, \quad \tau_{yz} = \tau'_{yz}.\end{array}\right\} \qquad (64.1)$$

Here $\bar{\sigma}_x, \bar{\sigma}_y, \bar{\tau}_{xy}$ are the stresses averaged with respect to the thickness, which are caused by the longitudinal forces alone; $\sigma'_x, \sigma'_y, \tau'_{xy}$ are the stresses proportional to z and determined from Eqs. (61.12) and (61.9). In order to obtain an equation which, in the case given, must be satisfied by the deflection w, we introduce the quantities T_x, T_y, S_{xy}, S_{yx}, i.e., the longitudinal and tangential forces per unit length:

$$\left.\begin{array}{c}T_x = \int\limits_{-h/2}^{h/2} \sigma_x \, dz = h\bar{\sigma}_x, \\ T_y = h\bar{\sigma}_y, \quad S_{yx} = S_{xy} = h\bar{\tau}_{xy}.\end{array}\right\} \qquad (64.2)$$

It is obvious that T_x, T_y and S_{xy}, in the absence of volume

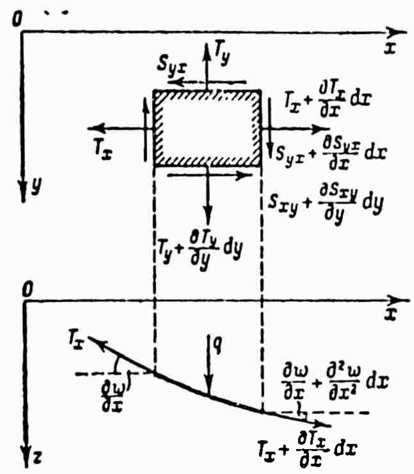

Fig. 127

forces, satisfy the equilibrium equations

$$\begin{aligned}\frac{\partial T_x}{\partial x} + \frac{\partial S_{xy}}{\partial y} &= 0,\\ \frac{\partial S_{xy}}{\partial x} + \frac{\partial T_y}{\partial y} &= 0;\end{aligned} \tag{64.3}$$

in order to determine them the plane problem must be solved for the plate.

Let us separate an element of the plate in the form of a rectangular parallelepiped with the sides dx, dy and h and consider its equilibrium. Besides the forces and moments shown in Fig. 126 longitudinal forces will act on the element, which are represented separately in Figs. 127 and 128. Taking into account that in the deformation the element is curved and the forces T_x, T_y, S_{xy} in the deformed plate will not lie in the xy-plane, we obtain the component of these forces in the z-direction:

$$Z = \left(T_x \frac{\partial^2 w}{\partial x^2} + 2S_{xy} \frac{\partial^2 w}{\partial x\, \partial y} + T_y \frac{\partial^2 w}{\partial y^2}\right) dx\, dy, \tag{64.4}$$

or per unit area

$$\bar{Z} = T_x \frac{\partial^2 w}{\partial x^2} + 2S_{xy} \frac{\partial^2 w}{\partial x\, \partial y} + T_y \frac{\partial^2 w}{\partial y^2}. \tag{64.5}$$

This force must be added to the load q in the equations of §62, and we obtain for a nonorthotropic plate

$$\begin{aligned}D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^3\, \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2\, \partial y^2} + 4D_{26} \frac{\partial^4 w}{\partial x\, \partial y^3} +\\ + D_{22} \frac{\partial^4 w}{\partial y^4} = q + T_x \frac{\partial^2 w}{\partial x^2} + 2S_{xy} \frac{\partial^2 w}{\partial x\, \partial y} + T_y \frac{\partial^2 w}{\partial y^2}\end{aligned} \tag{64.6}$$

and for an orthotropic plate

$$D_1 \frac{\partial^4 w}{\partial x^4} + 2D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} = \\ = q + T_x \frac{\partial^2 w}{\partial x^2} + 2S_{xy} \frac{\partial^2 w}{\partial x \partial y} + T_y \frac{\partial^2 w}{\partial y^2}. \tag{64.7}$$

The problem of the bending of a plate where longitudinal forces are taken into account becomes much more complex when it is not based on the assumption that the deflection is small compared to the thickness. In this case the deflection and the stress functions are determined from a system of two nonlinear equations.

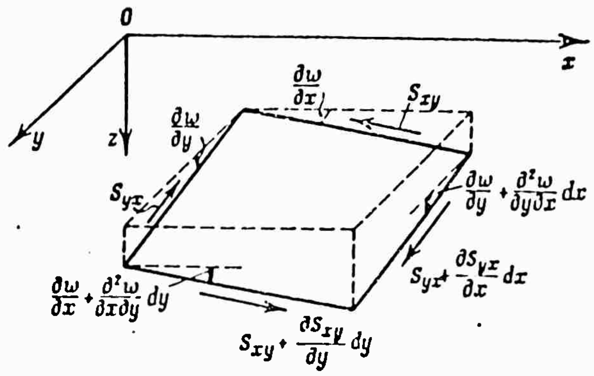

Fig. 128

Such a system for an isotropic plate was obtained by Karman.* Let us give a brief derivation of the corresponding system for an orthotropic plate.

Let us suppose that the stress components are determined by Eqs. (64.1) and the strain components also consist of two parts:

$$\varepsilon_x = \bar{\varepsilon}_x + e'_x, \quad \varepsilon_y = \bar{\varepsilon}_y + e'_y, \quad \gamma_{xy} = \bar{\gamma}_{xy} + \gamma'_{xy}. \tag{64.8}$$

The quantities $\bar{e}_x, \bar{e}_y, \bar{\gamma}_{xy}$ are the strain components in the median surface; they depend not only on the displacement u and v but also on the deflection w. From the general expressions (1.5), which are expanded in series where we retain the first powers of the derivatives with respect to u and v and the second powers of the derivatives with respect to w, we obtain

$$\left.\begin{aligned} \bar{e}_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \\ \bar{e}_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \\ \bar{\gamma}_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y}. \end{aligned}\right\} \tag{64.9}$$

Eliminating u and v by means of differentiation we obtain

$$\frac{\partial^2 \bar{\varepsilon}_x}{\partial y^2} + \frac{\partial^2 \bar{\varepsilon}_y}{\partial x^2} - \frac{\partial^2 \bar{\gamma}_{xy}}{\partial x \partial y} = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2}. \tag{64.10}$$

The components e'_x, e'_y, γ'_{xy} depend on the bending of the plate and are determined from Eqs. (61.2). The total stresses $\sigma_x, \sigma_y, \tau_{xy}$

are reduced with respect to the thickness of the plate to the longitudinal forces T_x, T_y, S_{xy} and the moments M_x, M_y, H_{xy}, and the stresses τ_{xz}, τ_{yz} are reduced to the crosscut forces N_x, N_y [see (61.14) and (61.15)2. The stresses $\bar{\sigma}_x$, $\bar{\sigma}_y$, $\bar{\tau}_{xy}$ satisfy the equilibrium equations

$$\frac{\partial \bar{\sigma}_x}{\partial x} + \frac{\partial \bar{\tau}_{xy}}{\partial y} = 0, \quad \frac{\partial \bar{\tau}_{xy}}{\partial x} + \frac{\partial \bar{\sigma}_y}{\partial y} = 0, \tag{64.11}$$

from which it follows that can be expressed in terms of the stress function F:

$$\bar{\sigma}_x = \frac{\partial^2 F}{\partial y^2}, \quad \bar{\sigma}_y = \frac{\partial^2 F}{\partial x^2}, \quad \bar{\tau}_{xy} = -\frac{\partial^2 F}{\partial x \, \partial y}. \tag{64.12}$$

These stresses are linked with the strains by the equations of the generalized Hooke's law

$$\left.\begin{aligned} \bar{\varepsilon}_x &= \frac{1}{E_1}(\bar{\sigma}_x - \nu_1 \bar{\sigma}_y), \\ \bar{\varepsilon}_y &= \frac{1}{E_2}(\bar{\sigma}_y - \nu_2 \bar{\sigma}_x), \\ \bar{\gamma}_{xy} &= \frac{1}{G}\bar{\tau}_{xy}. \end{aligned}\right\} \tag{64.13}$$

The crosscut forces and the longitudinal forces satisfy the equation

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} + q + T_x \frac{\partial^2 w}{\partial x^2} + 2S_{xy} \frac{\partial^2 w}{\partial x \, \partial y} + T_y \frac{\partial^2 w}{\partial y^2} = 0, \tag{64.14}$$

which is obtained from a consideration of the equilibrium of a rectangular plate element as shown in Fig. 126, taking into account the force components in the direction of the z-axis which are caused by the longitudinal forces (Figs. 127 and 128).

Let us substitute (64.13) in (64.10) and express the stress in terms of the function F; in addition to this, substitute the expressions for the (61.15) in Eq. (64.14) and give the longitudinal forces in terms of the function F.

We then obtain a system of two nonlinear equations which are satisfied by the stress and deflection functions:

$$\left.\begin{aligned} &\frac{1}{E_2} \cdot \frac{\partial^4 F}{\partial x^4} + \left(\frac{1}{G} - \frac{2\nu_1}{E_1}\right) \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{1}{E_1} \cdot \frac{\partial^4 F}{\partial y^4} = \\ &\qquad = \left(\frac{\partial^2 w}{\partial x \, \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2}, \\ &D_1 \frac{\partial^4 w}{\partial x^4} + 2D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} = \\ &\qquad = q + h\left(\frac{\partial^2 F}{\partial y^2} \cdot \frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \, \partial y} \cdot \frac{\partial^2 w}{\partial x \, \partial y} + \frac{\partial^2 F}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2}\right). \end{aligned}\right\} \tag{65.15}$$

These equations were derived by G.G. Rostovtsev.*

Owing to the fact that the first equation contains nonlinear terms, their integration entails great difficulties; at present

we do not know of any case of bending for which these equations would have been solved exactly.

In the case of small deflections the nonlinear term in the first equation can be neglected and the stress function is then determined independently of the deflection.

When a homogeneous plate lies on a massive elastic base and is bended by the load q, its deflection may be considered to be small relative to the thickness so that we can obtain the deflection equation on the basis of Winkler's assumption that the reaction R of the base at a given point of the plate is proportional to the deflection at this point:**

$$R = kw, \tag{64.16}$$

k is the elastic coefficient of the base or else, the bed coefficient. The equation of a plate on an elastic base is obtained in the same way as that for a plate which does not lie on an elastic base; it must only be taken into account that in the given case the load $q - kw$ act on the element. As the result we obtain the equation for the nonorthotropic plate

$$\begin{gathered} D_{11}\frac{\partial^4 w}{\partial x^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3\,\partial y} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \\ + 4D_{26}\frac{\partial^4 w}{\partial x\,\partial y^3} + D_{22}\frac{\partial^4 w}{\partial y^4} + kw = q. \end{gathered} \tag{64.17}$$

For an orthotropic plate

$$D_1\frac{\partial^4 w}{\partial x^4} + 2D_3\frac{\partial^4 w}{\partial x^2\,\partial y^2} + D_2\frac{\partial^4 w}{\partial y^4} + kw = q. \tag{64.18}$$

When a plate lying on an elastic base is exposed to the action of both a load q and longitudinal forces, the deflection equation is obtained from (64.6) and (64.7) when we add the term kw to the left-hand sides.

§65. DETERMINATION OF THE RIGIDITIES OF BENDING AND TORSION OF PLATES AND OF CORRUGATED AND STRESSED EDGES

The fundamental quantities which characterize the elastic properties of a homogeneous orthotropic plate on bending are its principal rigidities D_1, D_2 and D_k and the Poisson coefficients ν_1, ν_2. In order to calculate the rigidities of bending and torsion of an orthotropic homogeneous plate we must know its thickness and the main elastic constants $E_1, E_2, \nu_1, \nu_2, G$.

Note that the rigidity can be represented in the form

$$D_1 = \frac{E_1 J}{1 - \nu_1\nu_2}, \quad D_2 = \frac{E_2 J}{1 - \nu_1\nu_2}, \quad D_k = GJ, \tag{65.1}$$

where $J = \frac{h^3}{12}$ are the moments of inertia of beam sections cut out of the plate in such a way that the directions of their axes coincide with the principal directions. For plates of constant

thickness the cross sections of all these beams are rectangles whose height is equal to the thickness h of the plate, with unit base area.

In practice, plates of isotropic material corrugated and reinforced by crimp or often provided with stiffening ribs may be considered approximately as homogeneous and orthotropic. Let us give examples of determination of the main rigidities of some of these plates.

1. Corrugated plates. A corrugated plate of isotropic material, i.e., a plate which is rippled in one direction, may be considered approximately as orthotropic and homogeneous (of course, if the number of corrugation waves along the side is high enough, or, in other words, if the corrugation wavelength is small compared with the side length of the plate).

The determination of the rigidity of the corrugated plate (or a plate reinforced by a close arrangement of parallel stiffening ribs) is reduced to the calculation of the moments of inertia per unit length, for the fundamental cross sections [see Eq. (65.1)].

Let us consider a corrugated plate as shown in Fig. 129. The principal directions of it are the direction of corrugation and the directions perpendicular to it (x and y in Fig. 29). We introduce the denotations: l is the chord of a half-wave, s the length of arc of a half-wave, h the plate thickness, E, ν Young's modulus and Poisson's coefficient of the material. Let us assume the cross-sectional profile of the corrugation to be sinusoidal

$$z = H \sin \frac{\pi x}{l}. \tag{65.2}$$

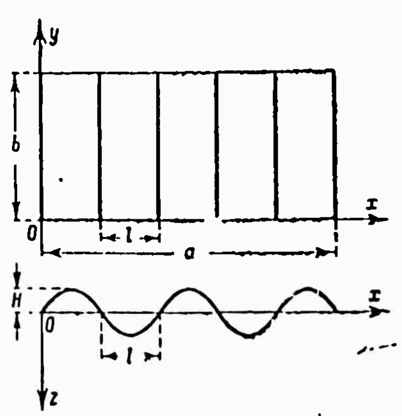

Fig. 129

According to Seidel, the approximate formulas for the rigidity have the form*

$$D_1 = \frac{l'}{s} \cdot \frac{Eh^3}{12(1-\nu^2)}, \quad D_2 = EJ, \quad D_3 = \frac{s}{l} \cdot \frac{Eh^3}{12(1+\nu)}, \tag{65.3}$$

where J is the mean moment of inertia of a plate cross section in the xz-plane, equal to

$$J = 0,5hH^2\left[1 - \frac{0,81}{1+2,5\left(\frac{H}{2l}\right)^2}\right]. \tag{65.4}$$

2. Plate reinforced by a close arrangement of stiffening ribs. For an isotropic plate which is reinforced on two sides by parallel stiffening ribs (Fig. 130), the rigidity is determined according to the formulas

$$\left.\begin{aligned} D_1 = D_3 &= \frac{Eh^3}{12(1-\nu^2)}, \\ D_2 &= \frac{Eh^3}{12(1-\nu^2)} + \frac{E'J}{d}, \end{aligned}\right\} \tag{65.5}$$

where J is the moment of inertia of a rib cross section with respect to the axis lying in the median surface, E, ν are Young's modulus and Poisson's coefficient for the plate material, E' is Young's modulus of the rib material, d is the distance between the ribs (the ribs are supposed to be equal and arranged at like spacings).

When the ribs are arranged only on one side of the plate, the moments of inertia of the cross sections must be calculated relative to the lines passing through the centers of gravity of the cross sections which will not lie in the median surface.

For a plate which has been reinforced in two orthogonal directions by ribs arranged symmetrically on both sides of the median surface, the rigidity can be calculated according to the formulas

$$\left.\begin{aligned} D_1 &= \frac{Eh^3}{12(1-\nu^2)} + \frac{E'J_1}{d_1}, \\ D_2 &= \frac{Eh^3}{12(1-\nu^2)} + \frac{E''J_2}{d_2}, \\ D_3 &= \frac{Eh^3}{12(1-\nu^2)}. \end{aligned}\right\} \tag{65.6}$$

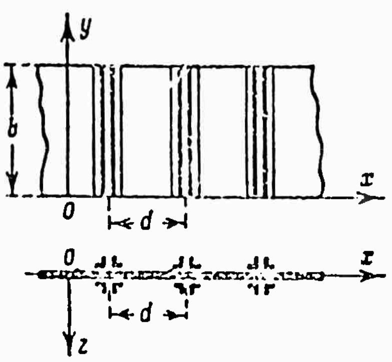

Fig. 130

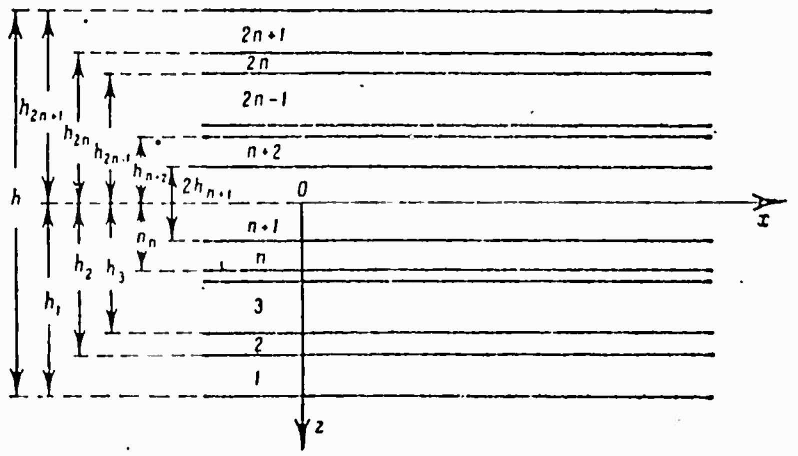

Fig. 131

The rib axes are assumed to be parallel to the principal directions; E', E'' are Young's moduli of the materials of which the ribs parallel to the axes y and x consists, J_1 and J_2 are the moments of inertia of the cross sections of these ribs relative to the lines passing through the centers of gravity of the cross sections and which, owing to the symmetry, lie in the median surfaces, d_1, d_2 are the distances between the ribs parallel to the y-axis and the x-axis.

The formulas for the plate's rigidities in the case of reinforcement by corrugation and when the plate is considered to be orthotropic are contained in the book by S.N. Kan and I.A. Sverdlov mentioned previously (pages 255-257).

§66. DETERMINATION OF THE RIGIDITIES AND OF THE REDUCED MODULI OF LAMINATED PLATES

An inhomogeneous plate which consists of elastic anisotropic layers glued or soldered together can in a series of case be considered as homogeneous and anisotropic. First of all this holds true for plates of symmetrical structure which consists of an odd number of homogeneous layers. Knowing the elastic constants of each layer we can determine the rigidities of bending and torsion for all plates, and also the reduced elastic moduli for the plates deformed by forces which were acting on their median surfaces.

Let us consider a multilayer plate consisting of an odd number of homogeneous anisotropic layers arranged symmetrically with respect to the middle layer. More precisely, two layers of equal thicknesses and of the same elastic properties are attached to either side of the middle layer; another two equal layers are attached to the outer surfaces of the former and so on such that the whole plate is a body symmetrical in both geometrical and elastic respects (relative to the mid-plane).

We shall suppose that the adjacent layers are prevented from sliding on the contact surfaces by means of gluing or soldering. We restrict ourselves to plates consisting of orthotropic layers whose planes of elastic symmetry are parallel to one another, one of them being parallel to the mid-plane.*

We assume the mid-plane of the middle layer which, at the same time, is the mid-plane of the plate as a whole, lying in the xy-plane of coordinates and the directions of x and y being coincident with the principal directions of elasticity of the layer.

We introduce the following denotations: $2n + 1$ is the number of layer (we count them from bottom to top so that the lowest layer is the first, the middle layer is number $n + 1$ and the top layer is the $2n + 1$st layer); h is the total thickness of the plate, $h_1, h_2, \ldots, h_{n+1}, \ldots, h_{2n}, h_{2n+1}$ are the distances from the mid-plane to the surfaces of the various layers (Fig. 131); $\delta_1 = h_1 - h_2, \delta_2 = h_2 - h_3, \ldots, \delta_n = h_n - h_{n+1}, \delta_{n+1} = 2h_{n+1}$ denote the layer thicknesses from the first to the $n + 1$st including; $E_1^{(m)}, E_2^{(m)}, G^{(m)}, \nu_1^{(m)}, \nu_2^{(m)}$ are the main Young's moduli, the shearing modulus and Poisson's coefficients of the orthotropic layer number m; $E^{(m)}, \nu^{(m)}$ is Young's

modulus and Poisson's coefficient of the isotropic layer number m; $\sigma_x^{(m)}, \sigma_y^{(m)}, \tau_{xy}^{(m)}, \tau_{xz}^{(m)}, \tau_{yz}^{(m)}$ are the stress components in this layer, $w(x, y)$ is the deflection of the median surface. Owing to the symmetry $E_1^{(2n+1)} = E_1^{(1)}, E_1^{(2n)} = E_1^{(2)}$ etc.; the same conditions are satisfied by the other elastic constants.

Investigating the generalized plane state of stress of a multilayer plate we introduce in our considerations the reduced moduli and Poisson coefficients of the plate in a plane state of stress, which are denoted in the following by $\bar{E}_1, \bar{E}_2, \bar{G}, \bar{\nu}_1, \bar{\nu}_2$. Moreover, considering (in the next section) the bending of a plate which consists of analogous orthotropic layers with different orienrations of the principal directions, we introduce the new denotations $E_1', E_2', G', \nu_1', \nu_2'$, the reduced moduli and Poisson coefficients for bending.

Consider a multilayer plate bent by a normal load q, which is distributed on one of the plane surfaces, and by the moments m and the forces p distributed along the edge. We want to determine the rigidities of bending and torsion and the stresses in each layer, based on the supposition that for the multilayer plate the hypothesis of the straight normals remains applicable, according to which the relative elongations $\varepsilon_x^{(m)}, \varepsilon_y^{(m)}$ and the shear $\gamma_{xy}^{(m)}$ can be expressed in terms of the deflection of the median surface in the following way (see §61):

$$\varepsilon_x^{(m)} = -z\frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_y^{(m)} = -z\frac{\partial^2 w}{\partial y^2}, \quad \gamma_{xy}^{(m)} = -2z\frac{\partial^2 w}{\partial x\,\partial y}. \tag{66.1}$$

The results are the following.

The plate which is bent as a whole and considered homogeneous and orthotropic, possesses the rigidities D_1, D_2, D_k and the Poisson coefficients ν_1' and ν_2', which are determined by the formulas

$$\left.\begin{aligned}
D_1 &= \frac{2}{3}\left[\sum_{m=1}^{n}\frac{E_1^{(m)}}{1-\nu_1^{(m)}\nu_2^{(m)}}\left(h_m^3 - h_{m+1}^3\right) + \frac{E_1^{(n+1)}h_{n+1}^3}{1-\nu_1^{(n+1)}\nu_2^{(n+1)}}\right],\\
D_2 &= \frac{2}{3}\left[\sum_{m=1}^{n}\frac{E_2^{(m)}}{1-\nu_1^{(m)}\nu_2^{(m)}}\left(h_m^3 - h_{m+1}^3\right) + \frac{E_2^{(n+1)}h_{n+1}^3}{1-\nu_1^{(n+1)}\nu_2^{(n+1)}}\right],\\
D_k &= \frac{2}{3}\left[\sum_{m=1}^{n}G^{(m)}\left(h_m^3 - h_{m+1}^3\right) + G^{(n+1)}h_{n+1}^3\right],\\
\nu_1' &= \frac{2}{3D_2}\left[\sum_{m=1}^{n}\frac{E_2^{(m)}\nu_1^{(m)}}{1-\nu_1^{(m)}\nu_2^{(m)}}\left(h_m^3 - h_{m+1}^3\right) + \frac{E_2^{(n+1)}\nu_1^{(n+1)}h_{n+1}^3}{1-\nu_1^{(n+1)}\nu_2^{(n+1)}}\right],\\
\nu_2' &= \nu_1'\frac{D_2}{D_1}.
\end{aligned}\right\} \tag{66.2}$$

The moments and crosscut forces are determined according to Eqs. (61.14) and (61.15) and the deflection w satisfies Eq. (62.3). The boundary conditions on the edge of a multilayer plate do not differ from the boundary conditions on the edge of a homogeneous

plate given in §63.

The stress components, however, will not be connected with the moments and crosscut forces by the same simple relations (61.12) which hold good in the case of the homogeneous thin sheet. The stress components in the layer number m are determined by the formulas

$$\left.\begin{aligned}\sigma_x^{(m)} &= -z\frac{E_1^{(m)}}{1-\nu_1^{(m)}\nu_2^{(m)}}\left(\frac{\partial^2 w}{\partial x^2}+\nu_2^{(m)}\frac{\partial^2 w}{\partial y^2}\right),\\ \sigma_y^{(m)} &= -z\frac{E_2^{(m)}}{1-\nu_1^{(m)}\nu_2^{(m)}}\left(\frac{\partial^2 w}{\partial y^2}+\nu_1^{(m)}\frac{\partial^2 w}{\partial x^2}\right),\\ \tau_{xy}^{(m)} &= -z\cdot 2G^{(m)}\frac{\partial^2 w}{\partial x\,\partial y};\end{aligned}\right\} \tag{66.3}$$

$$\left.\begin{aligned}\tau_{xz}^{(m)} = \frac{z^2}{2}\cdot\frac{\partial}{\partial x}\left[\frac{E_1^{(m)}}{1-\nu_1^{(m)}\nu_2^{(m)}}\cdot\frac{\partial^2 w}{\partial x^2}+\left(\frac{E_1^{(m)}\nu_2^{(m)}}{1-\nu_1^{(m)}\nu_2^{(m)}}+2G^{(m)}\right)\frac{\partial^2 w}{\partial y^2}\right]-\\ -\frac{h^2}{2}\cdot\frac{\partial}{\partial x}\left[C_{11}^{(m)}\frac{\partial^2 w}{\partial x^2}+\left(C_{12}^{(m)}+2C_{66}^{(m)}\right)\frac{\partial^2 w}{\partial y^2}\right],\\ \tau_{yz}^{(m)} = \frac{z^2}{2}\cdot\frac{\partial}{\partial y}\left[\left(\frac{E_1^{(m)}\nu_2^{(m)}}{1-\nu_1^{(m)}\nu_2^{(m)}}+2G^{(m)}\right)\frac{\partial^2 w}{\partial x^2}+\frac{E_2^{(m)}}{1-\nu_1^{(m)}\nu_2^{(m)}}\cdot\frac{\partial^2 w}{\partial y^2}\right]-\\ -\frac{h^2}{2}\cdot\frac{\partial}{\partial y}\left[\left(C_{12}^{(m)}+2C_{66}^{(m)}\right)\frac{\partial^2 w}{\partial x^2}+C_{22}^{(m)}\frac{\partial^2 w}{\partial y^2}\right].\end{aligned}\right\} \tag{66.4}$$

The denotations are the following:

$$\left.\begin{aligned}&C_{11}^{(1)}=\frac{E_1^{(1)}}{1-\nu_1^{(1)}\nu_2^{(1)}},\quad C_{22}^{(1)}=\frac{E_2^{(1)}}{1-\nu_1^{(1)}\nu_2^{(1)}},\\ &C_{12}^{(1)}=\frac{E_1^{(1)}\nu_2^{(1)}}{1-\nu_1^{(1)}\nu_2^{(1)}},\quad C_{66}^{(1)}=G^{(1)},\\ &C_{11}^{(m)}=\frac{1}{h^2}\left[\sum_{k=1}^{m-1}\frac{E_1^{(k)}}{1-\nu_1^{(k)}\nu_2^{(k)}}\left(h_k^2-h_{k+1}^2\right)+\frac{E_1^{(m)}h_m^2}{1-\nu_1^{(m)}\nu_2^{(m)}}\right],\\ &C_{22}^{(m)}=\frac{1}{h^2}\left[\sum_{k=1}^{m-1}\frac{E_2^{(k)}}{1-\nu_1^{(k)}\nu_2^{(k)}}\left(h_k^2-h_{k+1}^2\right)+\frac{E_2^{(m)}h_m^2}{1-\nu_1^{(m)}\nu_2^{(m)}}\right],\\ &C_{12}^{(m)}=\frac{1}{h^2}\left[\sum_{k=1}^{m-1}\frac{E_1^{(k)}\nu_2^{(k)}}{1-\nu_1^{(k)}\nu_2^{(k)}}\left(h_k^2-h_{k+1}^2\right)+\frac{E_1^{(m)}\nu_2^{(m)}h_m^2}{1-\nu_1^{(m)}\nu_2^{(m)}}\right],\\ &C_{66}^{(m)}=\frac{1}{h^2}\left[\sum_{k=1}^{m-1}G^{(k)}\left(h_k^2-h_{k+1}^2\right)+G^{(m)}h_m^2\right]\\ &\qquad (m=2,\,3,\,\ldots,\,n+1).\end{aligned}\right\} \tag{66.5}$$

The stresses in the symmetrical layers are distributed symmetrically.

If the plate consists of isotropic layers arranged symmetrically relative to the middle layer, it is bent like an isotropic plate with the rigidity D and the Poisson coefficient ν' which are determined by the formulas

$$
\left.
\begin{aligned}
D &= \frac{2}{3}\left[\sum_{m=1}^{n}\frac{E^{(m)}}{1-(\nu^{(m)})^2}\left(h_m^3-h_{m+1}^3\right)+\frac{E^{(n+1)}h_{n+1}^3}{1-(\nu^{(n+1)})^2}\right],\\
\nu' &= \frac{2}{3D}\left[\sum_{m=1}^{n}\frac{E^{(m)}\nu^{(m)}}{1-(\nu^{(m)})^2}\left(h_m^3-h_{m+1}^3\right)+\frac{E^{(n+1)}\nu^{(n+1)}h_{n+1}^3}{1-(\nu^{(n+1)})^2}\right].
\end{aligned}
\right\} \quad (66.6)
$$

The following formulas are obtained for the stress components in layer number m:

$$
\left.
\begin{aligned}
\sigma_x^{(m)} &= -z\frac{E^{(m)}}{1-(\nu^{(m)})^2}\left(\frac{\partial^2 w}{\partial x^2}+\nu^{(m)}\frac{\partial^2 w}{\partial y^2}\right),\\
\sigma_y^{(m)} &= -z\frac{E^{(m)}}{1-(\nu^{(m)})^2}\left(\frac{\partial^2 w}{\partial y^2}+\nu^{(m)}\frac{\partial^2 w}{\partial x^2}\right),\\
\tau_{xy}^{(m)} &= -z\frac{E^{(m)}}{1+\nu^{(m)}}\cdot\frac{\partial^2 w}{\partial x\,\partial y};
\end{aligned}
\right\} \quad (66.7)
$$

$$
\left.
\begin{aligned}
\tau_{xz}^{(m)} &= \frac{1}{2}\left[z^2\frac{E^{(m)}}{1-(\nu^{(m)})^2}-h^2C^{(m)}\right]\frac{\partial}{\partial x}(\nabla^2 w),\\
\tau_{yz}^{(m)} &= \frac{1}{2}\left[z^2\frac{E^{(m)}}{1-(\nu^{(m)})^2}-h^2C^{(m)}\right]\frac{\partial}{\partial y}(\nabla^2 w).
\end{aligned}
\right\} \quad (66.8)
$$

Here

$$
\left.
\begin{aligned}
\nabla^2 &= \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\\
C^{(1)} &= \frac{E^{(1)}}{1-(\nu^{(1)})^2},\\
C^{(m)} &= \frac{1}{h^2}\left[\sum_{k=1}^{m-1}\frac{E^{(k)}}{1-(\nu^{(k)})^2}\left(h_k^2-h_{k+1}^2\right)+\frac{E^{(m)}h_m^2}{1-(\nu^{(m)})^2}\right]\\
&(m=2,\ 3,\ \ldots,\ n+1).
\end{aligned}
\right\} \quad (66.9)
$$

In particular, for a plate which consists of three orthotropic plates of like thicknesses $h/3$, which are arranged symmetrically, we obtain the rigidity formulas

$$
\left.
\begin{aligned}
D_1 &= \frac{1}{12}\left(\frac{h}{3}\right)^3\left(\frac{26E_1^{(1)}}{1-\nu_1^{(1)}\nu_2^{(1)}}+\frac{E_1^{(2)}}{1-\nu_1^{(2)}\nu_2^{(2)}}\right),\\
D_2 &= \frac{1}{12}\left(\frac{h}{3}\right)^3\left(\frac{26E_2^{(1)}}{1-\nu_1^{(1)}\nu_2^{(1)}}+\frac{E_2^{(2)}}{1-\nu_1^{(2)}\nu_2^{(2)}}\right),\\
D_k &= \frac{1}{12}\left(\frac{h}{3}\right)^3\left(26G^{(1)}+G^{(2)}\right).
\end{aligned}
\right\} \quad (66.10)
$$

These expressions can also be written in a more illustrative form when we use $D_1^{(1)}, D_2^{(1)}, D_k^{(1)}$ to denote the rigidities of the outermost layers, which are considered as isolated plates, and $D_1^{(2)}, D_2^{(2)}, D_k^{(2)}$ are the rigidities of the middle layer. Instead of (66.10) we have

$$\left.\begin{aligned} D_1 &= 26D_1^{(1)} + D_1^{(2)}, \\ D_2 &= 26D_2^{(1)} + D_2^{(2)}, \\ D_k &= 26D_k^{(1)} + D_k^{(2)}. \end{aligned}\right\} \quad (66.11)$$

If the whole layer is isotropic,

$$\left.\begin{aligned} D &= 26D^{(1)} + D^{(2)}, \\ \nu' &= \frac{1}{D}\left(26D^{(1)}\nu^{(1)} + D^{(2)}\nu^{(2)}\right), \end{aligned}\right\} \quad (66.12)$$

where $D^{(1)}, \nu^{(1)}$ denote the rigidity and Poisson's coefficient of the outer layers, $D^{(2)}, \nu^{(2)}$ is the same for the middle layer.

Let us now consider the generalized plane state of stress of a multilayer symmetrical plate. An approximate expression of the reduced moduli for the plane state of stress is obtained under the supposition that the strain components $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ are the same for all layers and that they do not depend on z, while the stress components in the given layer do not vary along the thickness of it. The stresses averaged with respect to the thickness of the whole plate are given by the formulas

$$\left.\begin{aligned} \bar{\sigma}_x &= \frac{1}{h}\left[2\sum_{m=1}^{n}\sigma_x^{(m)}\delta_m + \sigma_x^{(n+1)}\delta_{n+1}\right], \\ \bar{\sigma}_y &= \frac{1}{h}\left[2\sum_{m=1}^{n}\sigma_y^{(m)}\delta_m + \sigma_y^{(n+1)}\delta_{n+1}\right], \\ \bar{\tau}_{xy} &= \frac{1}{h}\left[2\sum_{m=1}^{n}\tau_{xy}^{(m)}\delta_m + \tau_{xy}^{(n+1)}\delta_{n+1}\right]. \end{aligned}\right\} \quad (66.13)$$

Taking these expressions into account and using the equations of the generalized Hooke's law for each layer we obtain the relations

$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{\bar{E}_1}\left(\bar{\sigma}_x - \bar{\nu}_1\bar{\sigma}_y\right), \\ \varepsilon_y &= \frac{1}{\bar{E}_2}\left(\bar{\sigma}_y - \bar{\nu}_2\bar{\sigma}_x\right), \\ \gamma_{xy} &= \frac{1}{\bar{G}}\bar{\tau}_{xy}. \end{aligned}\right\} \quad (66.14)$$

The reduced moduli and Poisson coefficients entering these equations are determined by the formulas

$$\left.\begin{aligned}
\bar{E}_1 &= \frac{1-\bar{\nu}_1\bar{\nu}_2}{h}\left(2\sum_{m=1}^{n}\frac{E_1^{(m)}\delta_m}{1-\nu_1^{(m)}\nu_2^{(m)}}+\frac{E_1^{(n+1)}\delta_{n+1}}{1-\nu_1^{(n+1)}\nu_2^{(n+1)}}\right),\\
\bar{E}_2 &= \frac{1-\bar{\nu}_1\bar{\nu}_2}{h}\left(2\sum_{m=1}^{n}\frac{E_2^{(m)}\delta_m}{1-\nu_1^{(m)}\nu_2^{(m)}}+\frac{E_2^{(n+1)}\delta_{n+1}}{1-\nu_1^{(n+1)}\nu_2^{(n+1)}}\right),\\
\bar{G} &= \frac{1}{h}\left(2\sum_{m=1}^{n}G^{(m)}\delta_m+G^{(n+1)}\delta_{n+1}\right),\\
\bar{\nu}_1 &= \frac{2\sum\limits_{m=1}^{n}\frac{E_2^{(m)}\nu_1^{(m)}\delta_m}{1-\nu_1^{(m)}\nu_2^{(m)}}+\frac{E_2^{(n+1)}\nu_1^{(n+1)}\delta_{n+1}}{1-\nu_1^{(n+1)}\nu_2^{(n+1)}}}{2\sum\limits_{m=1}^{n}\frac{E_2^{(m)}\delta_m}{1-\nu_1^{(m)}\nu_2^{(m)}}+\frac{E_2^{(n+1)}\delta_{n+1}}{1-\nu_1^{(n+1)}\nu_2^{(n+1)}}},\\
\bar{\nu}_2 &= \bar{\nu}_1\frac{\bar{E}_2}{\bar{E}_1}.
\end{aligned}\right\} \tag{66.15}$$

A.L. Rabinovich studied in detail a series of particular cases of multilayer plates which are of interest in practice.* Apart from plates which consist of layers whose moduli and thicknesses are given with a certain arbitrariness, A.L. Rabinovich considered plates consisting of veneer sheets and delta wood, which were bent by loads and forces acting in the median surface.

§67. RIGIDITIES AND REDUCED MODULI OF PLATES CONSISTING OF SIMILAR LAYERS

All formulas of the preceding section become simpler when the layers are of equal thicknesses as in the case given:

$$\left.\begin{aligned}
h_1-h_2=h_2-h_3=\ldots=h_n-h_{n+1}=h/(2n+1),\\
\delta_1=\delta_2=\ldots=\delta_n=h/(2n+1).
\end{aligned}\right\} \tag{67.1}$$

If, moreover, all layers are of one and the same orthotropic material and are arranged in such a way that in adjacent layers analogous principal directions are orthogonal, all summations in the formulas of §66 are easy to carry out so that the final formulas will not contain any sums.

Let us denote by E_1, E_2, G, ν_1 and ν_2 Young's moduli, the shearing modulus and Poisson's coefficients of the outer layers for the principal directions x and y and by λ the ratio of Young's moduli

$$\lambda = E_2/E_1. \tag{67.2}$$

For all layers with odd numbers we have

$$\begin{gathered}
E_1^{(m)}=E_1,\quad E_2^{(m)}=E_2,\quad G^{(m)}=G,\quad \nu_1^{(m)}=\nu_1,\quad \nu_2^{(m)}=\nu_2\\
(m=1,\ 3,\ 5,\ \ldots,\ 2n+1).
\end{gathered} \tag{67.3}$$

For all layers with even numbers

$$E_1^{(m)} = E_2, \quad E_2^{(m)} = E_1, \quad G^{(m)} = G, \quad \nu_1^{(m)} = \nu_2, \quad \nu_2^{(m)} = \nu_1$$
$$(m = 2,\ 4,\ 6,\ \ldots,\ 2n). \tag{67.4}$$

Substituting the values of (67.1)-(67.4) in Eqs. (66.2) we obtain expressions for the rigidities of the plate given. Introducing the reduced moduli for bending E_1', E_2', G and the reduced Poisson coefficients ν_1', ν_2' we can write the rigidity formulas in the same way as for a homogeneous orthotropic plate*

$$\left.\begin{array}{c} D_1 = \dfrac{E_1' h^3}{12(1 - \nu_1'\nu_2')}, \quad D_2 = \dfrac{E_2' h^3}{12(1 - \nu_1'\nu_2')}, \quad D_k = \dfrac{G' h^3}{12}, \\ D_3 = D_1\nu_2' + 2D_k. \end{array}\right\} \tag{67.5}$$

Here

$$\left.\begin{array}{l} E_1' = \dfrac{E_1}{2(2n+1)^3} \cdot \dfrac{1 - \nu_1'\nu_2'}{1 - \nu_1\nu_2} \{(2n+1)^3(1+\lambda) + \\ \qquad + [3(2n+1)^2 - 2](1-\lambda)\}, \\ E_2' = \dfrac{E_2}{2(2n+1)^3} \cdot \dfrac{1 - \nu_1'\nu_2'}{\lambda(1 - \nu_1\nu_2)} \{(2n+1)^3(1+\lambda) - \\ \qquad - [3(2n+1)^2 - 2](1-\lambda)\}, \\ G' = G, \\ \nu_1' = \dfrac{2(2n+1)^3\nu_2}{(2n+1)^3(1+\lambda) - [3(2n+1)^2 - 2](1-\lambda)}, \\ \nu_2' = \dfrac{2(2n+1)^3\nu_2}{(2n+1)^3(1+\lambda) + [3(2n+1)^2 - 2](1-\lambda)}, \\ E_1'\nu_2' = E_2'\nu_1' = E_1\nu_2 \dfrac{1 - \nu_1'\nu_2'}{1 - \nu_1\nu_2} = E_2\nu_1 \dfrac{1 - \nu_1'\nu_2'}{1 - \nu_1\nu_2}. \end{array}\right\} \tag{67.6}$$

The moduli given for a plane state of stress and the reduced Poisson coefficients for such a plate are determined from formulas obtained from (66.15):

$$\left.\begin{array}{l} \bar{E}_1 = E_1 \dfrac{1 - \bar{\nu}_1\bar{\nu}_2}{1 - \nu_1\nu_2} \cdot \dfrac{n + 1 + \lambda n}{2n+1}, \quad \bar{E}_2 = E_1 \dfrac{1 - \bar{\nu}_1\bar{\nu}_2}{1 - \nu_1\nu_2} \cdot \dfrac{n + \lambda + \lambda n}{2n+1}, \\ \bar{G} = G, \quad \bar{\nu}_1 = \nu_2 \dfrac{2n+1}{n + \lambda + \lambda n}, \quad \bar{\nu}_2 = \nu_2 \dfrac{2n+1}{n + 1 + \lambda n}. \end{array}\right\} \tag{67.7}$$

The given moduli and Poisson coefficients in the case of bending can be given in terms of the reduced moduli and coefficients in a plane state of stress. We arrive at the result:*

$$\left.\begin{array}{l} E_1' = \bar{E}_1 \dfrac{1 - \nu_1'\nu_2'}{1 - \bar{\nu}_1\bar{\nu}_2} \cdot \dfrac{(2n+1)^3(1+\lambda) + [3(2n+1)^2 - 2](1-\lambda)}{2(2n+1)^3(n + 1 + \lambda n)}, \\ E_2' = \bar{E}_2 \dfrac{1 - \nu_1'\nu_2'}{1 - \bar{\nu}_1\bar{\nu}_2} \cdot \dfrac{(2n+1)^3(1+\lambda) - [3(2n+1)^2 - 2](1-\lambda)}{2(2n+1)^3(n + \lambda + \lambda n)}, \\ G' = \bar{G} = G, \\ \nu_1' = \bar{\nu}_1 \dfrac{2(2n+1)^3(n + \lambda + \lambda n)}{(2n+1)^3(1+\lambda) - [3(2n+1)^2 - 2](1-\lambda)}, \\ \nu_2' = \nu_1' \dfrac{E_2'}{E_1'}. \end{array}\right\} \tag{67.8}$$

By way of example we consider a three-layer wooden veneer sheet flued together with a bakelite glue. The numerical values of the reduced moduli and coefficients in a plane state of stress were already given in §11. The reduced Young's moduli and modulus of shear for the principal directions, taken from the "Spravochnik aviakonstruktora" [Handbook for the Aircraft Designer] are equal to $1.2 \cdot 10^5$, $0.6 \cdot 10^5$ and $0.07 \cdot 10^5$ kg/cm² and the Poisson coefficients calculated from tabulated values are equal to 0.071 and 0.036. For birchwood the ratio between the smaller and the higher Young's moduli is equal to about 0.05.* Considering this value as basic and using Eqs. (67.8) and (67.5) on the assumption that $n=1,\ 2n+1=3,$ we obtain the following results.

Case 1. The x-axis is parallel to the fibers of the sheet.

$$\left.\begin{array}{lll} \bar{E}_1 = 1{,}2 \cdot 10^5, & \bar{E}_2 = 0{,}6 \cdot 10^5, & \bar{G} = 0{,}07 \cdot 10^5, \\ \bar{\nu}_1 = 0{,}071, & \bar{\nu}_2 = 0{,}036, & \lambda = 0{,}05; \\ E_1' = 1{,}69 \cdot 10^5, & E_2' = 0{,}14 \cdot 10^5, & G' = 0{,}07 \cdot 10^5, \\ \nu_1' = 0{,}31, & \nu_2' = 0{,}026, & \dfrac{E_1'}{E_2'} = \dfrac{\nu_1'}{\nu_2'} = 12{,}1; \end{array}\right\} \quad (67.9)$$

$$\left.\begin{array}{lll} D_1 = 1{,}70 \cdot 10^5 \dfrac{h^3}{12}, & D_2 = 0{,}14 \cdot 10^5 \dfrac{h^3}{12}, & \dfrac{D_1}{D_2} = 12{,}1, \\ D_3 = 0{,}183 \cdot 10^5 \dfrac{h^3}{12}, & \dfrac{D_3}{D_2} = 1{,}307, & D_k = 0{,}07 \cdot 10^5 \dfrac{h^3}{12}. \end{array}\right\} \quad (67.10)$$

The complex parameters, i.e., the roots of the equation

$$\mu^4 + 2\frac{D_3}{D_2}\mu^2 + \frac{D_1}{D_2} = 0, \quad (67.11)$$

will not be purely imaginary as in the case of the plane problem. Solving Eq. (67.11) we obtain

$$\mu_1 = 1{,}04 + 1{,}55i, \quad \mu_2 = -1{,}04 + 1{,}55i. \quad (67.12)$$

Case 2. The x-axis is perpendicular to the fibers of the sheet.

$$\left.\begin{array}{lll} \bar{E}_1 = 0{,}6 \cdot 10^5, & \bar{E}_2 = 1{,}2 \cdot 10^5, & \bar{G} = 0{,}07 \cdot 10^5, \\ \bar{\nu}_1 = 0{,}036, & \bar{\nu}_2 = 0{,}071, & \lambda = 20; \\ E_1' = 0{,}14 \cdot 10^5, & E_2' = 1{,}69 \cdot 10^5, & G' = 0{,}07 \cdot 10^5, \\ \nu_1' = 0{,}026, & \nu_2' = 0{,}31, & \dfrac{E_1'}{E_2'} = \dfrac{\nu_1'}{\nu_2'} = 0{,}0824; \end{array}\right\} \quad (67.13)$$

$$\left.\begin{array}{lll} D_1 = 0{,}14 \cdot 10^5 \dfrac{h^3}{12}, & D_2 = 1{,}70 \cdot 10^5 \dfrac{h^3}{12}, & \dfrac{D_1}{D_2} = 0{,}0824, \\ D_3 = 0{,}183 \cdot 10^5 \dfrac{h^3}{12}, & \dfrac{D_3}{D_2} = 0{,}108, & D_k = 0{,}07 \cdot 10^5 \dfrac{h^3}{12}. \end{array}\right\} \quad (67.14)$$

For complex parameters we obtain the following values

$$\mu_1 = 0{,}299 + 0{,}414i, \quad \mu_2 = -0{,}299 + 0{,}414i. \quad (67.15)$$

Considering numerical examples in his paper, A.L. Rabinovich

used somewhat different initial data assuming for Young's modulus and the modulus of shear in a three-layer birch veneer $1.3 \cdot 10^5$ and $0.08 \cdot 10^5$ kg/cm^2. The numerical values obtained for the reduced moduli in the case of bending were also different from our values but this difference was relatively small (see page 22 of the paper referred to).

It must be noted that the reduced Young's moduli in a plane state of stress may differ essentially in their values from the reduced moduli in the case of bending for the same laminated material as can be seen from the numerical example considered. When for a three-layer birch veneer the ratio of the first moduli for the principal directions is equal to two, the ratio of the second moduli is considerably higher: it is equal to about 12.

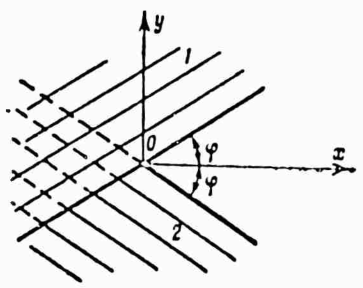

Fig. 132

Considering (in the following chapters) some particular cases of bending and rigidity, we give in most cases the results of calculations for a plate with these reduced moduli and rigidities as obtained above for the three-layer birch veneer [see (67.9)-(67.15)]. For the sake of brevity we shall call in the following "veneer." Certain authors (Ya.I. Sekerzh-Zen'kovich, L.I. Balabukh et al.), in papers on the rigidity of anisotropic plates published until 1940, used other values for the reduced moduli and coefficients of a three-layer birch veneer, namely

$$\left.\begin{array}{l} E_1' = 1{,}4 \cdot 10^5 \mathrm{kg/cm^2} E_2' = \dfrac{E_1'}{12},\ G' = 0{,}12 \cdot 10^5 \mathrm{kg/cm^2} \\ \nu_1' = 0{,}46,\quad \nu_2' = \dfrac{\nu_1'}{12} \end{array}\right\} \qquad (67.16)$$

(the x-axis is parallel to the fibers of the sheet). Discussing the contents of these papers we shall give the results of calculations carried out by the authors for plates of a material for which the constants have the values given in Eq. (67.16).

A nonhomogeneous plate consisting of an even number of analogous orthotropic layers, under well-known conditions, will behave on bending just as a homogeneous and orthotropic plate. Consider a given plate which consists of two equally thick orthotropic layers glued together; they are assumed to display the same elastic properties and are joined in such a way that the equivalent principal directions of elasticity of these two layers make an angle of 2φ (Fig. 132).

The investigations show that such a plate behaves on bending just as a uniform and orthotropic layer. Its principal axes of elasticity, x and y, are in the directions of the bisectrices of the angles made by equivalent principal directions of the layers. Formulas (61.14) and (61.15) and Eq. (62.3) apply to such a plate and the rigidities are given by Eqs. (67.5) where

$$\left.\begin{aligned}
E_1' &= \frac{1-\nu_1'\nu_2'}{1-\nu_1\nu_2}\{E_1\cos^4\varphi + 2[E_1\nu_2 + 2G(1-\nu_1\nu_2)]\times\\
&\qquad\times\sin^2\varphi\cos^2\varphi + E_2\sin^4\varphi\},\\
E_2' &= \frac{1-\nu_1'\nu_2'}{1-\nu_1\nu_2}\{E_1\sin^4\varphi + 2[E_1\nu_2 + 2G(1-\nu_1\nu_2)]\times\\
&\qquad\times\sin^2\varphi\cos^2\varphi + E_2\cos^4\varphi\},\\
G' &= G + \frac{E_1+E_2-2E_1\nu_2-4G(1-\nu_1\nu_2)}{1-\nu_1\nu_2}\sin^2\varphi\cos^2\varphi,\\
\nu_1' &= \frac{E_1\nu_2 + [E_1+E_2-2E_1\nu_2-4G(1-\nu_1\nu_2)]\sin^2\varphi\cos^2\varphi}{E_1\cos^4\varphi + 2[E_1\nu_2+2G(1-\nu_1\nu_2)]\sin^2\varphi\cos^2\varphi + E_2\sin^4\varphi},\\
\nu_2' &= \nu_1'\frac{E_2'}{E_1'},
\end{aligned}\right\} \qquad (67.17)$$

$E_1, E_2, G, \nu_1, \nu_2$ being the main elastic constants of each layer.*

§68. DETERMINATION OF THE RIGIDITIES OF PLATES WHOSE ELASTIC MODULI VARY AS FUNCTIONS OF THE THICKNESS

Using the hypothesis of the straight normals it is not difficult to determine the rigidities for plates whose elastic moduli vary symmetrically with the thickness (that is, the values of the moduli are the same at points which are at equal distances on either side of the median surface). In the present book, §66, we discussed in detail the bending problems of such plates. We shall now restrict ourselves to the most important results obtained for orthotropic and isotropic plates with variable moduli.

Consider a plate of constant thickness h which is orthotropic but at the same time nonhomogeneous: its moduli are the same for all points in a plane parallel to the mid-plane, but they vary with the thickness in a symmetrical manner. At each point we have three planes of elastic symmetry, one of them being parallel to the mid-plane, and at different points the corresponding planes of elastic symmetry have one and the same direction. We assume the mid-plane coincident with the xy-plane and, as usual, the axes x and y agree with the principal directions of elasticity (Fig. 133). We denote by $E_1, E_2, G, \nu_1, \nu_2$ the principal Young's moduli and Poisson's coefficients. In the case given these quantities will be even functions of z:

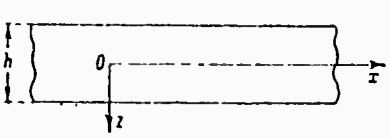

Fig. 133

$$E_1(-z)=E_1(z),\quad E_2(-z)=E_2(z),\quad G(-z)=G(z),$$
$$\nu_1(-z)=\nu_1(z),\quad \nu_2(-z)=\nu_2(z).$$

We shall consider these functions to be given. As regards the continuity of these functions we make in no way any suppositions: the functions may be unsteady and may possess discontinuities.

The results are the following. The nonhomogeneous plate is bent as a homogeneous orthotropic plate; the moments and crosscut forces in it are calculated from Eqs. (61.14) and (61.15), the deflection w of the mid-plane is determined from Eq. (62.3).

The rigidities are obtained from the formulas

$$\left.\begin{aligned}
D_1 &= 2\int_0^{h/2} \frac{E_1 z^2}{1-\nu_1\nu_2}\,dz,\\
D_2 &= 2\int_0^{h/2} \frac{E_2 z^2}{1-\nu_1\nu_2}\,dz,\\
D_k &= 2\int_0^{h/2} G z^2\,dz,\\
\nu_1' &= \frac{2}{D_3}\int_0^{h/2} \frac{E_2 \nu_1 z^2}{1-\nu_1\nu_2}\,dz,\\
\nu_2' &= \nu_1' \frac{D_3}{D_1}
\end{aligned}\right\} \qquad (68.1)$$

(ν_1', ν_2' are quantities corresponding to Poisson's coefficients for a homogeneous plate).

The stress components are determined by the formulas

$$\left.\begin{aligned}
\sigma_x &= -\frac{E_1 z}{1-\nu_1\nu_2}\left(\frac{\partial^2 w}{\partial x^2} + \nu_2 \frac{\partial^2 w}{\partial y^2}\right),\\
\sigma_y &= -\frac{E_2 z}{1-\nu_1\nu_2}\left(\frac{\partial^2 w}{\partial y^2} + \nu_1 \frac{\partial^2 w}{\partial x^2}\right),\\
\tau_{xy} &= -2Gz\frac{\partial^2 w}{\partial x\,\partial y};
\end{aligned}\right\} \qquad (68.2)$$

$$\left.\begin{aligned}
\tau_{xz} &= \frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial x^2}\int_{-h/2}^{z} \frac{E_1 z}{1-\nu_1\nu_2}\,dz + \frac{\partial^2 w}{\partial y^2}\int_{-h/2}^{z}\left(\frac{E_1\nu_2}{1-\nu_1\nu_2} + 2G\right) z\,dz\right],\\
\tau_{yz} &= \frac{\partial}{\partial y}\left[\frac{\partial^2 w}{\partial x^2}\int_{-h/2}^{z}\left(\frac{E_1\nu_2}{1-\nu_1\nu_2} + 2G\right) z\,dz + \frac{\partial^2 w}{\partial y^2}\int_{-h/2}^{z} \frac{E_2 z}{1-\nu_1\nu_2}\,dz\right].
\end{aligned}\right\} \qquad (68.3)$$

In particular, for a nonhomogeneous plate possessing the properties of isotropy, for which

$$E_1 = E_2 = E(z), \quad \nu_1 = \nu_2 = \nu(z), \quad G = \frac{E}{2(1+\nu)}, \qquad (68.4)$$

we obtain the following result: a plate is bent like an isotropic one when its rigidity D and the Poisson coefficient ν are equal to

$$\left.\begin{aligned}
D &= 2\int_0^{h/2} \frac{E z^2}{1-\nu^2}\,dz,\\
\nu' &= \frac{2}{D}\int_0^{h/2} \frac{E\nu z^2}{1-\nu^2}\,dz.
\end{aligned}\right\} \qquad (68.5)$$

For the stress components we obtain the formulas

$$
\left.\begin{aligned}
\sigma_x &= -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right),\\
\sigma_y &= -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right),\\
\tau_{xy} &= -\frac{Ez}{1+\nu}\cdot\frac{\partial^2 w}{\partial x\,\partial y};
\end{aligned}\right\}\tag{68.6}
$$

$$
\left.\begin{aligned}
\tau_{xz} &= \frac{\partial}{\partial x}(\nabla^2 w)\cdot\int\limits_{-h/2}^{z}\frac{Ez}{1-\nu^2}\,dz,\\
\tau_{yz} &= \frac{\partial}{\partial y}(\nabla^2 w)\cdot\int\limits_{-h/2}^{z}\frac{Ez}{1-\nu^2}\,dz.
\end{aligned}\right\}\tag{68.7}
$$

Example. We have a plate which is isotropic but nonhomogeneous; its Poisson coefficient ν is a constant quantity for any point of the plate and Young's modulus is a square function of the thickness:

$$E = E_0 + E_2\left(\frac{z}{h}\right)^2. \tag{68.8}$$

This plate is bent as a homogeneous isotropic one with Poisson's coefficient ν and a rigidity calculated from

$$D = \frac{E_0 h^3}{12(1-\nu^2)}\left(1 + 0{,}15\,\frac{E_2}{E_0}\right). \tag{68.9}$$

§69. CALCULATION OF THE RIGIDITIES FOR ARBITRARY DIRECTIONS

The rigidities of an anisotropic plate, D_{ij}, are quantities which depend on the directions of the coordinate axes chosen, i.e., the D_{ij} are changed on a transition from the x, y, z system of coordinates to the system x', y', z'. Let us consider the formulas for the recalculation of rigidities for the transition from one system of coordinates to another (analogous formulas were given in §9 for the elastic constants).

Assume in the x, y, z system of coordinates the rigidities of a generally nonorthotropic plate equal to D_{11}, D_{22}, D_{12}, D_{66}, D_{16} and D_{26} while in the x', y', z' system which has been turned about the z-axis through an angle of φ, the rigidities are equal to D'_{11}, D'_{22}, D'_{12}, D'_{66}, D'_{16} and D'_{26}. To derive the rigidity recalculation formulas we consider the expression for the potential energy of deformation per unit volume; if in Eq. (61.21) the derivatives of the deflections are replaced by the strain components ε_x, ε_y, γ_{xy}, we obtain:

in the x, y, z system

$$\overline{V} = \frac{6}{h^3}\left[D_{11}\varepsilon_x^2 + 2D_{12}\varepsilon_x\varepsilon_y + D_{22}\varepsilon_y^2 + D_{66}\gamma_{xy}^2 + 2(D_{16}\varepsilon_x + D_{26}\varepsilon_y)\gamma_{xy}\right], \tag{69.1}$$

and in the x', y', z' system

$$\bar{V} = \frac{6}{h^3}\left[D'_{11}\varepsilon_x'^2 + 2D'_{12}\varepsilon'_x\varepsilon'_y + D'_{22}\varepsilon_y'^2 + D'_{66}\gamma_{xy}'^2 + \right. \\ \left. + 2\left(D'_{16}\varepsilon'_x + D'_{26}\varepsilon'_y\right)\gamma'_{xy}\right]. \tag{69.2}$$

The strain components $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ and $\varepsilon'_x, \varepsilon'_y, \gamma'_{xy}$ are linked by the relations*

$$\left.\begin{aligned} \varepsilon_x &= \varepsilon'_x\cos^2\varphi + \varepsilon'_y\sin^2\varphi - \gamma'_{xy}\sin\varphi\cos\varphi, \\ \varepsilon_y &= \varepsilon'_x\sin^2\varphi + \varepsilon'_y\cos^2\varphi + \gamma'_{xy}\sin\varphi\cos\varphi, \\ \gamma_{xy} &= 2\left(\varepsilon'_x - \varepsilon'_y\right)\sin\varphi\cos\varphi + \gamma'_{xy}\left(\cos^2\varphi - \sin^2\varphi\right). \end{aligned}\right\} \tag{69.3}$$

Substituting these expressions in (69.1) and setting it equal to (69.2) we obtain the transformation formulas sought:

$$\left.\begin{aligned} D'_{11} &= D_{11}\cos^4\varphi + 2(D_{12} + 2D_{66})\sin^2\varphi\cos^2\varphi + D_{22}\sin^4\varphi + \\ &\quad + 2(D_{16}\cos^2\varphi + D_{26}\sin^2\varphi)\sin 2\varphi, \\ D'_{22} &= D_{11}\sin^4\varphi + 2(D_{12} + 2D_{66})\sin^2\varphi\cos^2\varphi + D_{22}\cos^4\varphi - \\ &\quad - 2(D_{16}\sin^2\varphi + D_{26}\cos^2\varphi)\sin 2\varphi, \\ D'_{12} &= D_{12} + [D_{11} + D_{22} - 2(D_{12} + 2D_{66})]\sin^2\varphi\cos^2\varphi + \\ &\quad + (D_{26} - D_{16})\cos 2\varphi\sin 2\varphi, \\ D'_{66} &= D_{66} + [D_{11} + D_{22} - 2(D_{12} + 2D_{66})]\sin^2\varphi\cos^2\varphi + \\ &\quad + (D_{26} - D_{16})\cos 2\varphi\sin 2\varphi, \\ D'_{16} &= \frac{1}{2}[D_{22}\sin^2\varphi - D_{11}\cos^2\varphi + (D_{12} + 2D_{66})\cos 2\varphi]\sin 2\varphi + \\ &\quad + D_{16}\cos^2\varphi(\cos^2\varphi - 3\sin^2\varphi) + D_{26}\sin^2\varphi(3\cos^2\varphi - \sin^2\varphi), \\ D'_{26} &= \frac{1}{2}[D_{22}\cos^2\varphi - D_{11}\sin^2\varphi - (D_{12} + 2D_{66})\cos 2\varphi]\sin 2\varphi + \\ &\quad + D_{16}\sin^2\varphi(3\cos^2\varphi - \sin^2\varphi) + D_{26}\cos^2\varphi(\cos^2\varphi - 3\sin^2\varphi). \end{aligned}\right\} \tag{69.4}$$

Let us consider the case of an orthotropic plate. Let the directions of the axes x and y coincide with the principal directions and the main rigidities be given by D_1, D_2, D_k, $D_3 = D_1\nu_2 + 2D_k$.

When it is required to pass over to a new system of coordinates, x', y', z', whose axes make angles of φ with the former, the deflection equation in the new system will read

$$D'_{11}\frac{\partial^4 w}{\partial x'^4} + 4D'_{16}\frac{\partial^4 w}{\partial x'^3\,\partial y'} + 2\left(D'_{12} + 2D'_{66}\right)\frac{\partial^4 w}{\partial x'^2\,\partial y'^2} + \\ + 4D'_{26}\frac{\partial^4 w}{\partial x'\,\partial y'^3} + D'_{22}\frac{\partial^4 w}{\partial y'^4} = q, \tag{69.5}$$

and the expressions for the moments and crosscut forces are obtained in the form of (61.9) and (61.10) (where D_{ij} must be replaced by D'_{ij}). The rigidities of bending about the new axes, D'_{11} and D'_{22}, the rigidity of torsion, D'_{66}, and the rigidity D'_{12} are determined from the formulas resulting from (69.4):

$$
\left.\begin{aligned}
D'_{11} &= D_1 \cos^4 \varphi + 2D_3 \sin^2 \varphi \cos^2 \varphi + D_2 \sin^4 \varphi,\\
D'_{22} &= D_1 \sin^4 \varphi + 2D_3 \sin^2 \varphi \cos^2 \varphi + D_2 \cos^4 \varphi,\\
D'_{66} &= D_k + (D_1 + D_2 - 2D_3) \sin^2 \varphi \cos^2 \varphi,\\
D'_{12} &= D_2\nu_1 + (D_1 + D_2 - 2D_3) \sin^2 \varphi \cos^2 \varphi.
\end{aligned}\right\} \quad (69.6)
$$

The secondary rigidities D'_{16} and D'_{26} which vanish in the main system x, y, z, are equal to

$$
\left.\begin{aligned}
D'_{16} &= \frac{1}{2}(D_2 \sin^2 \varphi - D_1 \cos^2 \varphi + D_3 \cos 2\varphi) \sin 2\varphi,\\
D'_{26} &= \frac{1}{2}(D_2 \cos^2 \varphi - D_1 \sin^2 \varphi - D_3 \cos 2\varphi) \sin 2\varphi.
\end{aligned}\right\} \quad (69.7)
$$

Note that the expressions for $D'_{11} + D'_{22} + 2D'_{12}$ and $D'_{66} - D'_{12}$ remain unchanged whatever the system of coordinates, i.e., they are invariants:

$$
\left.\begin{aligned}
D'_{11} + D'_{22} + 2D'_{12} &= D_1 + D_2 + 2D_2\nu_1,\\
D'_{66} - D'_{12} &= D_k - D_2\nu_1.
\end{aligned}\right\} \quad (69.8)
$$

The complex bending parameters μ_1 and μ_2 transform in a transition to new axes according to formulas which agree precisely with the transformation formulas for complex parameters of the plane state of stress (see §10).

§70. THE BENDING OF A PLATE DISPLAYING CYLINDRICAL ANISOTROPY

Using the same suppositions and simplifications as applied in the development of the approximate theory of bending of homogeneous plates (thin sheets), it is easy also to develop a theory of bending of plates of curvilinear anisotropy and, in particular, of plates possessing cylindrical anisotropy. This type of plates will be considered briefly.

For simplicity the plate with cylindrical anisotropy is supposed to be orthotropic at the same time, with the planes of elastic symmetry being all radial planes passing through the axis g of anisotropy. The pole of anisotropy, the point of intersection of the axis of anisotropy and the mid-plane (which is assumed perpendicular to this axis), may lie within or without the plate. We let the pole of anisotropy coincide with the origin of the cylindrical system of coordinates, r, θ, z, the z-axis being in the direction of the axis of anisotropy and the x-axis, which is the polar axis, arbitrarily in the mid-plane. In this system of coordinates the equations of the generalized Hooke's law will have the form (3.3).

The equations of the theory of bending are derived in the same way as in the case of the homogeneous plate. Neglecting σ_z we can write the three equations of the generalized Hooke's law in the form:

$$\left.\begin{aligned}\varepsilon_r &= \frac{1}{E_r}(\sigma_r - \nu_r\sigma_\theta),\\ \varepsilon_\theta &= \frac{1}{E_\theta}(\sigma_\theta - \nu_\theta\sigma_r),\\ \gamma_{r\theta} &= \frac{1}{G_{r\theta}}\tau_{r\theta}.\end{aligned}\right\} \tag{70.1}$$

Here E_r, E_θ are Young's moduli for tension (compression) in the radial direction r and in the tangential direction θ; ν_r, ν_θ are the main Poisson's coefficients ($E_r\nu_\theta = E_\theta\nu_r$); $G_{r\theta}$ is the modulus of shear for the (principal) directions r and θ. On the basis of the hypothesis of the straight normals we obtain expressions for the displacements of points u_r and u_θ in the directions r and θ in terms of the deflection $w(r, \theta)$ of the mid-plane:

$$u_r = -z\frac{\partial w}{\partial r}, \quad u_\theta = -z\frac{1}{r}\cdot\frac{\partial w}{\partial \theta}. \tag{70.2}$$

We then determine $\varepsilon_r, \varepsilon_\theta, \gamma_{r\theta}$. From Eqs. (70.1) we find $\sigma_r, \sigma_\theta, \tau_{r\theta}$; from the equations of equilibrium [see the first and second equations of (1.4) where $R = \Theta = 0$] we obtain the components τ_{rz} and $\tau_{\theta z}$. The stresses σ_r, σ_θ are reduced to the bending moments M_r, M_θ, the stresses $\tau_{r\theta}$ to the torsion moment $H_{r\theta}$ and the stresses $\tau_{rz} = \tau_{zr}$, $\tau_{\theta z} = \tau_{z\theta}$ to the crosscut forces N_r and N_θ. A schematic representation of the stress distribution around a point in the plate is given in Fig. 134 (upper diagram); the moments and crosscut forces to which the stresses are reduced are shown schematically in the lower part of Fig. 134.

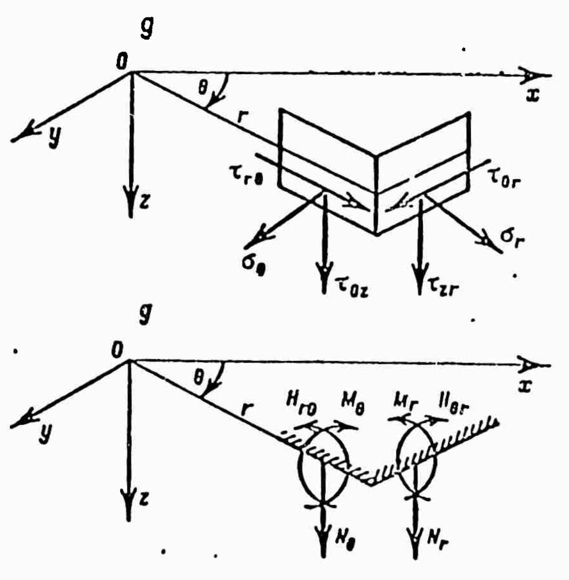

Fig 134

For the stress components, the moments and crosscut forces the following formulas are obtained:

$$\left.\begin{aligned} &\sigma_r = \frac{12M_r}{h^3}z, \quad \sigma_\theta = \frac{12M_\theta}{h^3}z, \quad \tau_{r\theta} = \frac{12H_{r\theta}}{h^3}z, \\ &\tau_{rz} = \frac{6N_r}{h^3}\left(\frac{h^2}{4} - z^2\right), \quad \tau_{\theta z} = \frac{6N_\theta}{h^3}\left(\frac{h^3}{4} - z^2\right); \end{aligned}\right\} \tag{70.3}$$

$$\left.\begin{aligned} &M_r = -D_r\left[\frac{\partial^2 w}{\partial r^2} + \nu_\theta\left(\frac{1}{r}\cdot\frac{\partial w}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 w}{\partial \theta^2}\right)\right], \\ &M_\theta = -D_\theta\left[\nu_r\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial w}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 w}{\partial \theta^2}\right], \\ &H_{\theta r} = H_{r\theta} = -2D_k\frac{\partial^2}{\partial r\,\partial \theta}\left(\frac{w}{r}\right); \end{aligned}\right\} \tag{70.4}$$

$$\left.\begin{aligned} N_r = &-\left[D_r\left(\frac{\partial^3 w}{\partial r^3} + \frac{1}{r}\cdot\frac{\partial^2 w}{\partial r^2}\right) + D_{r\theta}\frac{1}{r^2}\cdot\frac{\partial^2}{\partial \theta^2}\left(\frac{\partial w}{\partial r} - \frac{w}{r}\right) - \right. \\ &\left. - D_\theta\frac{1}{r^2}\left(\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right], \\ N_\theta = &-\left[D_{r\theta}\frac{1}{r}\cdot\frac{\partial^3 w}{\partial r^2\,\partial \theta} + D_\theta\frac{1}{r^2}\cdot\frac{\partial}{\partial \theta}\left(\frac{\partial w}{\partial r} + \frac{1}{r}\cdot\frac{\partial^2 w}{\partial \theta^2}\right)\right]. \end{aligned}\right\} \tag{70.5}$$

Here D_r, D_θ are the bending rigidities for the directions r, θ (i.e., about the axes θ and r passing through a given point), D_k is the rigidity of torsion

$$\left.\begin{aligned} &D_r = \frac{E_r h^3}{12(1-\nu_r\nu_\theta)}, \quad D_\theta = \frac{E_\theta h^3}{12(1-\nu_r\nu_\theta)}, \\ &D_k = \frac{G_{r\theta}h^3}{12}, \qquad D_{r\theta} = D_r\nu_\theta + 2D_k. \end{aligned}\right\} \tag{70.6}$$

Considering the equilibrium of a plate element bounded by three pairs of coordinate surfaces, and of the analogous element represented in Fig. 126, we obtain the following equations for the deflection $w(r, \theta)$:

$$\begin{aligned} &D_r\frac{\partial^4 w}{\partial r^4} + 2D_{r\theta}\frac{1}{r^2}\cdot\frac{\partial^4 w}{\partial r^2\,\partial \theta^2} + D_\theta\frac{1}{r^4}\cdot\frac{\partial^4 w}{\partial \theta^4} + 2D_r\frac{1}{r}\cdot\frac{\partial^3 w}{\partial r^3} - 2D_{r\theta}\frac{1}{r^3}\frac{\partial^3 w}{\partial r\,\partial \theta^2} - \\ &- D_\theta\frac{1}{r^2}\cdot\frac{\partial^2 w}{\partial r^2} + 2(D_\theta + D_{r\theta})\frac{1}{r^4}\cdot\frac{\partial^2 w}{\partial \theta^2} + D_\theta\frac{1}{r^3}\cdot\frac{\partial w}{\partial r} = q(r, \theta), \end{aligned} \tag{70.7}$$

where q is the intensity of normal loading distributed over the plane surface.*

In the case of an isotropic material

$$D_r = D_\theta = D_{r\theta} = D = \frac{Eh^3}{12(1-\nu^2)}.$$

Equation (70.7) takes the form

$$D\nabla^2\nabla^2 w = q(r, \theta), \tag{70.8}$$

where ∇^2 is the Laplace operator in polar coordinates,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial \theta^2}. \tag{70.9}$$

Manu-script Page No.	[Footnotes]
260*	Gehring, F., De aequationibus differentialibus quibus aequilibrium et motus laminae crystallinae definitur [Differential Equations Defining the Equilibrium and Motion of Crystalline Layers], Berlin 1860.
260**	Boussinesque, M.J., Compléments à une étude sur la théorie de l'équilibre et du mouvement des solides élastiques [Supplements to an Investigation into the Theory of Equilibrium and Motions of Elastic Solids] Journal de Math. pures et appl. [Journal of Pure and Applied Mathematics] Ser. 3, Vol. 5, 1879.
260***	See paper by M.T. Huber: 1) Teorja plyt [Theory of Plates] Lwow, 1921; 2) Einige Anwendungen der Biegungstheorie orthotroper Platten [Some Applications of the Theory of Bending of Orthotropic Plates] Zeitschr. f. Angew. Math. und Mech. [Journal of Applied Mathematics and Mechanics] Vol. 6, Fasc. 3, 1926; 3) Probleme der Statik technisch wichtiger orthotroper Platten [Problems of the Statics of Technically Important Orthotropic Plates] Warsaw, 1929.
266	The equations of the theory of bending for an anisotropic plate have been derived, for example, in the third of Huber's papers mentioned in the preceding section and in our own paper: "O nekotorykh voprosakh, svyazannykh s teoriyey izgiba tonkikh plit" [On Some Problems Connected with the Theory of Bending of Thin Plates] Prikladnaya matematika i mekhanika, novaya seriya [Applied Mathematics and Mechanics, New Series], Vol. II, No. 2, 1938, and for an isotropic plate in the books by B.G. Galerkin "Uprugiye tonkiye plity" [Elastic Thin Plates] Gosstroyizdat [State Publishers of Construction Engineering] 1933 and S.P. Timoshenko "Plastinki i obolochki" [Plates and Shells], OGIZ, Gostekhizdat, Moscow-Leningrad, 1948.
267	See, e.g., the book by S.P. Timoshenko, mentioned, pages 92-98.
268	See page 187 of our paper mentioned in the preceding section. The formulas given below are also derived in this paper.
269*	See our paper, page 191, mentioned previously.
269**	The determinant of these systems for unequal complex parameters is always nonzero (see footnote in §8).
270	See our paper mentioned in §62, where derivations are given for Eqs. (63.14) and the boundary conditions for functions of a complex variable and where these functions are investigated.

271* Lur'ye, A.I., K zadache o ravnovesii plastiny s opertym krayami [To the Equilibrium Problem of Plates with Supported Sides] Izv. Leningr. politekhn. in-ta [Bull. of the Leningrad Polytechnical Institute] Vol. XXXI, 1928, pages 305-320; see also his paper: Nekotoryye zadachi ob izgibe krugloy plastinki [Some Problems on the Bending of a Round Plate] Prikladnaya matematika i mekhanika [Applied Mathematics and Mechanics] Vol. IV, No. 1, 1940.

271** Fridman, M.M., 1) O nekotorykh zadachakh teorii izgiba tonkikh plit [On Some Problems of the Theory of Bending of Thin Plates] Prikladnaya matematika i mekhanika, Vol. V, No. 1, 1941; 2) Izgib tonkoy izotropnoy plity s krivolineynym otverstiyem [The Bending of a Thin Isotropic Plate with Curvilinear Aperture], Ibid. Vol. IX, No. 4, 1954; 3) Izgib tonkoy izotropnoy plity s vpayannoy rugloy isotropnoy shayboy iz uprugogo materiala [The Bending of a Thin Isotropic Plate with Soldered-in Round Isotropic Disk of an Elastic Material] Ibid. Vol. XIV, No. 4, 1950; 4) Izgib krugloy plity sosredotochennymi silami [The Bending of a Round Plate by Concentrated Forces] ibid., Vol. XV, No. 2, 1951; 5) Resheniye obshchey zadachi ob izgibe tonkoy uprugoy plity, opertoy vdol kraya [Solution of the General Problem on the Bending of a Thin Elastic Plate Supported Along the Edge], ibid., Vol. XVI, No. 4, 1952.

273 Karman, Th., Encyklopädie der mathematischen Wissenschaften [Encyclopedia of Mathematical Sciences] Vol. IV, 1910, page 349.

274 Rostovtsev, G.G., Raschet tonkoy ploskoy obshivki, podkreplennoy rebrami zhestkosti [Calculation of Thin Panel Reinforced by Stiffening Ribs] Trudy Leningr. in-ta inzhenerov grazhdanskogo voadushnogo flota [Transactions of the Leningrad Institute of Engineers of Civil Aviation] No. 20, 1940.

275 Timoshenko, S.P., Plastinki i obolochki [Plates and Shells] Gostekhizdat, Moscow, 1948, page 37, 243.

276 Cf. paper by E. Seydel, Schubknickversuche mit Wellblechtafeld [Shear-Breaking Tests with Corrugated Iron Sheets] DVL-Bericht [DVL Report] or the book by S.N. Kats and I.A. Sverdlov, Raschet samoleta na prochnost [Strength Calculation of Airplane] Oboroygiz, Moscow, 1910, page 254.

278 See paper by S.G. Lekhnitskiy, Izgib neodnorodnykh anizotropnykh tonkikh plit simmetrichnogo stroyeniya [Bending of Nonhomogeneous Anisotropic Thin Plates of Symmetrical Structure] Prikladnaya matematika i mekhanika, Vol. V, No. 1, 1941. In this paper the more general case where the layers are not orthotropic was also investigated.

283 Rabinovich, A.L., O raschete ortotropnykh sloistykh paneley na rastyazheniye, sdvig i izgib [On the Calculation of Orthotropic Laminated Panels as to Tension, Shear and Bending] Ministerstvo aviatsionnoy promyshlennosti SSSR [Ministry of Aviation Industry USSR] Trudy, No. 675, 1948.

284* In the same form the rigidity can be given for the more general case of a laminated plate, only the form of the reduced moduli will be more complex.

284** Equations (67.5) (67.8) agree essentially with the formulas obtained by A.L. Rabinovich in the paper referred to in §66 (pages 13-14, 17-18). A.L. Rabinovich used other notations.

285 See A.L. Rabinovich's paper mentioned, page 15.

287 See paper by C.B. Smith, Some New Types of Orthotropic Plates Laminated of Orthotropic Material, Journ. of Appl. Mech., Vol. 20, 1953, No. 2. In his paper Smith uses other denotations. This paper contains general considerations with respect to the determination of the rigidity of plates consisting of four and more (even number) plates.

290 These formulas are derived in the mechanics of continuous media.

293 This equation agrees essentially with an equation derived independently by Carrier in the paper: G.F. Carrier, The Bending of the Cylindrically Aelotropic Plate, Journ. of Appl. Mech., Vol. 11, 1944, No. 3.

Chapter 10

THE BENDING OF PLATES UNDER NORMAL LOAD

§71. THE SIMPLEST CASES OF BENDING

In this chapter we shall consider a series of concrete problems on the bending of homogeneous plates of rectangular, elliptic and round form, of strips and of a curvilinear-anisotropic round plate.

The simplest cases of bending of a uniform plate are: 1) pure bending; 2) pure torsion; 3) bending on a cylindrical surface. The deflections, moments crosscut forces and stresses are obtained in these cases in an elementary way, and we give the formulas without derivations. In all cases the plate is assumed to be homogeneous but not orthotropic so that the bending equation referred to the chosen axes x and y has the form of (62.2) and the expressions for the moments and crosscut forces have the form of (61.9) and (61.10).

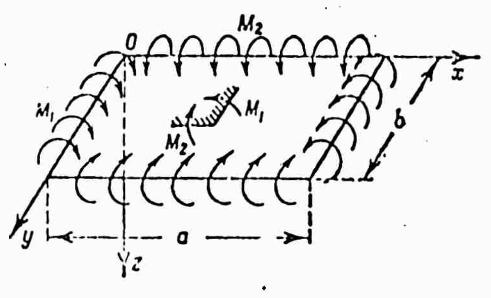

Fig. 135

1. Pure bending. A rectangular plate is bent by the moments M_1 (per unit length) distributed uniformly on two sides and the moments M_2 (per unit length) distributed uniformly on the two other sides.

Denoting the lengths of the sides by a and b and directing the axes along the sides (Fig. 135), we obtain

$$M_x = M_1, \quad M_y = M_2, \quad H_{xy} = N_x = N_y = 0; \tag{71.1}$$

$$w = Ax^2 + Bxy + Cy^2 + C_1x + C_2y + C_0, \tag{71.2}$$

where A, B, C are the constants determined from the equations

$$
\left.\begin{aligned}
AD_{11}+BD_{16}+CD_{12}&=-\frac{M_1}{2},\\
AD_{12}+BD_{26}+CD_{22}&=-\frac{M_2}{2},\\
AD_{16}+BD_{66}+CD_{26}&=0,
\end{aligned}\right\}\qquad (71.3)
$$

and C_1, C_2, C_0 are constants obtained from the conditions of fixing. For example, for a plate which is fixed at the three corners (0, 0), (a, 0), (0, b),

$$
w=A(x^2-ax)+Bxy+C(y^2-by). \qquad (71.4)
$$

Another case of pure bending is the bending of a plate of arbitrary form by the moments M (per unit length) which are distributed uniformly along the edge (Fig. 136).

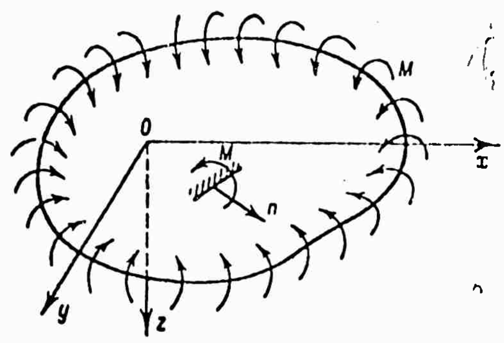

Fig. 136

In this case for an arbitrary element with the normal n

$$
M_n=M, \quad H_{tn}=0, \quad N_n=0. \qquad (71.5)
$$

2. Pure torsion. A rectangular plate is deformed by the torsional moment H (per unit length) distributed uniformly on all sides (Fig. 137). In this case

$$
M_x=M_y=0, \quad H_{xy}=H, \quad N_x=N_y=0. \qquad (71.6)
$$

The deflection w has the form (71.2) and the constants A, B, C are determined by the equations

$$
\left.\begin{aligned}
AD_{11}+BD_{16}+CD_{12}&=0,\\
AD_{12}+BD_{26}+CD_{22}&=0,\\
AD_{16}+BD_{66}+CD_{26}&=-\frac{H}{2}.
\end{aligned}\right\}\qquad (71.7)
$$

Another variant of pure torsion is the deformation of a plate hinged on two opposite sides, by the forces $2H$, applied to the free corners (Fig. 138). For this variant the same formulas (71.6), (71.7), (71.2) are applicable.

3. Bending on a cylindrical surface. A plate in the form of a long rectangle fastened uniformly on the long sides and arbitrarily on the short sides, is deformed by a load which does not

vary along the long sides. In this case at points remote from the short sides the curved surface of the plate will be similar to a cylindrical surface; for an infinitely long plate the surface will be precisely cylindrical.

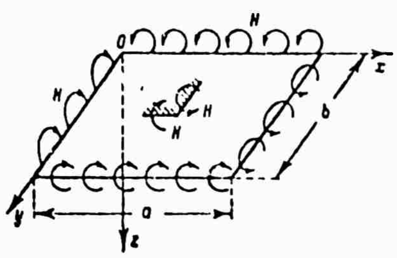

Fig. 137

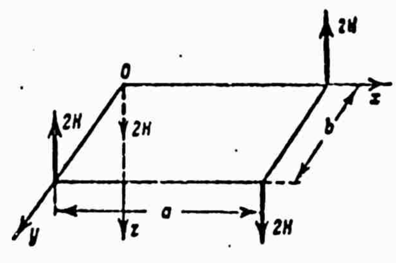

Fig. 138

Placing the origin of coordinates on the long side far away from the short sides and directing the x-axis along the long side (Fig. 139), the deflection $w(y)$ may be assumed to be a function of y alone.

Then

$$\left.\begin{array}{c} M_x = -D_{12}w'', \quad M_y = -D_{22}w'', \quad H_{xy} = -D_{20}w'', \\ N_x = -D_{20}w''', \quad N_y = -D_{22}w'''. \end{array}\right\} \qquad (71.8)$$

The deflection is determined from the equation

$$D_{22}w^{IV} = q(y). \qquad (71.9)$$

This equation coincides with the deflection equation of a beam with the rigidity D_{22} bent by an arbitrary normal load q; the quantities M_y and N_y are determined as the bending moment and the crosscut force in this beam. The fixing of the beam ends must correspond to the fixing of the long sides of the plate. In this way the problem on the bending of a plate on a cylindrical surface is reduced to the problem of the bending of a beam which can be solved within the framework of the elementary theory of bending. The unnecessary force factors M_x, N_{xy} and N_x which do not exist for the beam are easy to determine from the deflection obtained. For an orthotropic plate at which the principal directions are parallel to the sides, $H_{xy} = N_x = 0$.

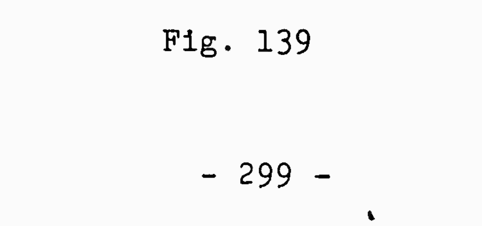

Fig. 139

§72. BENDING OF AN ORTHOTROPIC RECTANGULAR PLATE WITH HINGED SIDES

A rectangular orthotropic plate whose principal directions are parallel to the directions of the sides rests (on hinges) on all four sides and is bent by a normal load distributed according to an arbitrary law.

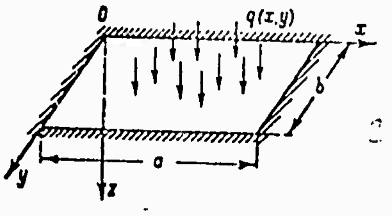

Fig. 140

We orient the x and y axes along the sides (Fig. 140) and denote by a, b the lengths of the sides of the plate and by c their ratio: $c = a/b$. The equation of the deflections will have the form of (62.3); when integrating it the boundary conditions to be taken into account are the following:

$$\left.\begin{array}{ll} \text{with } x=0,\ x=a & w=M_x=0; \\ \text{with } y=0,\ y=b & w=M_y=0. \end{array}\right\} \tag{72.1}$$

All these conditions will be satisfied when we use the solution of Eq. (62.3) in the form of a series

$$w=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \tag{72.2}$$

In order to determine the coefficients A_{mn} we expand the function $q(x, y)$ in a double Fourier series

$$q=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \tag{72.3}$$

where

$$a_{mn}=\frac{4}{ab}\int_0^a\int_0^b q\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy. \tag{72.4}$$

Substituting Expressions (72.2) and (72.3) in Eq. (62.3) and equating the coefficients of equal sines on the left-hand and the right-hand sides, we obtain the following expression for the deflection

$$w=\frac{b^4}{\pi^4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{D_1\left(\frac{m}{c}\right)^4+2D_3n^2\left(\frac{m}{c}\right)^2+D_2n^4}. \tag{72.5}$$

This solution is analogous to Navier's solution for an iso-

tropic plate*

$$w = \frac{b^4}{D\pi^4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \frac{\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{2}}{\left[\left(\frac{m}{c}\right)^2 + n^2\right]^2} \tag{72.6}$$

(D is the rigidity of the isotropic plate). In Eq. (72.5) we must substitute the values of the coefficients a_{mn} which depend on the law of load distribution. Thus, for the case of a load distributed uniformly over the whole area, we obtain

$$\left.\begin{aligned} a_{mn} &= \frac{16p}{\pi^2} \cdot \frac{1}{mn} \text{ for } m,\ n = 1,\ 3,\ 5,\ \ldots; \\ a_{mn} &= 0 \text{ for all other } m \text{ and } n \end{aligned}\right\} \tag{72.7}$$

where p is the load per unit area.

For a concentrated force P which attacks at the point $(\xi,\ \eta)$,

$$\begin{gathered} a_{mn} = \frac{4P}{ab} \sin \frac{m\pi\xi}{a} \sin \frac{n\pi\eta}{b} \\ (m = 1,\ 2,\ 3,\ \ldots,\ n = 1,\ 2,\ 3,\ \ldots). \end{gathered} \tag{72.8}$$

The moments and crosscut forces are determined from Eqs. (61.14)-(61.15) and are likewise double series.

A solution in the form of a double series is of theoretical interest but not suitable for application in practice. Though the series expressing the deflection, moments and crosscut forces are virtually always convergent, the convergence is so slow that many terms of these series must be taken for a calculation. Therefore, in cases where one can do it without the double series, one prefers to use simple series for the solutions, which converge much better (we shall consider such solutions below).

When on the sides of a plate, which is bent by an arbitrary load q (72.3), the normal forces p_1 and p_2 are distributed uniformly, we shall have an equation of the form (64.7) where we must substitute

$$T_x = p_1, \quad T_y = p_2, \quad S_{xy} = 0.$$

Also in this case the deflection is easy to determine in the form of a series

$$\begin{gathered} w = \frac{b^4}{\pi^4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \times \\ \times \frac{\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}}{D_1 \left(\frac{m}{c}\right)^4 + 2D_3 n^2 \left(\frac{m}{c}\right)^2 + D_2 n^4 + \left(\frac{b}{\pi}\right)^2 \left[p_1 \left(\frac{m}{c}\right)^2 + p_2 n^2\right]}. \end{gathered} \tag{72.9}$$

If we are concerned with tensile forces, $p_1 > 0$ and $p_2 > 0$; the denominators of all terms of the series are positive and higher than the corresponding denominators of the series (72.5). From this it is clear that an adding of tensile forces distributed

along the sides will reduce the deflection; the plate behaves as if it were more rigid. If the forces are compressive, $p_1 < 0$, $p_2 < 0$ and the denominators of Eq. (72.9) are differences of positive quantities. An addition of compressive forces increases the deflection or, in other words, reduces the rigidity of the plate. With compressive forces we may also encounter cases where one or several denominators of Eq. (72.9) are vanishing and the deflection under the action of the load q therefore becomes theoretically infinitely large. This indicates that a plate compressed by the forces p_1 and p_2 alone is in unstable equilibrium.

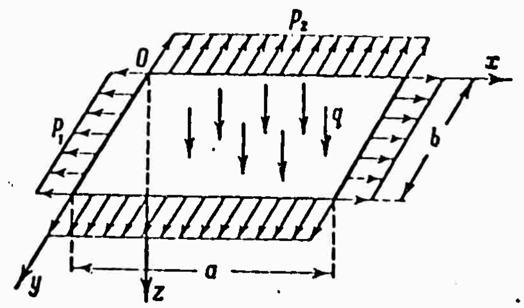

Fig. 141

The stability problems of the plate are also considered in Chapters 13-16.

§73. THE BENDING OF AN ORTHOTROPIC RECTANGULAR PLATE WITH TWO RESTING SIDES

A rectangular orthotropic plate with two opposite sides supported and the other two sides fixed in some way is bent by a normal load q. We consider a case where the load is constant along the supported sides. A more general case of load will be discussed in §76. For a given plate we can obtain a solution in the form of simple series which is a generalization of the well-known Morris-Levi solution for the case of an orthotropic plate.*

Let us place the origin of coordinates in the middle of the supported side and direct the x-axis along the supported side and the y-axis perpendicular to it (Fig. 142a).

The deflection equation will have the form

$$D_1 \frac{\partial^4 w}{\partial x^4} + 2D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} = q(y), \tag{73.1}$$

where $q(y)$ is a given function. The function w defined by this equation must satisfy the boundary conditions for the supported sides

$$\text{with } y = 0,\ y = b \quad w = \frac{\partial^2 w}{\partial y^2} + \nu_1 \frac{\partial^2 w}{\partial x^2} = 0 \tag{73.2}$$

and four conditions for the other sides.

We shall seek a solution in the form of a sum

$$w = w_0(y) + w_1(x, y). \tag{73.3}$$

Here $w_0(y)$ is a function satisfying the equation

$$D_2 w_0^{IV} = q(y) \tag{73.4}$$

and the conditions

$$w_0(0) = w_0''(0) = 0, \quad w_0(b) = w_0''(b) = 0; \tag{73.5}$$

this is the deflection of a beam of length b and rigidity D, which rests on its ends and is loaded by the load q (Fig. 142b). The function w_1 satisfies the homogeneous equation

$$D_1 \frac{\partial^4 w_1}{\partial x^4} + 2D_3 \frac{\partial^4 w_1}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w_1}{\partial y^4} = 0, \tag{73.6}$$

and Conditions (73.2) and is chosen so that it satisfies the conditions on the sides $x = \pm a/2$.

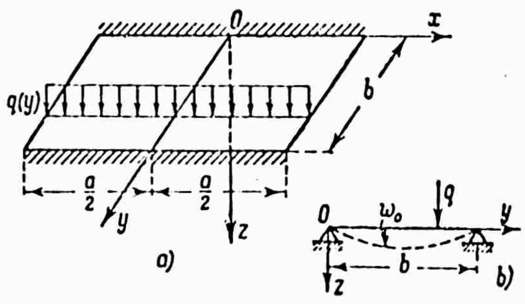

Fig. 142

In many cases of loads the function w_0 can be determined in a finite form by the methods of the theory of materials resistance, but in order to satisfy the conditions on the sides $x = \pm a/2$ it must be represented in the form of a Fourier sine series. This series will have the form

$$w_0 = \frac{b^4}{D_2 \pi^4} \sum_{n=1}^{\infty} \frac{a_n}{n^4} \sin \frac{n\pi y}{b}, \tag{73.7}$$

where

$$a_n = \frac{2}{b} \int_0^b q \sin \frac{n\pi y}{b} \, dy \tag{73.8}$$

are the Fourier series expansion coefficients of the function $q(y)$ representing the law of load distribution. The function w_1 is sought in the form of a series

$$w_1 = \sum_{n=1}^{\infty} X_n(x) \sin\frac{n\pi y}{b}. \qquad (73.9)$$

The total deflection $w = w_0 + w_1$ will satisfy the conditions on the resting sides and we have only to take care that the conditions on the sides $x = \pm a/2$, are satisfied which is always possible.

For the functions X_n we obtain the equation

$$D_1 X_n^{IV} - 2D_3\left(\frac{n\pi}{b}\right)^2 X_n'' + D_2\left(\frac{n\pi}{b}\right)^4 X_n = 0. \qquad (73.10)$$

The form of the function X_n depends on the roots s_1, s_2 of the characteristic equation

$$D_1 s^4 - 2D_3 s^2 + D_2 = 0. \qquad (73.11)$$

These roots are connected with the complex parameters μ_1, μ_2 of bending: $s_1 = i/\mu_1$, $s_2 = i/\mu_2$.

According to the relations between the quantities of rigidity of a plate we may distinguish three cases.

Case 1. The roots of Eq. (73.11) are real and unequal:

$$\pm s_1, \quad \pm s_2 \quad (s_1 > 0, \ s_2 > 0).$$

Case 2. The roots of Eq. (73.11) are real and pairwise equal:

$$\pm s \quad (s > 0).$$

Case 3. The roots of Eq. (73.11) are complex:

$$s \pm ti, \quad -s \pm ti \quad (s > 0, \ t > 0).$$

In Case 1

$$X_n = A_n \operatorname{ch}\frac{n\pi s_1 x}{b} + B_n \operatorname{sh}\frac{n\pi s_1 x}{b} + C_n \operatorname{ch}\frac{n\pi s_2 x}{b} + D_n \operatorname{sh}\frac{n\pi s_2 x}{b}. \qquad (73.12)$$

In Case 2

$$X_n = (A_n + B_n x) \operatorname{ch}\frac{n\pi s x}{b} + (C_n + D_n x) \operatorname{sh}\frac{n\pi s x}{b}. \qquad (73.13)$$

In Case 3

$$X_n = \left(A_n \cos\frac{n\pi t x}{b} + B_n \sin\frac{n\pi t x}{b}\right) \operatorname{ch}\frac{n\pi s x}{b} + \left(C_n \cos\frac{n\pi t x}{b} + D_n \sin\frac{n\pi t x}{b}\right) \operatorname{sh}\frac{n\pi s x}{b}, \qquad (73.14)$$

where A_n, B_n, C_n, D_n are arbibrary constants.

In this way we obtain the following expressions for the deflection. In Case 1

$$w = \sum_{n=1}^{\infty}\left(\frac{a_n b^4}{D_2 \pi^4 n^4} + A_n \operatorname{ch}\frac{n\pi s_1 x}{b} + B_n \operatorname{sh}\frac{n\pi s_1 x}{b} + C_n \operatorname{ch}\frac{n\pi s_2 x}{b} + D_n \operatorname{sh}\frac{n\pi s_2 x}{b}\right)\sin\frac{n\pi y}{b}. \tag{73.15}$$

In Case 2

$$w = \sum_{n=1}^{\infty}\left[\frac{a_n b^4}{D_2 \pi^4 n^4} + (A_n + B_n x)\operatorname{ch}\frac{n\pi s x}{b} + (C_n + D_n x)\operatorname{sh}\frac{n\pi s x}{b}\right]\sin\frac{n\pi y}{b}. \tag{73.16}$$

In Case 3

$$w = \sum_{n=1}^{\infty}\left[\frac{a_n b^4}{D_2 \pi^4 n^4} + \left(A_n \cos\frac{n\pi t x}{b} + B_n \sin\frac{n\pi t x}{b}\right)\operatorname{ch}\frac{n\pi s x}{b} + \left(C_n \cos\frac{n\pi t x}{b} + D_n \sin\frac{n\pi t x}{b}\right)\operatorname{sh}\frac{n\pi s x}{b}\right]\sin\frac{n\pi y}{b}. \tag{73.17}$$

In each case the denominator contains four arbitrary constants and as many conditions as we have on the sides $x = \pm a/2$; these conditions can be satisfied with an arbitrary fixing of the sides.

In the following it is not necessary to analyze all three cases (1, 2, 3) in detail. It is sufficient to consider Case 1; the solutions for the other cases are obtained by means of a limiting transition with $s_1 = s_2 = s$, or by a separation of the real part of the complex expression which is obtained when we put $s_1 = s + ti$, $s_2 = s - ti$.

In the case of four supported sides we obtain

$$w = w_0(y) + \frac{b^4}{D_2 \pi^4}\sum_{n=1}^{\infty}\frac{a_n}{n^4(s_1^2 - s_2^2)} \times \left(\frac{s_2^2 \operatorname{ch}\frac{n\pi s_1 x}{b}}{\operatorname{ch}\frac{n\pi s_1 c}{2}} - \frac{s_1^2 \operatorname{ch}\frac{n\pi s_2 x}{b}}{\operatorname{ch}\frac{n\pi s_2 c}{2}}\right)\sin\frac{n\pi y}{b}. \tag{73.18}$$

For a plate whose sides $x = \pm a/2$ are fixed

$$w = w_0(y) + \frac{b^4}{D_2 \pi^4}\sum_{n=1}^{\infty}\frac{a_n}{n^4} \times \frac{s_2 \operatorname{sh}\frac{n\pi s_2 c}{2}\operatorname{ch}\frac{n\pi s_1 x}{b} - s_1 \operatorname{sh}\frac{n\pi s_1 c}{2}\operatorname{ch}\frac{n\pi s_2 x}{b}}{s_1 \operatorname{sh}\frac{n\pi s_1 c}{2}\operatorname{ch}\frac{n\pi s_2 c}{2} - s_2 \operatorname{sh}\frac{n\pi s_2 c}{2}\operatorname{ch}\frac{n\pi s_1 c}{2}}\sin\frac{n\pi y}{b}. \tag{73.19}$$

Knowing the expression for the deflections, we determine from Eqs. (63.14) and (61.15) the moments and crosscut forces and from them the stresses.

Series of the type (73.18) are much more suitable for the calculation than the double series of the last section of the book; they are usually converging rapidly. For the calculation of the deflections and stresses at given points in an accuracy

sufficient for practice we can restrict ourselves to a few terms of the series.

§74. THE BENDING OF A RECTANGULAR ORTHOTROPIC PLATE BY A LOAD DISTRIBUTED UNIFORMLY

When a load q is distributed over a plate as considered in the preceding section in a uniform manner (Fig. 143),

$$w_0 = \frac{q}{24D_2}(y^4 - 2by^3 + b^3y); \tag{74.1}$$

$$a_n = \frac{4q}{\pi n} \text{ with } n = 1, 3, 5, \ldots; \quad a_n = 0 \text{ with } n = 2, 4, 6, \ldots \tag{74.2}$$

The series (73.18) and (73.19) will only contain terms with odd n.

Let us consider two cases in greater detail.

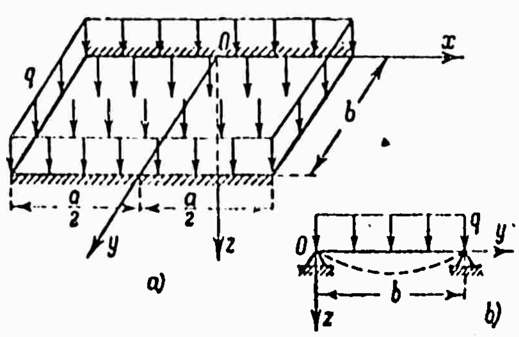

Fig. 143

1. Plate with four supported sides. The moments and crosscut forces are determined from Eqs. (61.14) and (61.15). For the deflections and bending moments we obtain the following expressions:

$$w = \frac{q}{24D_2}(y^4 - 2by^3 + b^3y) + \\ + \frac{4qb^4}{D_2\pi^5} \cdot \frac{1}{s_1^2 - s_2^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^5}\left(\frac{s_2^2 \operatorname{ch}\frac{n\pi s_1 x}{b}}{\operatorname{ch}\frac{n\pi s_1 c}{2}} - \frac{s_1^2 \operatorname{ch}\frac{n\pi s_2 x}{b}}{\operatorname{ch}\frac{n\pi s_2 c}{2}}\right)\sin\frac{n\pi y}{b}; \tag{74.3}$$

$$
\left.
\begin{aligned}
M_x &= -\frac{q\nu_1}{2}(y^2 - by) + \\
&+ \frac{4qb^2}{\pi^3}\cdot\frac{D_1}{D_2}\cdot\frac{1}{s_1^2 - s_2^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^3}\left[s_2^2(\nu_2 - s_1^2)\frac{\operatorname{ch}\frac{n\pi s_1 x}{b}}{\operatorname{ch}\frac{n\pi s_1 c}{2}} - \right. \\
&\qquad \left. - s_1^2(\nu_2 - s_2^2)\frac{\operatorname{ch}\frac{n\pi s_2 x}{b}}{\operatorname{ch}\frac{n\pi s_2 c}{2}}\right]\sin\frac{n\pi y}{b}, \\
M_y &= -\frac{q}{2}(y^2 - by) + \\
&+ \frac{4qb^2}{\pi^3}\cdot\frac{1}{s_1^2 - s_2^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^3}\left[s_2^2(1 - \nu_1 s_1^2)\frac{\operatorname{ch}\frac{n\pi s_1 x}{b}}{\operatorname{ch}\frac{n\pi s_1 c}{2}} - \right. \\
&\qquad \left. - s_1^2(1 - \nu_1 s_2^2)\frac{\operatorname{ch}\frac{n\pi s_2 x}{b}}{\operatorname{ch}\frac{n\pi s_2 c}{2}}\right]\sin\frac{n\pi y}{b}.
\end{aligned}
\right\}
\qquad (74.4)
$$

The maximum deflection and highest bending moments are obtained in the center and can be represented in the following way

$$
w_{max} = \frac{5qb^4}{384D_2}\alpha; \qquad (74.5)
$$

$$
(M_x)_{max} = \frac{qb^2}{8}\beta', \quad (M_y)_{max} = \frac{qb^2}{8}\beta. \qquad (74.6)
$$

α, β' and β are dimensionless correction coefficients which take into account the influence of the lateral sides $x = \pm a/2$. When the side ratio $\sigma = a/b$ is high these coefficients can be taken equal to $\alpha = 1,\ \beta' = \nu_2,\ \beta = 1,$ and then we shall have a bending on a cylindrical surface.

The determination of α, β' and β with finite σ in Case 1 is obvious from Eq. (74.3)-(74.4). We give the values of the coefficients for Case 3.

$$\left.\begin{aligned}
\alpha &= 1 - \frac{1{,}004}{st}\sum_{n=1,3,5,\ldots}^{\infty}\frac{(-1)^{\frac{n-1}{2}}}{n^{3}\delta_n}\times\\
&\times\left[2st\,\mathrm{ch}\,\frac{n\pi sc}{2}\cos\frac{n\pi tc}{2} + (s^2 - t^2)\,\mathrm{sh}\,\frac{n\pi sc}{2}\sin\frac{n\pi tc}{2}\right],\\
\beta' &= \nu_1 + \frac{1{,}032}{st}\sum_{n=1,3,5,\ldots}^{\infty}\frac{(-1)^{\frac{n-1}{2}}}{n^{3}\delta_n}\times\\
&\times\left[(s^2 + t^2 - \nu_2 s^2 + \nu_2 t^2)\,\mathrm{sh}\,\frac{n\pi sc}{2}\sin\frac{n\pi tc}{2} -\right.\\
&\qquad\left. - 2\nu_2 st\,\mathrm{ch}\,\frac{n\pi sc}{2}\cos\frac{n\pi tc}{2}\right],\\
\beta &= 1 + \frac{1{,}032}{st}\sum_{n=1,3,5,\ldots}^{\infty}\frac{(-1)^{\frac{n-1}{2}}}{n^{3}\delta_n}\times\\
&\times\left[(\nu_1 s^2 + \nu_1 t^2 - s^2 + t^2)\,\mathrm{sh}\,\frac{n\pi sc}{2}\sin\frac{n\pi tc}{2} -\right.\\
&\qquad\left. - 2st\,\mathrm{ch}\,\frac{n\pi sc}{2}\cos\frac{n\pi tc}{2}\right],\\
\delta_n &= \mathrm{ch}\,n\pi sc + \cos n\pi ts.
\end{aligned}\right\}\qquad(74.7)$$

As shown in §67, the parameters μ_1, μ_2 and, consequently, also s_1, s_2 are found to be complex for such a material as veneer. For a veneer plate the series entering Eqs. (74.7) converge satisfactorily.

The formulas for Case 2 are obtained in the way shown above by means of a limiting transition with $t = 0$.

The bending of a plate with four supported sides, under the action of a uniformly distributed load was studied in detail by Huber for the case where the rigidity satisfies the condition $D_3 = \sqrt{D_1 D_2}$ (Case 2). Huber gives Tables for the calculation of the maximum deflection w_{max}, the maximum moments $(M_x)_{max}$, $(M_y)_{max}$, and other quantities, which are of interest for strength calculations, such as 1) the maximum crosscut forces $(N_x)_{max}$, $(N_y)_{max}$, obtained in the middles of the sides; 2) the maximum values of the support reactions $(R_1)_{max}$, $(R_2)_{max}$, obtained for these points; 3) the total pressure $\overline{R}_1$ of the plate on the support $x = \pm a/2$ and $\overline{R}_2$ on the support $y = 0$, $y = b$; 4) the reaction R at the corners of the plate. All these quantities are considered as functions of the ratio

$$s = \frac{a}{b}\sqrt[4]{\frac{D_2}{D_1}}\quad (a \geqslant b);\qquad(74.8)$$

they are determined from the formulas

$$w_{max} = \frac{qb^4}{D_2}\varphi;\qquad(74.9)$$

$$\left.\begin{aligned}
(M_x)_{max} &= \left(\mu_{11} + \mu_{22}\nu_2\sqrt{\frac{D_1}{D_2}}\right)\frac{qa^2}{s^2},\\
(M_y)_{max} &= \left(\mu_{22} + \mu_{11}\nu_1\sqrt{\frac{D_2}{D_1}}\right)qb^2
\end{aligned}\right\}\qquad(74.10)$$

(in the middle of the plate $x=0,\ y=b/2$);

$$\left.\begin{aligned}(N_x)_{\max}&=\left[\mu_{111}+\mu_{122}\left(\nu_2+\frac{2D_k}{D_1}\right)\sqrt{\frac{D_1}{D_2}}\right]\frac{qa}{\varepsilon},\\(R_1)_{\max}&=\left[\mu_{111}+\mu_{122}\left(\nu_2+\frac{4D_k}{D_1}\right)\sqrt{\frac{D_1}{D_2}}\right]\frac{qa}{\varepsilon}\end{aligned}\right\}\tag{74.11}$$

(at the points $x=\pm a/2,\ y=b/2$, i.e., in the middles of sides b);

$$\left.\begin{aligned}(N_y)_{\max}&=\left[\mu_{222}+\mu_{112}\left(\nu_1+\frac{2D_k}{D_1}\right)\sqrt{\frac{D_2}{D_1}}\right]qb,\\(R_2)_{\max}&=\left[\mu_{222}+\mu_{112}\left(\nu_1+\frac{4D_k}{D_1}\right)\sqrt{\frac{D_2}{D_1}}\right]qb\end{aligned}\right\}\tag{74.12}$$

(at the points $x=0,\ y=0$ and $x=0,\ y=b$, i.e., in the middles of the sides a);

$$\left.\begin{aligned}\bar{R}_1&=\left[\mu'_{111}+\mu'_{122}\left(\nu_2+\frac{4D_k}{D_1}\right)\sqrt{\frac{D_1}{D_2}}\right]\frac{qab}{\varepsilon},\\\bar{R}_2&=\left[\mu'_{222}+\mu'_{112}\left(\nu_1+\frac{4D_k}{D_1}\right)\sqrt{\frac{D_2}{D_1}}\right]qab;\end{aligned}\right\}\tag{74.13}$$

$$R=\mu'_{12}\frac{4D_k}{D_2}\sqrt[4]{\frac{D_2}{D_1}}qb^2.\tag{74.14}$$

We shall give Huber's table (on page 310) for the coefficients $\mu_{11},\mu_{22},\ldots,\mu'_{12}$ for a series of values of the ratio ε.*

2. Plate with two supported and two fixes sides. Deflection and bending moments are determined from the equations

$$w=\frac{q}{24D_2}(y^4-2by^3+b^3y)+\frac{4qb^4}{D_2\pi^5}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^5\Delta_n}\left(s_2\operatorname{sh}\frac{n\pi s_2c}{2}\operatorname{ch}\frac{n\pi s_1x}{b}-\right.$$
$$\left.-s_1\operatorname{sh}\frac{n\pi s_1c}{2}\operatorname{ch}\frac{n\pi s_2x}{b}\right)\sin\frac{n\pi y}{b};\tag{74.15}$$

$$\left.\begin{aligned}M_x&=\frac{q\nu_1}{2}(by-y^2)+\\&+\frac{4qb^2}{\pi^3}\cdot\frac{D_1}{D_2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^3\Delta_n}\left[\left(\nu_2-s_1^2\right)s_2\operatorname{sh}\frac{n\pi s_2c}{2}\operatorname{ch}\frac{n\pi s_1x}{b}-\right.\\&\qquad\left.-\left(\nu_2-s_2^2\right)s_1\operatorname{sh}\frac{n\pi s_1c}{2}\operatorname{ch}\frac{n\pi s_2x}{b}\right]\sin\frac{n\pi y}{b},\\M_y&=\frac{q}{2}(by-y^2)+\\&+\frac{4qb^2}{\pi^3}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^3\Delta_n}\left[\left(1-\nu_1s_1^2\right)s_2\operatorname{sh}\frac{n\pi s_2c}{2}\operatorname{ch}\frac{n\pi s_1x}{b}-\right.\\&\qquad\left.-\left(1-\nu_1s_2^2\right)s_1\operatorname{sh}\frac{n\pi s_1c}{2}\operatorname{ch}\frac{n\pi s_2x}{b}\right]\sin\frac{n\pi y}{b},\\\Delta_n&=s_1\operatorname{sh}\frac{n\pi s_1c}{2}\operatorname{ch}\frac{n\pi s_2c}{2}-s_2\operatorname{sh}\frac{n\pi s_2c}{2}\operatorname{ch}\frac{n\pi s_1c}{2}.\end{aligned}\right\}\tag{74.16}$$

TABLE 14

Values of the Coefficients in Eqs. (74.9)-(74.14)

c	φ	μ_{11}	μ_{23}	μ_{111}	μ_{123}	μ_{222}
1	0,00107	0,0368	0,0368	0,219	0,119	0,219
1,5	0,00772	0,0280	0,0728	0,199	0,161	0,335
2	0,01013	0,0174	0,0961	0,189	0,181	0,410
2,5	0,01150	0,0099	0,1100	0,188	0,183	0,453
3	0,01223	0,0055	0,1172	0,187	0,185	0,476
5	0,01297	0,0001	0,1245	0,186	0,186	0,499
∞	0,01302	0,0000	0,1250	0,186	0,186	0,500

c	μ_{113}	μ'_{111}	μ'_{122}	μ'_{222}	μ'_{112}	μ'_{12}
1	0,119	0,157	0,093	0,157	0,093	0,0166
1,5	0,089	0,145	0,122	0,211	0,081	0,0611
2	0,055	0,138	0,133	0,299	0,066	0,0663
2,5	0,031	0,137	0,134	0,338	0,054	0,0670
3	0,017	0,136	0,135	0,365	0,045	0,0675
5	0,001	0,136	0,136	0,419	0,027	0,0679
∞	0,000	0,136	0,136	0,500	0,000	0,0679

The maximum bending is obtained in the center and may be described by the formula

$$w_{max} = \frac{5qb^4}{384D_2}\alpha. \tag{74.17}$$

As regards the bending moments, they reach their highest values in an isotropic plate in the midpoints of the fixed sides; in the case of an orthotropic plate we cannot exclude the possibility that the bending moments reach their highest values in the center. Denoting by M_{x0}, M_{y0} the bending moments in the center and by M_{xb}, M_{yb} the bending moments at the midpoints of the fixed sides, we can represent these quantities by the formulas

$$\left.\begin{aligned} M_{x0} &= \frac{qb^2}{8}\beta', & M_{y0} &= \frac{qb^2}{8}\beta, \\ M_{xb} &= -\frac{qb^2}{8}\beta_1', & M_{yb} &= -\frac{qb^2}{8}\beta_1. \end{aligned}\right\} \tag{74.18}$$

With high ratios $c = a/b$

$$\alpha = 1,\quad \beta' = \nu_1,\quad \beta = 1,\quad \beta_1' = \sqrt{\frac{D_1}{D_2}},\quad \beta_1 = \nu_1\sqrt{\frac{D_2}{D_1}}.$$

We restrict ourselves to giving the expressions for the coefficients $\alpha, \beta', \beta, \beta_1'$ and β_1 only for Case 3:

$$
\left.
\begin{aligned}
\alpha &= 1 - 2{,}008 \sum_{n=1,3,5,\ldots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^3 \Delta_n} \times \\
&\quad \times \left(s \operatorname{ch} \frac{n\pi sc}{2} \sin \frac{n\pi tc}{2} + t \operatorname{sh} \frac{n\pi sc}{2} \cos \frac{n\pi tc}{2} \right), \\
\beta' &= \nu_1 - 2{,}064 \frac{D_1}{D_2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^3 \Delta_n} \times \\
&\quad \times \left[(\nu_2 - s^2 - t^2) s \operatorname{ch} \frac{n\pi sc}{2} \sin \frac{n\pi tc}{2} + \right. \\
&\quad \left. + (\nu_2 + s^2 + t^2) t \operatorname{sh} \frac{n\pi sc}{2} \cos \frac{n\pi tc}{2} \right], \\
\beta &= 1 - 2{,}064 \sum_{n=1,3,5,\ldots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^3 \Delta_n} \times \\
&\quad \times \left[(1 - \nu_1 s^2 - \nu_1 t^2) s \operatorname{ch} \frac{n\pi sc}{2} \sin \frac{n\pi tc}{2} + \right. \\
&\quad \left. + (1 + \nu_1 s^2 + \nu_1 t^2) t \operatorname{sh} \frac{n\pi sc}{2} \cos \frac{n\pi tc}{2} \right], \\
\beta_1' &= 1{,}032 \sqrt{\frac{D_1}{D_2}} \sum_{n=1,3,5,\ldots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^3 \Delta_n} (t \operatorname{sh} n\pi sc - s \sin n\pi tc), \\
\beta_1 &= 1{,}032 \nu_1 \sqrt{\frac{D_2}{D_1}} \sum_{n=1,3,5,\ldots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^3 \Delta_n} (t \operatorname{sh} n\pi sc - s \sin n\pi tc), \\
\Delta_n &= t \operatorname{sh} n\pi sc + s \sin n\pi tc.
\end{aligned}
\right\}
\qquad (74.19)
$$

In the case of a veneer plate the terms of the series appearing in Eqs. (74.19) decrease rapidly enough so that in calculations, with accuracies sufficient for practice, we can be satisfied with a few terms. The convergence of the series become worse as the ratio c decreases.

§75. THE BENDING OF A STRIP WITH SUPPORTED SIDES

Another problem which is very interesting for practice is the problem of the bending of a long rectangular orthotropic plate with supported sides, under the action of a load distributed over a limited part of its surface (in the theory such a plate is considered as an infinite band with supported sides). This problem was solved by M.T. Huber in both the general form and for many particular cases. We shall here give the fundamental results obtained by Huber, without entering into details as to their bases and statements.*

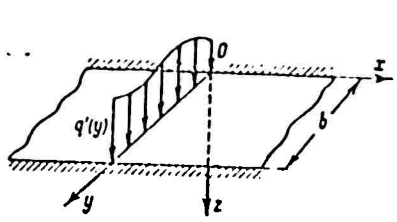

Fig. 144

1. Strip loaded along a straight line section. An infinite band of width b with supported sides is subjected to the action of a load distributed arbitrarily on a straight line section perpendicular to the sides. The principal directions of elasticity are assumed parallel and perpendicular to the directions of the sides. When the x-axis agrees with the side and the y-axis with the loaded section as shown in Fig. 144, we expand the given load $q'(y)$ (per unit length) in a Fourier series; we obtain

$$q' = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi y}{b}. \tag{75.1}$$

The expressions for the deflections will depend on the solutions of the characteristic equation (73.11) and, for positive x, they may be written as follows (for parts of the plate on the right-hand side of the loaded section):

in Case 1 (see §73)

$$w = \frac{b^3}{2\pi^3 \sqrt{D_1 D_2}} \cdot \frac{1}{s_1^2 - s_2^2} \times$$

$$\times \sum_{n=1}^{\infty} \frac{a_n}{n^3} \left(s_1 e^{-\frac{n\pi s_1 x}{b}} - s_2 e^{-\frac{n\pi s_2 x}{b}} \right) \sin \frac{n\pi y}{b}; \tag{75.2}$$

in Case 2

$$w = \frac{b^3}{4\pi^3 \sqrt{D_1 D_2}\, s} \sum_{n=1}^{\infty} \frac{a_n}{n^3} \left(1 + \frac{n\pi s x}{b} \right) e^{-\frac{n\pi s x}{b}} \sin \frac{n\pi y}{b} \tag{75.3}$$

in Case 3

$$w = \frac{b^3}{4\pi^3 \sqrt{D_1 D_2}} \sum_{n=1}^{\infty} \frac{a_n}{n^3} \left(\frac{1}{s} \cos \frac{n\pi t x}{b} + \frac{1}{t} \sin \frac{n\pi t x}{b} \right) e^{-\frac{n\pi s x}{b}} \sin \frac{n\pi y}{b}. \tag{75.4}$$

The expressions for the deflection on the left-hand side of the loaded section are obtained when x is replaced by $-x$.

In each particular case given we must find the coefficients a_n of the Fourier series and substitute them in Eqs. (75.2)-(75.4). Knowing the expressions for the deflections, we can determine the bending and torsional moments and the crosscut forces from Eqs. (61.14)-(61.15) and from them we obtain the stresses.

Note that the formulas for the cases 2 and 3 are obtained from Eqs. (75.2) by means of a limiting transition; in the following we shall restrict ourselves to giving only the deflection expressions for Case 1.

If load q' is distributed uniformly on a section of length b_1 of the y-axis, the midpoint of this section being at the distance η from the origin of coordinates (Fig. 145), we have

$$a_n = \frac{4q'}{\pi n} \sin \frac{n\pi b_1}{2b} \sin \frac{n\pi\eta}{b}; \tag{75.5}$$

$$w = \frac{2q' b^3}{\pi^4 \sqrt{D_1 D_2}} \cdot \frac{1}{s_1^2 - s_2^2} \times$$

$$\times \sum_{n=1}^{\infty} \frac{1}{n^4} \sin\frac{n\pi b_1}{2b} \sin\frac{n\pi\eta}{b}\left(s_1 e^{-\frac{n\pi s_2 x}{b}} - s_2 e^{-\frac{n\pi s_1 x}{b}}\right)\sin\frac{n\pi y}{b} \tag{75.6}$$

(for $x \geqslant 0$).

In particular, if the plate is loaded by load q' distributed uniformly along the width ($b_1 = b$, $\eta = b/2$, Fig. 146), we obtain for parts on the right-hand side of the loaded section

$$w = -\frac{2q'b^3}{\pi^4\sqrt{D_1 D_2}} \cdot \frac{1}{s_1^2 - s_2^2} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^4}\left(s_1 e^{-\frac{n\pi s_2 x}{b}} - s_2 e^{-\frac{n\pi s_1 x}{b}}\right)\sin\frac{n\pi y}{b}. \tag{75.7}$$

An investigation of this case results in the following.*

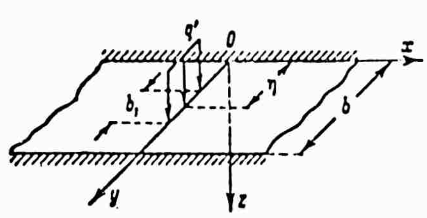

Fig. 145

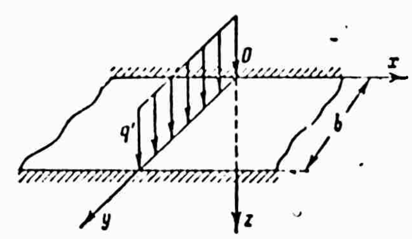

Fig. 146

The maximum deflection is obtained at the point $x = 0$, $y = b/2$, corresponding to the midpoint of the loaded section and is determined by the formula (independent of the case of solution considered)

$$w_{max} = \frac{q'b^3}{\pi^4 D_2} \cdot \frac{0{,}939}{\sqrt{\frac{D_3}{2D_2} + \frac{1}{2}\sqrt{\frac{D_1}{D_2}}}}. \tag{75.8}$$

The maximum bending moments for this point are

$$\left.\begin{aligned} (M_x)_{max} &= 0{,}0929 \frac{\sqrt{\frac{D_1}{D_2}} + \nu_1}{\sqrt{\frac{D_3}{2D_2} + \frac{1}{2}\sqrt{\frac{D_1}{D_2}}}}\, q'b, \\ (M_y)_{max} &= 0{,}0929 \frac{\nu_1\sqrt{\frac{D_2}{D_1}} + 1}{\sqrt{\frac{D_3}{2D_2} + \frac{1}{2}\sqrt{\frac{D_1}{D_2}}}}\, q'b. \end{aligned}\right\} \tag{75.9}$$

If the load is applied not to the whole length b of the section but only a part of it, of the length b_1 (with the center at the point $x = 0$, $y = b/2$), the maximum moments are determined by formulas which differ from (75.9) only by a numerical coefficient after the equality sign. The coefficient exceeds 0.0929 and depends on the ratio b_1/b; the smaller this ratio, the higher the coefficient or the stress concentration. Thus, according to

Huber, we obtain with the ratios $b_1/b = 1/9, 1/20, 1/21$ values of 0.2736, 0.3370 and 0.3517 for the coefficient.

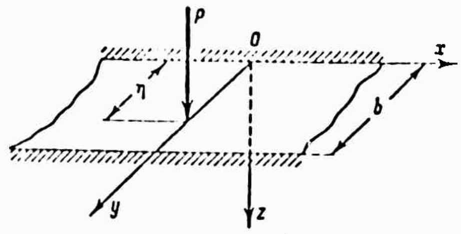

Fig. 147

2. Strip loaded by a concentrated force (Fig. 147). The concentrated force may be considered as a limiting case of a load distributed along an infinitesimal section, with a finite resultant. Denoting by P the magnitude of the force, assuming $q'b_1 = P$ and carrying out the limiting transition with $b_1 = 0$ we obtain from (75.6) the deflection formula for the right-hand side of the point to which the force is applied:

$$w = \frac{Pb^2}{\pi^3 \sqrt{D_1 D_2}} \cdot \frac{1}{s_1^2 - s_2^2} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi\eta}{b} \left(s_1 e^{-\frac{n\pi s_2 x}{b}} - s_2 e^{-\frac{n\pi s_1 x}{b}} \right) \sin \frac{n\pi y}{b}. \quad (75.10)$$

The deflection decreases rapidly as we remove from the point of attack of the force. Huber shows that in practice the deflection may be considered vanishing even at a finite distance from the point of attack of the force, which is given by

$$x = 1{,}5 b_{np}, \quad (75.11)$$

where b_{pr} is the reduced width which, according to Huber, is determined in the following way:

in Case 1

$$b_{np} = b \sqrt{\frac{D_3}{D_2} + \sqrt{\left(\frac{D_3}{D_2}\right)^2 - \frac{D_1}{D_2}}}; \quad (75.12)$$

in Case 2

$$b_{np} = b \sqrt{\frac{D_1}{D_2}}; \quad (75.13)$$

in Case 3

$$b_{np} = \frac{b}{\sqrt{\frac{D_3}{2D_1} + \frac{1}{2}\sqrt{\frac{D_2}{D_1}}}}. \quad (75.14)$$

When the force is applied to a point on the axis of the strip, $\eta = b/2$, the deflection under the fo-ce is equal to*

$$w_{max} = 0{,}01696\,\frac{Pb^3}{D_3}\cdot\frac{1}{\sqrt{\frac{D_3}{2D_2}+\frac{1}{2}\sqrt{\frac{D_1}{D_2}}}}. \qquad (75.15)$$

An investigation shows that all formulas for a strip with supported sides, which is bent by a concentrated force, can be transformed such as if they would not contain infinite series. This is a consequence of the fact that the second derivatives of the deflection w of the plate considered contain series which can be summed up. In the case of real unequal s_1 and s_2 (see §73) the expressions for the second derivatives of the deflection are

$$\frac{\partial^2 w}{\partial x^2} = \frac{s_1^2\varphi - s_2^2\psi}{s_1^2 - s_2^2}, \quad \frac{\partial^2 w}{\partial y^2} = \frac{\psi - \varphi}{s_1^2 - s_2^2}, \qquad (75.16)$$

where

$$\left.\begin{aligned} \varphi &= -\frac{P}{4\pi s_1 D_1}\ln\frac{\operatorname{ch}\frac{\pi s_1 x}{b} - \cos\frac{\pi}{b}(\eta - y)}{\operatorname{ch}\frac{\pi s_1 x}{b} - \cos\frac{\pi}{b}(\eta + y)}, \\ \psi &= -\frac{P}{4\pi s_2 D_1}\ln\frac{\operatorname{ch}\frac{\pi s_2 x}{b} - \cos\frac{\pi}{b}(\eta - y)}{\operatorname{ch}\frac{\pi s_2 x}{b} - \cos\frac{\pi}{b}(\eta + y)}; \end{aligned}\right\} \qquad (75.17)$$

$$\frac{\partial^2 w}{\partial x\,\partial y} = -\frac{P}{2\pi D_1 (s_1^2 - s_2^2)} \times$$

$$\times\left\{\operatorname{arctg}\frac{e^{-\frac{\pi s_2 x}{b}}\sin\frac{\pi}{b}(\eta + y)}{1 - e^{-\frac{\pi s_2 x}{b}}\sin\frac{\pi}{b}(\eta + y)} + \operatorname{arctg}\frac{e^{-\frac{\pi s_2 x}{b}}\sin\frac{\pi}{b}(\eta - y)}{1 - e^{-\frac{\pi s_2 x}{b}}\sin\frac{\pi}{b}(\eta - y)} - \right.$$

$$\left. - \operatorname{arctg}\frac{e^{-\frac{\pi s_1 x}{b}}\sin\frac{\pi}{b}(\eta + y)}{1 - e^{-\frac{\pi s_1 x}{b}}\sin\frac{\pi}{b}(\eta + y)} - \operatorname{arctg}\frac{e^{-\frac{\pi s_1 x}{b}}\sin\frac{\pi}{b}(\eta - y)}{1 - e^{-\frac{\pi s_1 x}{b}}\sin\frac{\pi}{b}(\eta - y)}\right\}. \qquad (75.18)$$

Substituting Expressions (75.16)-(75.18) in Eqs. (61.14) we obtain the bending moments and torsional moments. Knowing the expressions for the second derivatives of the deflection, we can, by means of integration, also determine the deflections w themselves; these are rather complicated expressions but do not contain any series.*

From the concentrated force it is easy to pass over to a load distributed over the area of an arbitrary figure S on a plane strip. The solution is obtained in the form of (75.10) or, in a final form, as obtained by summation over the deflections caused by infinitesimal forces; it is derived in the following way. Let $P\delta(x, y; \eta)$ be the deflection at the point (x, y) distant from the point $(0, \eta)$ of attack of the concentrated force; the deflection at the point (x, y) caused by a force applied to point (ξ, η) is represented by a function $P\delta(x - \xi, y; \eta)$. When the plate is subject to the action of a load $p(x, y)$ distributed over a certain area S, the load $p(\xi, \eta)d\xi d\eta$, which falls on the element $d\xi d\eta$ of surface area, may be considered as a concentrated force; the deflection due to this force at point (x, y) will be equal to

$$p(\xi, \eta)\delta(x - \xi, y; \eta)\,d\xi\,d\eta.$$

The deflection caused by the whole load is obtained by integrating the preceding expression over the area S:

$$w = \iint p(\xi, \eta)\delta(x - \xi, y; \eta)\,d\xi\,d\eta. \qquad (75.19)$$

Certain cases of distributed load were studied in detail by Huber; in particular, he studied the case of a load distributed uniformly on the area of a rectangle.*

§76. APPLICATION OF THE THEORY OF BENDING OF AN ORTHOTROPIC STRIP

With the help of a solution for an orthotropic strip bent by a given normal load, we can obtain a solution of the problem on the bending of a rectangular orthotropic plate with two, three or four supported sides, under the action of an arbitrary load (in §§73-74 we only considered a load which did not vary along the supported sides). This solution is obtained in the following way.** As we wish to determine the deflection, moments and cross-cut forces in a plate of length a and width b where side a is supported, we consider a fictitious auxiliary plate in the form of an infinite strip of the same width b with supported sides. By means of straight lines perpendicular to the sides we cut the strip in a series of rectangular sections of side a whose area is equal to the plate investigated; we then load these rectangles by loads which are connected with the load acting on the given plate in a certain definite manner. This definite distribution of load on the sections will at the boundaries of the sections produce the same conditions as exist on the sides of the plate investigated; we shall show this below.

The solution of the problem for a strip loaded arbitrarily may thus be considered to be solved [it is obtained from the solution of the case of a concentrated force, see Eq. (75.19)] and we can therefore also obtain the solution for a rectangular plate with arbitrary distribution of the load. The problem is reduced to a summation of the deflection of the strip as caused by a load repeated periodically which is easy to carry out. Let us consider three basic cases of fixing of the sides of a rectangular plate which is bent by a given load.

1. Two sides supported, two sides fixed. On the sides b of the plate the following conditions are assumed satisfied:

$$w = 0, \quad \frac{\partial w}{\partial x} = 0. \qquad (76.1)$$

Considering an auxiliary strip divided into sections corresponding to the given plate, we shall distribute the load in the following way: one of the sections, we shall call it the first, is loaded by q (equal to the load acting on the plate investigated); the adjacent second section is loaded by q^* distributed symmetrically with respect to the loads q relative to the line of intersection of the fields; the third one is again loaded by q, the fourth by q^* and so on as shown in Fig. 148a. Under the action of these forces the strip is bent and the cross-sectional line of

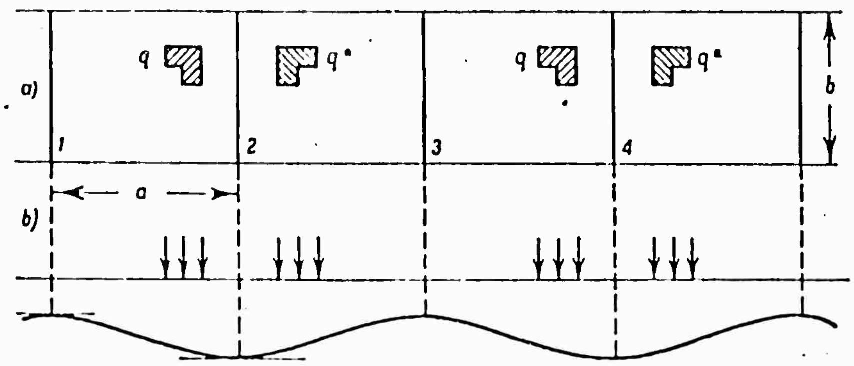

Fig. 148

the surface will have the form shown in 148b; by virtue of the symmetry the second condition of (76.1) will be satisfied at the lines of intersection. We then distribute on the lines of intersection the loads q_1' and q_2' in upward direction, which are represented in the form of series with indeterminate coefficients

$$q_1' = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi y}{b}, \quad q_2' = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{b}. \tag{76.2}$$

The total bending at an arbitrary point of the strip is represented in the form of a sum of the deflection w_1 from the loads q and q^* and the deflection w_2 from the loads q_1' and q_2':

$$w = w_1 + w_2. \tag{76.3}$$

It is required that the condition $w = 0$ is satisfied on the lines of intersection and we determine the coefficients a_n and b_n and the deflection w itself. The boundary conditions for the first section will here be exactly the same as the conditions on the edges of the given plate and the deflection of this section will therefore be equal to the deflection in the corresponding site of the plate.

2. All sides supported. In this case the conditions satisfied on the sides b are

$$w = 0, \quad \frac{\partial^2 w}{\partial x^2} + \nu_2 \frac{\partial^2 w}{\partial y^2} = 0. \tag{76.4}$$

We load the first field of the auxiliary strip by the loads q acting on the plate investigated, and the second field by $-q^*$ symmetrically to the first relative to the line of intersection, but of opposite direction; the third field is again loaded by q and so on (Fig. 149a). The cross-sectional line of the bent surface will have the form shown in Fig. 149b. It is obvious that the first conditions (76.4) are satisfied on the boundaries of the fields. It is easy to show that also the second conditions will be fulfilled. In fact, to the lines of intersection correspond the displaced points of the cross section of the bent sur-

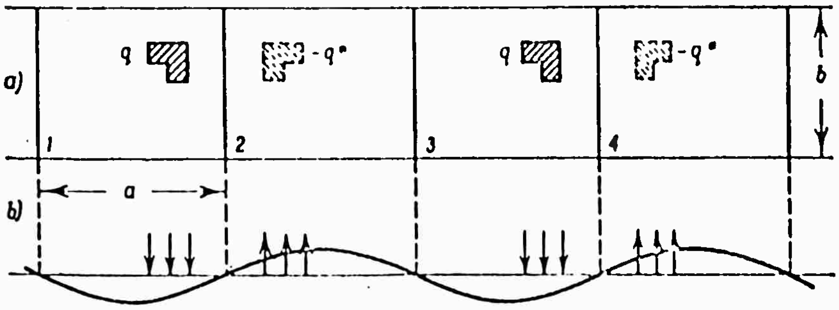

Fig. 149

face at which the second derivative of w with respect to x is vanishing; the derivative $\partial^2 w/\partial y^2$ vanishes at the same points where $w=0$ so that on the intersecting lines $\frac{\partial^2 w}{\partial x^2}+\nu_2\frac{\partial^2 w}{\partial y^2}=0.$ Hence it is clear that the first field can be considered as an isolated plate with supported sides.

3. Three sides supported, one fixed. If one of the sides b of the plate is supported and the other, opposite side is fixed, Conditions (76.4) must be satisfied on the former and Conditions (76.1) on the latter. In this case the first field of the auxiliary strip is loaded by the same forces q as are acting on the plate; the second field is loaded by the forces q^* which are symmetrical relative to the line of intersection; the third field is loaded by $-q$ distributed just as q but directed oppositely; the fourth field is loaded by $-q^*$ symmetrically with respect to the line of intersection, the fifth one by q and so on (Fig. 150a). Considering the cross section of the bent surface (Fig. 150b) we see that on one of the lines bounding the first field Conditions (76.4) are satisfied, on the other line only one of the conditions, $\partial w/\partial x=0$. In order to achieve the fulfillment of all conditions, we distribute on the lines of intersection alternately the loads q_1' and q_2' which we shall represent in the form of series (76.2). The total deflection of the strip consists of the deflection w_1 under the action of the loads q, q^*, $-q$, $-q^*$ and the deflection w_2 under the action of the loads q_1' and q_2':

$$w=w_1+w_2. \tag{76.5}$$

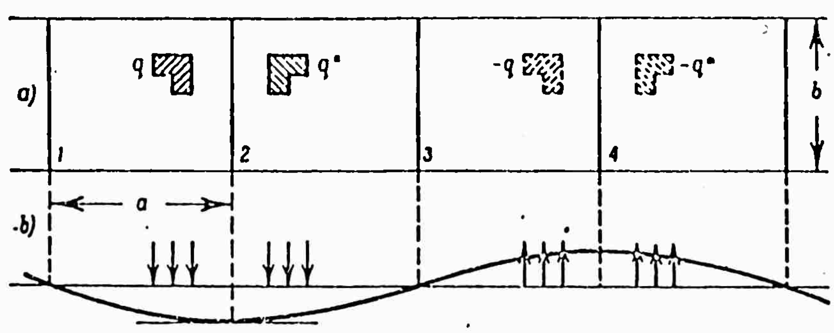

Fig. 150

We determine the unknown coefficients a_n and b_n on conditi that on the lines of intersection of the fields the deflection equal to zero.

In this way Huber obtained a solution to the problem of th bending of a plate with supported sides under the action of a c centrated force P and under the action of forces distributed in checkered order, and he also studied a series of cases of bendi of plates with two, three and four sides supported, under the a tion of loads distributed in various ways. As it is not possible here to deal with all these cases we shall give Huber's solution only for the case of a concentrated force attacking at point (ξ, η) on a plate where all four sides are resting on a support (Fig. 151).

The form of the expression obtained for the deflection depends on which of the three cases, 1 2 or 3, is considered. In Case 1 we obtain:*

for $0 \leqslant x < \xi$

$$w = w(x, a-\xi, y) = \frac{2Pb^2}{\pi^3 \sqrt{D_1 D_2}} \cdot \frac{1}{s_1^2 - s_2^2} \times$$
$$\times \sum_{n=1}^{\infty} \frac{\sin\frac{n\pi\eta}{b}}{n^3} \left[\frac{s_1 \operatorname{sh} \frac{n\pi s_2 (a-\xi)}{b} \operatorname{sh} \frac{n\pi s_2 x}{b}}{\operatorname{sh} \frac{n\pi s_2 a}{b}} - \right.$$
$$\left. - \frac{s_2 \operatorname{sh} \frac{n\pi s_1 (a-\xi)}{b} \operatorname{sh} \frac{n\pi s_1 x}{b}}{\operatorname{sh} \frac{n\pi s_1 a}{b}} \right] \sin \frac{n\pi y}{b}; \qquad (76.6)$$

for $\xi < x \leqslant a$

$$w = w(a-x, \xi, y).$$

In the case of equal and complex roots of the characteristic equation (73.11) the formulas for w are obtained easily from Eqs. (76.6) by means of a limiting transition or a separation of the real part of the complex expression.

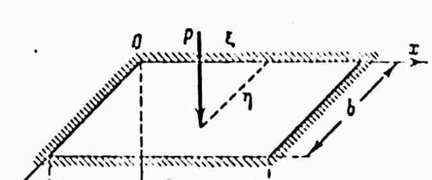

Fig. 151

Using the solution for an orthotropic strip bent by a concentrated force, W. Nowacki considered a series of other problems on the bending of a rectangular orthotropic plate and also a semistrip, infinite in one direction, and a strip with transverse incision.**

Solutions of problems on the bending of an orthotropic semistrip with supported sides by a concentrated force or moment were obtained by Z. Cywinsky and J. Mossakowski*** who also used the solution for a strip bent by a force (in a finite form).

Let us here also recall the papers by Z. Kaczkowski who obtained solutions to a series of particular problems on the bending of anisotropic plates in the form of a parallelogram, rectangle and equilateral triangle, lying on an elastic support and deformed by a normal load and forces acting in the median surfaces.*

§77. THE BENDING OF A RECTANGULAR ORTHOTROPIC PLATE REINFORCED BY PARALLEL STIFFENING RIBS

Let us consider a rectangular orthotropic plate whose principal directions are parallel to its sides, which is reinforced by elastic rods (stiffening ribs) parallel to one side. We suppose the ribs to be connected rigidly with the plate on their whole length and the sides of the plate on which the ends of the ribs are fixed to be supported while the two other sides are fixed arbitrarily or are free; the ends of the ribs are also resting on a support and cannot be turned. The principal axes of inertia of the rib cross sections are assumed parallel and perpendicular to the mid-plane. We consider the bending of such a plate by a normal load distributed both on the sections between the ribs and on the ribs themselves.

The solution of an analogous problem for an isotropic plate with ribs was obtained by A.S. Lokshin** (for the case of equal ribs arranged at equal spacings) and by A.P. Filippov who used another method.*** Some static and dynamic problems for isotropic plates with ribs were dealt with in the papers by W. Nowacki.****

Let us give the fundamental results for an orthotropic plate.***** We place the origin of coordinates in one of the corners of the plate, directing the x-axis along the side parallel to the ribs and the y-axis along the side which is perpendicular to the ribs (Fig. 152). Let us introduce the following denotations:

a) for quantities referring to the ribs: N is the number of ribs, η_k the distance between the front edge of the plate and rib number k; EJ_k is the rigidity of bending in planes parallel to xy; C_k is the rigidity of torsion; $W_k(x)$ and $\theta_k(x)$ are the deflections and the angles of twist; Q_k are the loads acting on a rib (per unit length); M_k and N_k are the bending moments and the crosscut forces in the ribs:

$$M_k = -EJ_k W_k'', \quad N_k = -EJ_k W_k''' \qquad (k = 1, 2, \ldots, N); \tag{77.1}$$

b) for the quantities referring to the plate: a is the length of the side parallel to the ribs; b is the length of the side perpendicular to the ribs; D_1, D_2, D_k are the rigidities of bending and torsion for the principal directions; $D_3 = D_1\nu_2 + 2D_k$; ν_2 is Poisson's coefficients; $w_k(x, y)$ are the deflections of the sections between neighboring ribs and between the outer sides a and between the outer sides a and the outermost ribs,

M_{xk}, M_{yk}, H_k, N_{xk}, N_{yk} are the bending moments, the torsional moment and the crosscut forces for these sections; $q_k(x, y)$ is the normal load (per unit area) distributed on the sections of the plate ($k = 1, 2, \ldots, N, N+1$).

The bending and torsional moments and the crosscut forces for each section of the plate are linked with the deflection by Eqs. (61.14) and (61.15) and the deflections satisfy the equations

$$D_1 \frac{\partial^4 w_k}{\partial x^4} + 2D_3 \frac{\partial^4 w_k}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w_k}{\partial y^4} = q_k(x, y) \quad (k = 1, 2, \ldots, N+1). \tag{77.2}$$

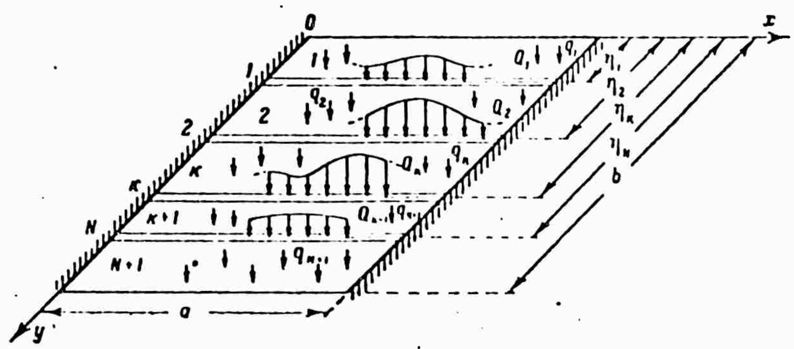

Fig. 152

The conditions on the sides $y = 0$ and $y = b$ depend on the way of fixing and on the sides $x = 0$ and $x = a$ have the form

$$w_k = 0, \quad M_{xk} = 0. \tag{77.3}$$

Since, according to agreement, the ribbing cannot be separated from the plate, the deflections and angles of twist of the ribs are expressed in terms of the deflections of the plate sections in the following way:

$$\left.\begin{aligned} W_k(x) &= w_k(x, \eta_k) = w_{k+1}(x, \eta_k), \\ \theta_k(x) &= \left(\frac{\partial w_k}{\partial y}\right)_{y=\eta_k} = \left(\frac{\partial w_{k+1}}{\partial y}\right)_{y=\eta_k}, \end{aligned}\right\} \tag{77.4}$$

where at the ends of the ribs, i.e., at the points $x = 0$, $y = \eta_k$ and $x = a$, $y = \eta_k$,

$$W_k = 0, \quad \theta_k = 0, \quad M_k = 0. \tag{77.5}$$

Moreover, on the lines of contact of adjacent plate sections $y = \eta_k$ the following equilibrium conditions must be satisfied (the width of the ribs is neglected)

$$\left.\begin{array}{c} M_{y,k+1}-M_{yk}=C_k\frac{\partial^4 w_k}{\partial x^2\partial y},\\ N_{y,k+1}+\frac{\partial H_{k+1}}{\partial x}-N_{yk}-\frac{\partial H_k}{\partial x}=EJ_k\frac{\partial^4 w_k}{\partial x^4}-Q_k\\ (k=1,\ 2,\ \ldots,\ N). \end{array}\right\} \tag{77.6}$$

The problem is reduced to determining the functions $w_k(x,\ y)$ satisfying Eqs. (77.2) and Conditions (77.3), (77.4), (77.6) and also the conditions on the sides $y = 0$ and $y = b$. If one of these sides rests on a support, on it $w_k=M_{yk}=0$; but if it is fixed $w_k=\frac{\partial w_k}{\partial y}=0$ ($k=1$ or $k = N + 1$).

Expanding the loads q_k and Q_k in Fourier sine series we obtain

$$\left.\begin{array}{c} q_k=\sum_{m=1}^{\infty} q_{km}(y)\sin\beta x,\\ Q_k=\sum_{m=1}^{\infty} Q_{km}\sin\beta x\\ \left(\beta=\frac{m\pi}{a}\right). \end{array}\right\} \tag{77.7}$$

We shall seek solutions to Eqs. (77.2) in the form of series

$$w_k=\sum_{m=1}^{\infty} f_{km}(y)\sin\beta x. \tag{77.8}$$

These expressions satisfy Conditions (77.2) on the supported sides.

For functions f_{km} we obtain the differential equations

$$D_2 f_{km}^{IV}-2D_3\beta^2 f''_{km}+D_1\beta^4 f_{km}=q_{km}(y) \tag{77.9}$$

and the conditions on the lines $y=\eta_k$ ($k=1,\ 2,\ \ldots,\ N$):

$$\left.\begin{array}{l} f_{k+1,m}=f_{km},\quad f'_{k+1,m}=f'_{km},\\ f''_{k+1,m}-f''_{km}=\frac{C_k\beta^2}{D_2}f'_{km},\\ f'''_{k+1,m}-f'''_{km}=\frac{Q_{km}}{D_2}-\frac{EJ_k}{D_2}\beta^4 f_{km}. \end{array}\right\} \tag{77.10}$$

Moreover, the functions f_{1m} and $f_{N+1,\ m}$ must satisfy the conditions on the sides $y = 0$ and $y = b$. When these sides are supported, on them $f_{km}=f''_{km}=0$; on the rigidly fixed sides we have $f_{km}=f'_{km}=0$ ($k=1$ or $k = N + 1$).

It is not difficult to notice the analogy between this problem and the problem on the bending of an arbitrary beam. With m given and constant, equal to an arbitrary integer, the functions $f_{km}(y)$ may be considered as the deflections of sections of a beam of length b with a rigidity equal to D_2. Equations (77.9) show that this beam lies on a massive elastic support with the elastic

coefficients $D_1\beta^4$ and is bent by a normal distributed load q_m and extended by an axial force equal to $T = 2D_3\beta^2$. Conditions (77.10) show that apart from the distributed load, the beam is subject to the action of concentrated moments M_{km} and concentrated forces P_{km} applied to the points $y = \eta_k$ (Fig. 153a). The moments are proportional to the angles of slope of the bent axis f'_{km}, the forces are linear functions of the deflections at the points of their application, i.e.,

$$\left.\begin{aligned} M_{km} &= -C_k\beta^2 f'_{km}(\eta_k), \\ P_{km} &= Q_{km} - EJ_k\beta^4 f_{km}(\eta_k). \end{aligned}\right\} \qquad (77.11)$$

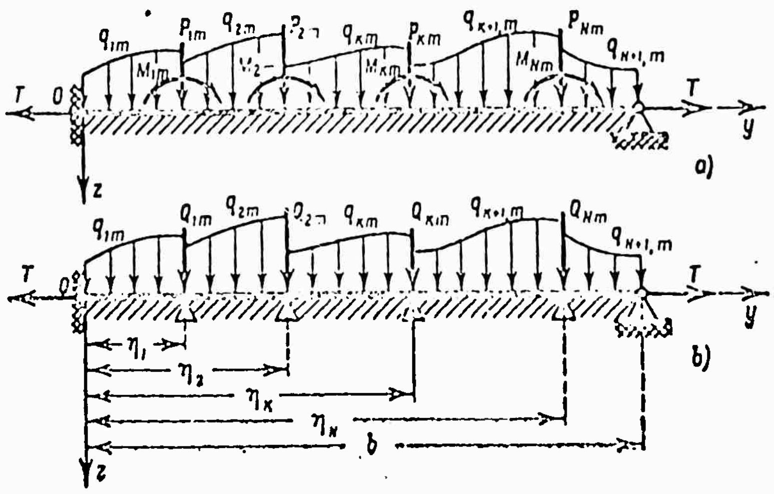

Fig. 153

This may be interpreted in the following way: at the points $y = \eta_k$ there are elastically rotating and at the same time elastically pliable supports at which the rigidities in turning and setting are equal to $C_k\beta^2$ and $EJ_k\beta^4$, respectively. Finally the functions $f_{km}(y)$ represent the deflections of beam sections with N intermediate supports which are elastically pivoted and elastically pliable, and this beam is assumed loaded by given distributed loads q_m and given concentrated forces Q_{km} applied to the supports (Fig. 153b). The ends of the beam are fixed in agreement with the fixings of the plate's sides $y = 0$ and $y = b$.

Having established this analogy we can use the methods of calculating multiple-span beams in order to determine the unknown functions f_{km}. Let us consider one of these methods.

We introduce in the consideration functions of the influence of the force and moment for the beam on the elastic support, by the tensile force T $\delta(\eta, y)$ and $\Delta(\eta, y)$.

The function $\delta(\eta, y)$ is the deflection of the beam at point y caused by a concentrated force equal to unity and applied to point η. Analogously, $\Delta(\eta, y)$ is the deflection of this beam caused by a concentrated unit moment applied to point η (Fig. 154). Knowing these functions we can write an expression for the deflection of the beam at an arbitrary point in the form

$$f_m(y) = \sum_{k=1}^{N} [P_{km}\delta(\eta_k, y) + M_{km}\Delta(\eta_k, y)] + F_m(y). \qquad (77.12)$$

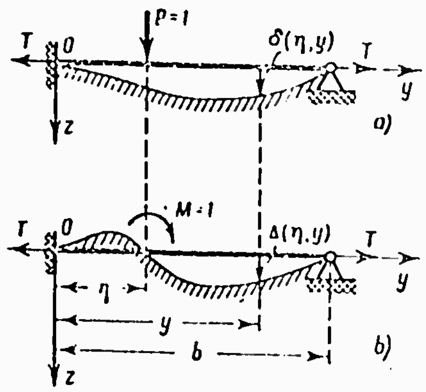

Fig. 154

F_m is here a particular solution of the nonhomogeneous equation of the type (77.9) depending on the distribution of the load q_m (the values taken for the sections of the beam are equal to $q_{1m}, q_{2m}, \ldots, q_{N+1, m}$).

The coefficients P_{km} and M_{km} are determined from Conditions (77.11) which yield a system of $2N$ equations corresponding to the number of unknowns. Introducing the abbreviations

$$\left.\begin{aligned} &\delta_{ij} = \delta(\eta_i, \eta_j), && \Delta_{ij} = \Delta(\eta_i, \eta_j), \\ &\delta'_{ij} = \left[\frac{\partial\delta(\eta_i, y)}{\partial y}\right]_{y=\eta_j}, && \Delta'_{ij} = \left[\frac{\partial\Delta(\eta_i, y)}{\partial y}\right]_{y=\eta_j}, \end{aligned}\right\} \qquad (77.13)$$

we can write this system in the following way

$$\left.\begin{aligned} &\sum_{i=1}^{N} P_{im}\delta'_{ik} + \sum_{i=1}^{N} M_{im}\left(\Delta'_{ik} - \frac{\delta}{C_k\beta^3}\right) = F'_m(\eta_k), \\ &\sum_{i=1}^{N} P_{im}\left(\delta_{ki} + \frac{\delta}{EJ_k\beta^4}\right) + \sum_{i=1}^{N} M_{im}\Delta_{ki} = \frac{Q_{km}}{EJ_k\beta^4} - F_m(\eta_k) \\ &(k = 1, 2, \ldots, N). \end{aligned}\right\} \qquad (77.14)$$

The symbol δ denotes here a quantity vanishing if $i \neq m$ and equal to unity if $i = m$.

Having determined the unknown coefficients we obtain the following expressions for the deflections of plate and ribs:

$$w = \sum_{m=1}^{\infty} \left\{ \sum_{i=1}^{N} [P_{im}\delta(\eta_i, y) + M_{im}\Delta(\eta_i, y)] + F_m(y) \right\} \sin\beta x; \qquad (77.15)$$

$$W_k = \sum_{m=1}^{\infty} \left[\sum_{i=1}^{N} (P_{im}\delta_{ik} + M_{im}\Delta_{ik}) + F_m(\eta_k) \right] \sin\beta x \qquad (77.16)$$

$$(k = 1, 2, \ldots, N).$$

When we neglect the rigidities of torsion of the ribbing, assuming $C_k = 0$, all equations and expressions become simpler. All moments M_{km} will be vanishing and for the forces P_{km} a system of N equations is obtained.

This method can be used successfully in all those cases where the number of ribs is small (not more than three). With a great number of ribs one must have recourse to other methods of calculation of multiple-span beams, since Eqs. (77.14) become very complicated.

§78. FUNCTIONS OF INFLUENCE FOR AN ORTHOTROPIC PLATE WITH SUPPORTED SIDES

The influence functions introduced in the preceding section are represented in the form

$$\delta(\eta, y) = \frac{1}{4\beta^3\sqrt{D_1D_2}}\, g(\eta, y), \quad \Delta(\eta, y) = \frac{1}{4\beta^3\sqrt{D_1D_2}}\, h(\eta, y), \qquad (78.1)$$

where $g(\eta, y)$ and $h(\eta, y)$ are nondimensional functions.

In many cases it is expedient to seek the function $g(\eta, y)$ in the form of two analytical expressions, one of them describes the form of the curved axis on the left-hand side of the point of attack of the force and the other that on the right-hand side:

$$g(\eta, y) = \begin{cases} g_1(y) & \text{with } 0 \leqslant y \leqslant \eta, \\ g_2(y) & \text{with } \eta \leqslant y \leqslant b. \end{cases} \qquad (78.2)$$

Both functions satisfy the equation

$$D_2 g^{IV} - 2D_3\beta^2 g'' + D_1\beta^4 g = 0 \qquad (78.3)$$

and the conditions

$$\left.\begin{aligned} &g_2(\eta) = g_1(\eta), && g_2'(\eta) = g_1'(\eta), \\ &g_2''(\eta) = g_1''(\eta), \quad g_2'''(\eta) && g_1'''(\eta) = 4\beta^3\sqrt{\frac{D_1}{D_2}}. \end{aligned}\right\} \qquad (78.4)$$

The first three conditions express the continuity of the beam deflections, their first derivatives and the bending moments while the fourth condition expresses the fact that in the transition through the point of attack of the concentrated force the crosscut force undergoes a discontinuity, i.e., a jump, equal to unity:

$$D_2\delta'''(\eta, \eta+0) - D_2\delta'''(\eta, \eta-0) = 1. \qquad (78.5)$$

In addition to this, the function g_1 must satisfy the conditions at the end of $y = 0$ and g_2 the conditions at the end of $y = b$ corresponding to the fixing of the plate's sides $y = 0$ and $y = b$.

The function h can also be represented in terms of various analytical expressions on either side of the point of attack of the moment. Considering the unit moment as the limiting case of equal and opposite forces Q attacking at the points η and $\eta + d\eta$ and satisfying the condition $Q\,d\eta = 1$, it is easy to recognize the following simple relation linking the functions δ and Δ:

$$\Delta(\eta, y) = -\frac{\partial\delta(\eta, y)}{\partial\eta}. \qquad (78.6)$$

Hence it follows that

$$h(\eta, y) = -\frac{1}{\beta}\cdot\frac{\partial g(\eta, y)}{\partial\eta}. \qquad (78.7)$$

The form of the functions g and h depends on the roots of the equation

$$D_2u^4 - 2D_3u^2 + D_1 = 0. \qquad (78.8)$$

These roots are connected with the complex parameters of bending by means of the simple relations $u_1 = -i\mu_1$, $u_2 = -i\mu_2$; they are the reciprocals of the quantities s_1, s_2 introduced in §73. Here too, three types of roots may be encountered.

Case 1. The roots of Eq. (78.8) are real and unequal:

$$\pm u_1, \ \pm u_2 \ (u_1 > 0, \ u_2 > 0).$$

Case 2. The roots are real and equal:

$$\pm u \ (u > 0).$$

Case 3. The roots are complex:

$$u \pm vi, \quad -u \pm vi \quad (u > 0, \ v > 0).$$

For a plate with four supported sides which corresponds to a beam supported at the ends, the function g may be represented in the form of a series

$$g(\eta, y) = \frac{8(md)^3}{\pi}\sqrt{\frac{D_1}{D_2}}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi\eta}{b}\sin\frac{n\pi y}{b}}{\left[(md)^2\frac{D_3}{D_2}+n^2\right]^2+\left[\frac{D_1}{D_2}-\left(\frac{D_3}{D_2}\right)^2\right](md)^4}. \qquad (78.9)$$

Here $d = b/a$.

For an isotropic plate we obtain the series

$$g(\eta, y) = \frac{8(md)^3}{\pi} \sum_{n=1}^{\infty} \frac{\sin\frac{n\pi\eta}{b} \sin\frac{n\pi y}{b}}{[(md)^2 + n^2]^2}. \qquad (78.10)$$

For a supported plate this function can also be determined in a finite form, when it is represented by the two analytical expressions g_1 and g_2 which will not be given here.*

The values of the functions g and h and their derivatives with respect to y, g' and h', with $\eta = \eta_i$ and $y = \eta_j$ will be abbreviated by g_{ij}, h_{ij}, g'_{ij} and $h'_{i,j}$.

In the following we shall tabulate the numerical values of the functions $g(\eta, y)$ for the plate considered in §67 with four supported sides, which is produced of an isotropic material and of veneer. In Table 15 we have compiled the values of g for an isotropic plate. Table 16 gives the values of g for a veneer plate in which the fibers of the sheet are perpendicular to side b, i.e., $D_1 > D_2$; in Table 17 the same is given for a plate whose sheet fibers are parallel to side a and, therefore, $D_1 < D_2$.

The values of the functions are calculated for the variables η and y taken at intervals equal to $b/8$. It is taken into account that the functions g for a supported plate are symmetrical relative to its variables so that $g_{ji} = g_{ij}$. Moreover, in this case the relation will hold good

$$g_{1n} = g_{7,8-n}, \; g_{2n} = g_{6,8-n}, \; \ldots, \; g_{7n} = g_{1,8-n}, \qquad (78.11)$$

for example,

$$g_{66} = g\left(\frac{5b}{8}, \frac{5b}{8}\right) = g_{33} = g\left(\frac{3b}{8}, \frac{3b}{8}\right),$$
$$g_{16} = g\left(\frac{b}{2}, \frac{3b}{4}\right) = g_{24} = g\left(\frac{b}{4}, \frac{b}{2}\right).$$

The functions g depend not only on η and y but also on the product md which represents a parameter. In our tables for each pair of values of η and y six values of g are given, which corresponds to $md = 0{,}5;\ 1;\ 1{,}5;\ 2;\ 2{,}5$ and 3. The numerical values of g are given in three to four decimal places.

Let us also give the form of the functions g for a plate in the form of an infinite strip of width a with supported sides, which corresponds to a beam of infinite length which extends on either side of a point taken as the origin of coordinates.

Case I

$$g(\eta, y) = \begin{cases} \dfrac{2}{u_1^2 - u_2^2}\left[u_1 e^{\beta u_2 (y-\eta)} - u_2 e^{\beta u_1 (y-\eta)}\right] \\ \text{for } -\infty < y \leqslant \eta; \\ \dfrac{2}{u_1^2 - u_2^2}\left[u_1 e^{\beta u_2 (\eta-y)} - u_2 e^{\beta u_1 (\eta-y)}\right] \\ \text{for } \eta \leqslant y < \infty. \end{cases} \qquad (78.12)$$

TABLE 15

Functions of g for a Supported Plate. Isotropic Material

η	y / md	$j=1$ $b/8$	2 $b/4$	3 $3b/8$	4 $b/2$	5 $5b/8$	6 $3b/4$	7 $7b/8$
$i=1$	0,5	0,0437	0,0691	0,0783	0,0719	0,0618	0,0119	0,0233
	1	0,180	0,259	0,258	0,223	0,172	0,115	0,0576
	1,5	0,328	0,413	0,318	0,259	0,171	0,105	0,0186
$b/8$	2	0,464	0,509	0,355	0,220	0,125	0,0651	0,0272
	2,5	0,581	0,557	0,319	0,165	0,0882	0,0339	0,0121
	3	0,677	0,570	0,266	0,115	0,0151	0,0157	0,0011
2	0,5		0,122	0,144	0,141	0,120	0,0861	
	1		0,438	0,480	0,430	0,310	0,230	
	1,5		0,676	0,664	0,522	0,367	0,222	
$b/4$	2	$g_{21}=g_{12}$	0,818	0,715	0,479	0,295	0,153	$g_{27}=g_{16}$
	2,5		0,899	0,697	0,397	0,215	0,0903	
	3		0,943	0,650	0,312	0,157	0,0197	
3	0,5			0,185	0,189	0,165		
	1			0,610	0,595	0,488		
	1,5			0,850	0,767	0,570		
$3b/8$	2	$g_{31}=g_{13}$	$g_{32}=g_{23}$	0,914	0,780	0,507	$g_{36}=g_{25}$	$g_{37}=g_{15}$
	2,5			0,977	0,731	0,109		
	3			0,988	0,665	0,316		
4	0,5				0,208			
	1				0,667			
	1,5				0,898			
$b/2$	2	$g_{41}=g_{11}$	$g_{42}=g_{21}$	$g_{43}=g_{31}$	0,971	$g_{45}=g_{31}$	$g_{46}=g_{21}$	$g_{47}=g_{14}$
	2,5				0,990			
	3				0,992			

Case 2

$$g(\eta,\ y)=\begin{cases}\dfrac{1}{u}e^{\beta u(y-\eta)}\left[1-\beta u(y-\eta)\right] \\ \text{for}\quad -\infty<y\leqslant\eta; \\ \dfrac{1}{u}e^{\beta u(\eta-y)}\left[1+\beta u(y-\eta)\right] \\ \text{for}\quad \eta\leqslant y<\infty.\end{cases} \tag{78.13}$$

Case 3

$$g(\eta,\ y)=\begin{cases}e^{\beta u(y-\eta)}\left[\dfrac{1}{u}\cos\beta v(y-\eta)-\dfrac{1}{v}\sin\beta v(y-\eta)\right] \\ \text{for}\quad -\infty<y\leqslant\eta; \\ e^{\beta u(\eta-y)}\left[\dfrac{1}{u}\cos\beta v(y-\eta)+\dfrac{1}{v}\sin\beta v(y-\eta)\right] \\ \text{for}\quad \eta\leqslant y<\infty.\end{cases} \tag{78.14}$$

For an infinite plate

$$\frac{1}{\beta}g'(\eta,\ y)=-\frac{1}{\beta}\cdot\frac{\partial g}{\partial\eta}=h(\eta,\ y), \tag{78.15}$$

which is easy to prove by differentiating Eqs. (78.12)-(78.14) with respect to y and η.

TABLE 16

Functions of g for a Supported Plate. Veneer. $D_1 > D_2$

η	y / md	$j = 1$ $b/8$	2 $b/4$	3 $3b/8$	4 $b/2$	5 $5b/8$	6 $3b/4$	7 $7b/8$
$i = 1$	0,5	0,113	0,171	0,182	0,166	0,133	0,092	0,016
	1	0,397	0,350	0,259	0,156	0,080	0,035	0,012
	1,5	0,465	0,386	0,178	0,053	0,007	−0,001	−0,003
$b/8$	2	0,565	0,337	0,087	0,005	−0,005	−0,003	−0,001
	2,5	0,617	0,265	0,0269	−0,006	−0,003	0,000	0,000
	3	0,610	0,187	0,031	−0,006	0,009	0,000	0,000
2	0,5		0,295	0,337	0,315	0,258	0,179	
	1		0,566	0,506	0,339	0,195	0,092	
	1,5		0,613	0,411	0,186	0,0651	0,001	
$b/4$	2	$g_{21} = g_{12}$	0,651	0,312	0,081	0,032	−0,006	$g_{27} = g_{16}$
	2,5		0,616	0,251	0,027	0,013	−0,002	
	3		0,643	0,183	0,003	0,061	0,000	
3	0,5			0,428	0,428	0,362		
	1			0,616	0,511	0,351		
	1,5			0,650	0,137	0,182		
$3b/8$	2	$g_{31} = g_{13}$	$g_{32} = g_{23}$	0,616	0,339	0,080	$g_{36} = g_{25}$	$g_{37} = g_{15}$
	2,5			0,614	0,254	0,028		
	3			0,614	0,182	0,002		
4	0,5				0,475			
	1				0,656			
	1,5				0,617			
$b/2$	2	$g_{41} = g_{11}$	$g_{43} = g_{21}$	$g_{43} = g_{34}$	0,615	$g_{45} = g_{34}$	$g_{46} = g_{24}$	$g_{47} = g_{14}$
	2,5				0,614			
	3				0,644			

In the following we give tables of the numerical values of the functions g, h and h'/β for isotropic and veneer strips. Table 18 gives the values of the functions for an isotropic plate while Table 19 gives them for two cases of venner plates. Everywhere $m = 1$.

TABLE 17

Functions of g for a Supported Plate. Veneer. $D_1 < D_2$

η	y / md	$j=1$ $b/8$	2 $b/4$	3 $3b/8$	4 $b/2$	5 $5b/8$	6 $3b/4$	7 $7b/8$
$i=1$	0,5	0,0170	0,0280	0,0326	0,0322	0,0276	0,0201	0,0106
	1	0,116	0,188	0,215	0,209	0,177	0,128	0,067
	1,5	0,296	0,461	0,507	0,474	0,389	0,274	0,141
$b/8$	2	0,500	0,730	0,748	0,652	0,501	0,338	0,168
	2,5	0,698	0,913	0,877	0,691	0,484	0,298	0,140
	3	0,892	1,106	0,919	0,637	0,388	0,209	0,088
2	0,5		0,0496	0,0601	0,0603	0,0523	0,0382	
	1		0,331	0,396	0,392	0,337	0,243	
	1,5		0,803	0,935	0,897	0,749	0,531	
$b/4$	2	$g_{21} = g_{12}$	1,248	1,383	1,252	0,995	0,673	$g_{27} = g_{16}$
	2,5		1,575	1,634	1,361	0,996	0,625	
	3		1,811	1,712	1,308	0,856	0,477	
3	0,5			0,0772	0,0802	0,0708		
	1			0,508	0,524	0,470		
	1,5			1,193	1,209	1,037		
$3b/8$	2	$g_{31} = g_{13}$	$g_{32} = g_{23}$	1,752	1,720	1,421	$g_{36} = g_{25}$	$g_{37} = g_{15}$
	2,5			2,060	1,932	1,502		
	3			2,200	1,951	1,395		
4	0,5				0,0878			
	1				0,575			
	1,5				1,334			
$b/2$	2	$g_{41} = g_{14}$	$g_{42} = g_{24}$	$g_{43} = g_{34}$	1,920	$g_{45} = g_{34}$	$g_{46} = g_{24}$	$g_{47} = g_{14}$
	2,5				2,200			
	3				2,288			

The values of the functions were calculated for an arbitrary point of attack of a unit force or moment η for the points of the beam $y = \eta + (l-1)\frac{a}{4}$, where $l = 1, 2, \ldots, 13$; i.e., the tables contain the quantities g_{ii}, h_{ii} and h'_{ii}/β. Other values of g_{ij} are obtained from data tabulated on the basis of the following relations

$$\left.\begin{array}{l} g_{11} = g_{22} = \ldots = g_{ii}, \\ g_{12} = g_{23} = \ldots = g_{i,\, i+1}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ g_{1n} = g_{2,\, n+1} = \ldots = g_{i,\, n+i-1}. \end{array}\right\} \qquad (78.16)$$

The same relations also exist for the functions h and h'/β.

TABLE 18

The Functions g, h, h'/β for an Infinite Strip. Isotropic Material

y	g	h	h'/β
η	1	0	−1
$\eta+\frac{a}{4}$	0,814	0,358	−0,0978
$\eta+\frac{a}{2}$	0,535	−0,327	0,119
$\eta+\frac{3a}{4}$	0,318	−0,223	0,129
$\eta+a$	0,179	−0,136	0,0925
$\eta+\frac{5a}{4}$	0,0971	−0,0774	0,0577
$\eta+\frac{3a}{2}$	0,0513	−0,0123	0,0334
$\eta+\frac{7a}{4}$	0,0266	−0,0225	0,0184
$\eta+2a$	0,0136	−0,0117	0,0099
$\eta+\frac{9a}{4}$	0,0069	−0,0060	0,0052
$\eta+\frac{5a}{2}$	0,0031	−0,0031	0,0027
$\eta+\frac{11a}{4}$	0,0017	−0,0015	0,0014
$\eta+3a$	0,0008	−0,0008	0,0007

Figure 155 shows the graphs of the functions g, h, h'/β for an isotropic strip. In Fig. 156 these functions are shown for a veneer strip in which the sheet fibers are perpendicular to the sides ($D_1 > D_2$) and in Fig. 157 the analogous graphs are shown for the case where the fibers are parallel to the sides ($D_1 < D_1$).

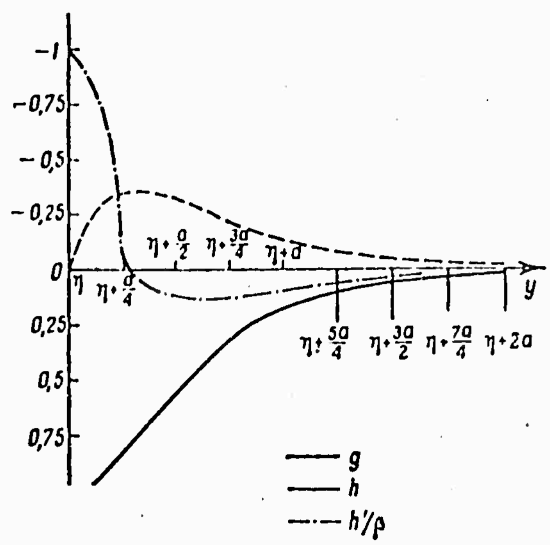

Fig. 155

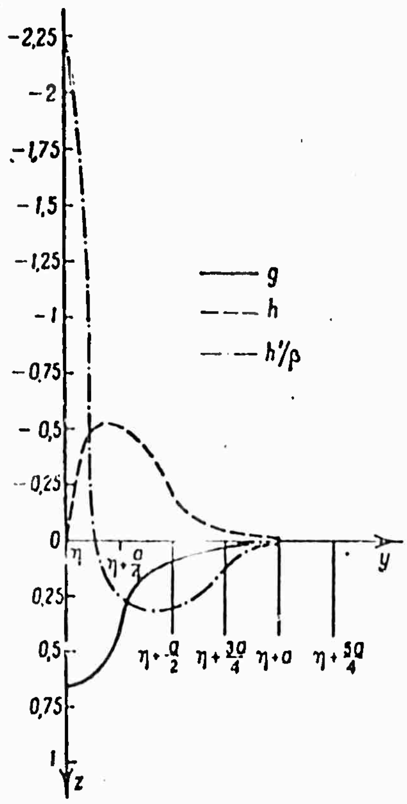

Fig. 156

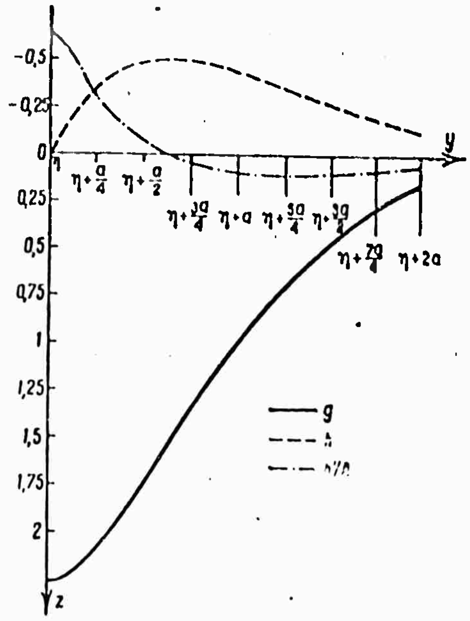

Fig. 157

TABLE 19

The Functions g, h, h'/β for an Infinite Strip. Veneer

y	$D_1 > D_2$			$D_1 < D_2$		
	g	h	h'/β	g	h	h'/β
η	0,645	0	— 2,241	2,252	0	— 0,618
$\eta+\frac{a}{4}$	0,338	— 0,466	0,267	2,091	— 0,355	— 0,287
$\eta+\frac{a}{2}$	0,0805	— 0,189	0,305	1,751	— 0,488	— 0,0707
$\eta+\frac{3a}{4}$	0,0030	— 0,0357	0,100	1,361	— 0,493	0,0455
$\eta+a$	— 0,0058	0,0021	0,0139	0,999	— 0,433	0,0977
$\eta+\frac{5a}{4}$	— 0,0026	0,0039	— 0,0032	0,692	— 0,349	0,111
$\eta+\frac{3a}{2}$	— 0,0006	0,0014	— 0,0025	0,452	— 0,261	0,104
$\eta+\frac{7a}{4}$	0,0000	0,0002	— 0,0007	0,276	— 0,188	0,0876
$\eta+2a$	0,0000	0,0000	— 0,0001	0,154	— 0,123	0,0683
$\eta+\frac{9a}{4}$	0,0000	0,0000	0,0000	0,0737	— 0,0804	0,0502
$\eta+\frac{5a}{2}$	0,0000	0,0000	0,0000	0,0246	— 0,0172	0,0348
$\eta+\frac{11a}{4}$	0,0000	0,0000	0,0000	— 0,0030	— 0,0218	0,0228
$\eta+3a$	0,0000	0,0000	0,0000	— 0,0163	— 0,0105	0,0140

§79. THE BENDING OF A PLATE WITH ONE RIB

By way of example we consider a rectangular orthotropic plate which is reinforced by a single rib directed along the axis of symmetry, $y = b/2$. The sides of the plate perpendicular to the rib are assumed supported, the two other sides fixed arbitrarily but likewise (for example, both sides supported or both sides free and the like).

Let us assume that the load Q acting on the rib is arbitrary and the loads q_1, q_2 acting on the plate are distributed symmetrically relative to the rib, i.e., $q_1(x, y) = q_2(x, b-y)$.

In the case given $N = 1$, $\eta = b/2$. The system of Eqs. (77.15) for the two unknowns M_{1m} and P_{1m} takes the form

$$\left.\begin{aligned} &P_{1m}\delta'_{11} + M_{1m}\left(\Delta'_{11} - \frac{1}{C\beta^2}\right) = F'_m\left(\frac{b}{2}\right), \\ &P_{1m}\left(\delta_{11} + \frac{1}{EJ\beta^4}\right) + M_{1m}\Delta_{11} = \frac{Q_m}{EJ\beta^4} - F_m\left(\frac{b}{2}\right), \end{aligned}\right\} \tag{79.1}$$

where Q_m is the Fourier series expansion coefficient of load Q, $\beta = m\pi/a$.

By virtue of the symmetry

$$F'_m\left(\frac{b}{2}\right) = 0, \quad \delta'_{11} = \Delta_{11} = 0. \tag{79.2}$$

Consequently,

$$M_{1m} = 0, \quad P_{1m} = \frac{Q_m - EJ\beta^4 F_m\left(\frac{b}{2}\right)}{1 + EJ\beta^4\delta_{11}}, \tag{79.3}$$

and we obtain an expression for the deflection of the plate

$$w = \sum_{m=1}^{\infty}\left[\frac{Q_m - EJ\beta^4 F_m\left(\frac{b}{2}\right)}{1 + EJ\beta^4\delta_{11}}\,\delta\left(\frac{b}{2}, y\right) + F_m(y)\right]\sin\beta x. \tag{79.4}$$

In order to obtain a final expression for the deflection, the given loads q_1 and q_2 must be expanded in Fourier series (77.7) and we must find a particular solution to Eq. (77.9).

Let us consider the case of an infinite strip with supported sides reinforced by a single rib which receives the whole external load of Q $(q_1 = q_2 = 0)$. We let the x-axis coincide with the axis of the rib and the y-axis with the side of the strip (Fig. 158). The load Q is assumed to be arbitrary function for y (which, of course, must satisfy the Dirichlet conditions as, otherwise, it could not be expanded in a Fourier series).

In the case of real and unequal roots u_1, u_2 the function g, which is proportional to the influence function δ, has the form of (78.12) and

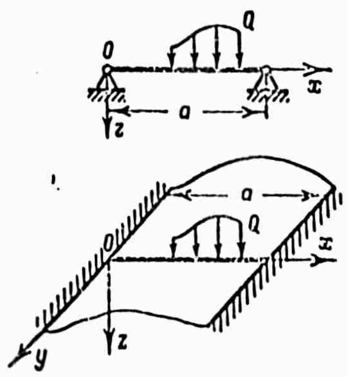

Fig. 158

$$\delta_{11} = \delta(0,\ 0) = \frac{1}{2\beta^3 \sqrt{D_1 D_2}\,(u_1 + u_2)}, \quad F_m = 0. \tag{79.5}$$

The deflection of the strip in front of the rib, i.e., with $-\infty < y < 0$ is determined from the formula

$$w = \frac{1}{u_1 - u_2} \sum_{m=1}^{\infty} \frac{Q_m (u_1 e^{\beta u_1 y} - u_2 e^{\beta u_2 y})}{EJ\beta^4 + 2\beta^3 \sqrt{D_1 D_2}\,(u_1 + u_2)} \sin \beta x. \tag{79.6}$$

The deflection of the strip behind the rib ($0 < y < \infty$) is obtained when in this expression y is replaced by $-y$. From (79.6), on the assumption that $y = 0$, we obtain an expression for the deflection of the rib

$$W = \frac{a^4}{EJ\pi^4} \sum_{m=1}^{\infty} \frac{Q_m}{m^4 \left[1 + \frac{2a \sqrt{D_1 D_2}\,(u_1 + u_2)}{EJ\pi m}\right]} \sin \frac{m\pi x}{a}. \tag{79.7}$$

The expression for the deflection of a beam with supported ends, which possesses the length a and the rigidity EJ and which is not fastened with a strip, i.e., it bridges over an empty space, loaded by Q, can be represented in the form of a Fourier series

$$\overline{W} = \frac{a^4}{EJ\pi^4} \sum_{m=1}^{\infty} \frac{Q_m}{m^4} \sin \frac{m\pi x}{a}. \tag{79.8}$$

A formula for the deflection of such a beam, which lies on a massive elastic support and which is bent by the load Q, can be obtained on the supposition that the support reaction is proportional to the deflection, and has the form

$$W = \frac{a^4}{EJ\pi^4} \sum_{m=1}^{\infty} \frac{Q_m}{m^4 \left(1 + \frac{ka^4}{EJ\pi^4 m^4}\right)} \sin \frac{m\pi x}{a}. \tag{79.9}$$

k is here the elastic coefficient of the support (bed coefficient).

Comparing Eqs. (79.7) and (79.9) we see that the strip with

which the beam is reinforced (the rib) need not be considered as an elastic base according to Winkler; it is obvious that the series entering these formulas are different in structure. If the load Q is distributed uniformly over the length of the rib, the coefficients of the Fourier series are equal to

$$Q_m = \begin{cases} \dfrac{4Q}{m\pi} & \text{for } m = 1, 3, 5, \ldots, \\ 0 & \text{for } m = 2, 4, 6, \ldots \end{cases} \tag{79.10}$$

The terms of the series (79.7) and (79.8) will decrease fast enough as m may only assume odd values. Retaining in these series only the first terms we obtain an approximate formula for the estimation of the deflection of the beam (rib) fastened by an elastic strip:

$$W(x) = \overline{W}(x)\,\frac{1}{1 + 2\dfrac{a\sqrt{D_1 D_2}}{EJ\pi}(u_1 + u_2)}. \tag{79.11}$$

We obtain formulas for the other two cases of roots of Eq. (78.8) from the above expressions when we assume $u_1 = u_2 = u$ or, in the case of complex roots, $u_1 = u + vi$, $u_2 = u - vi$. In particular, for an isotropic strip with Young's modulus E' and Poisson's coefficient ν' $(u_1 = u_2 = 1)$ we obtain the following approximate formula:

$$W(x) = \overline{W}(x)\,\frac{1}{1 + \dfrac{ah^3}{3\pi J}\cdot\dfrac{E'}{E(1 - \nu'^2)}}. \tag{79.12}$$

§80. THE BENDING OF HOMOGENEOUS ELLIPTIC AND CIRCULAR PLATES FIXED ON THEIR EDGES

Let us consider an elliptic homogeneous plate whose edge is fixed throughout its length, which is bent by a normal load q distributed uniformly over the whole surface area. This case of bending problem may be solved by elementary means in an exact way and we shall derive the solution. In the general case we shall assume the plate to be nonorthotropic.

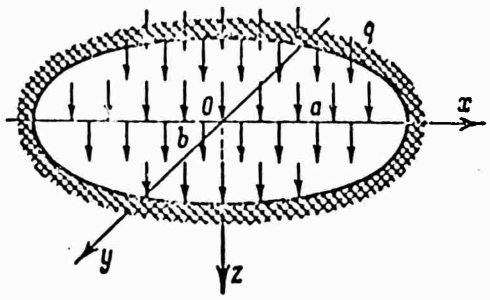

Fig. 159

We let the x and y axes coincide with the principal axes of the ellipse (Fig. 159) and denote by a and b the principal semi-axes of it, $c = a/b$. In the general case of the nonorthotropic plate the deflection equation will read

$$D_{11}\frac{\partial^4 w}{\partial x^4}+4D_{16}\frac{\partial^4 w}{\partial x^3\,\partial y}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\,\partial y^2}+4D_{26}\frac{\partial^4 w}{\partial x\,\partial y^3}+D_{22}\frac{\partial^4 w}{\partial y^4}=q. \tag{80.1}$$

On the right-hand side we have a constant quantity. The problem is reduced to the determination of a solution to Eq. (80.1) which satisfies the boundary conditions

$$w=0,\quad \frac{dw}{dn}=\frac{\partial w}{\partial x}\cos(n,\,x)+\frac{\partial w}{\partial y}\cos(n,\,y)=0 \tag{80.2}$$

(n is the direction of the inner normal to the contour of the plate). It is easy to show that this solution is a polynomial of fourth degree

$$w=\frac{qa^4}{64D'}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2, \tag{80.3}$$

where

$$D'=\frac{1}{8}\left[3D_{11}+2(D_{12}+2D_{66})c^2+3D_{22}c^4\right]. \tag{80.4}$$

Knowing the deflection, we obtain from Eqs. (61.9) the moments which, obviously, will be polynomials of degree two with respect to x and y, and from Eqs. (61.10) we obtain the crosscut forces which will be linear functions of x and y. It is also easy to find the points at which the stresses are highest. The deflection has its maximum value in the center

$$w_{\max}=\frac{qa^4}{64D'}. \tag{80.5}$$

Let us consider in greater detail the solution for a plate of orthotropic material in which the principal directions of elasticity are parallel to the directions of the principal axes of the ellipse. In this case

$$\left.\begin{aligned}&D_{11}=D_1,\quad D_{22}=D_2,\quad D_{12}+2D_{66}=D_3,\quad D_{16}=D_{26}=0,\\&D'=\frac{1}{8}(3D_1+2D_3c^2+3D_2c^4).\end{aligned}\right\} \tag{80.6}$$

The moments and crosscut forces are determined from the formulas

$$\left.\begin{aligned}&M_x=-\frac{qa^2D_1}{16D'}\left[(3+\nu_2c^2)\frac{x^2}{a^2}+(1+3\nu_2c^2)\frac{y^2}{b^2}-1-\nu_2c^2\right],\\&M_y=-\frac{qa^2D_2}{16D'}\left[(3\nu_1+c^2)\frac{x^2}{a^2}+(\nu_1+3c^2)\frac{y^2}{b^2}-\nu_1-c^2\right],\\&H_{xy}=\frac{qa^2D_k}{16D'}\cdot\frac{4c}{ab}\,xy;\end{aligned}\right\} \tag{80.7}$$

$$\left.\begin{aligned}&N_x=-\frac{qa}{8D'}\left[D_1(3+\nu_2c^2)+2D_kc^2\right]\frac{x}{a},\\&N_y=-\frac{qa}{8D'}\left[D_2(\nu_1+3c^2)+2D_kc\right]\frac{y}{b}.\end{aligned}\right\} \tag{80.8}$$

The bending moment reaches its maximum values at the ends of the major axis or at the ends of the minor axis.

If

$$D_1>D_2c^2,$$

we have

$$M_{max} = |M_x|_{\substack{x=a \\ y=0}} = \frac{qa^2}{8} \cdot \frac{D_1}{D'}; \tag{80.9}$$

but if

$$D_1 < D_2 c^2,$$

we have

$$M_{max} = |M_y|_{\substack{x=0 \\ y=b}} = \frac{qa^2}{8} \cdot \frac{D_2 c^2}{D'}. \tag{80.10}$$

The deflection of a plate of isotropic material with a rigidity $D = \frac{Eh^3}{12(1-\nu^2)}$ is obtained by means of a well-known formula obtained from (80.3):*

$$w = \frac{qa^4}{8D} \cdot \frac{1}{3+2c^2+3c^4} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2. \tag{80.11}$$

The solution for the homogeneous circular plate of radius a is a particular case of the solutions given above with $b = a$ and $c = 1$. Thus, for example, in the case of an orthotropic plate where the axes x and y coincide with the principal axes of elasticity,

$$D' = \frac{1}{8}(3D_1 + 2D_3 + 3D_2); \tag{80.12}$$

$$w = \frac{qa^4}{64D'}\left(1 - \frac{r^2}{a^2}\right)^2; \tag{80.13}$$

$$\left.\begin{aligned} M_x &= -\frac{qD_1}{16D'}[(1+\nu_2)(r^2 - a^2) + 2(x^2 + \nu_2 y^2)], \\ M_y &= -\frac{qD_2}{16D'}[(1+\nu_1)(r^2 - a^2) + 2(\nu_1 x^2 + y^2)], \\ H_{xy} &= -\frac{qD_k}{16D'} \cdot 4xy; \end{aligned}\right\} \tag{80.14}$$

$$\left.\begin{aligned} N_x &= -\frac{q}{8D'}(3D_1 + D_3)x, \\ N_y &= -\frac{q}{8D'}(D_3 + 3D_2)y \\ &(r = \sqrt{x^2 + y^2}). \end{aligned}\right\} \tag{80.15}$$

Note that the solutions given can be generalized to the case of a more complex load distributed according to the law of an integral algebraic function

$$q = \sum_{m=1}^{M} \sum_{n=1}^{N} q_{mn} x^m y^n. \tag{80.16}$$

In this case the expression for the deflection must be sought in the form of

$$w = \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} x^m y^n. \tag{80.17}$$

This function satisfies the boundary conditions (80.2) for

arbitrary values of the coefficients A_{mn} and the coefficients are all determined on the basis of Eq. (80.1).

Let, for example, an elliptic orthotropic plate be subject to the action of a normal load given in the form of a linear function of the variables x and y:

$$q = q_{00} + q_{10}\frac{x}{a} + q_{01}\frac{y}{b}, \tag{80.18}$$

where q_{00}, q_{10}, q_{01} are constant coefficients. After all transformations we obtain the following expression for the deflection

$$w = \frac{a^4}{64}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 \left[\frac{q_{00}}{D'} + \right. \\ \left. + \frac{8q_{10}}{3(5D_1 + 2D_3c^2 + D_2c^4)}\cdot\frac{x}{a} + \frac{8q_{01}}{3(D_1 + 2D_3c^2 + 5D_2c^4)}\cdot\frac{y}{b}\right]. \tag{80.19}$$

The problem of the bending of an elliptic plate whose edge rests on a support is more complicated. At present we only know an approximate solution for a circular orthotropic plate with given elastic constants bent uniformly by a distributed load; this solution was obtained by Okubo.*

§81. THE BENDING OF A CIRCULAR CURVILINEAR-ANISOTROPIC PLATE UNDER SYMMETRIC LOAD

Let us consider a circular plate of radius a and constant thickness h of a material which is orthotropic but possesses cylindrical anisotropy. It is supposed that the axis of anisotropy passes through the center and is perpendicular to the plane of the plate and that planes parallel to the mid-plane and also planes which pass through the axis of anisotropy are planes of elastic symmetry (in other words, in each point there are three principal directions of elasticity: the normal, radial and tangential directions). The plate, whose edge is fixed, supported or free on its whole extension, is subjected to the action of a normal load distributed symmetrically relative to the axis of anisotropy (axis of rotation); this load causes a bending with respect to the surface of revolution. We have to determine the deflection, the moments and all other quantities necessary for a strength calculation.

Let the point of intersection of the axis of anisotropy and the mid-plane, i.e., the pole of anisotropy, coincide with the origin and the axis of anisotropy with the z-axis of the cylindrical system of coordinates; the arbitrary radial direction x is taken in the direction of the polar axis from which the angle θ is measured (Fig. 160). In this system of coordinates the equations of the generalized Hooke's law will have the form (70.1) and the equations of the theory of bending of plates will have the form (70.3)-(70.7). The load q is a function of r and the deflection w, which determines the surface of revolution, is also a function of r alone. On the basis of this Eqs. (70.4)-(70.5) and Eq. (70.7) can be rewritten as follows:

$$\left.\begin{aligned}
M_r &= -D_r\left(w'' + \frac{\nu_\theta}{r} w'\right),\\
M_\theta &= -D_\theta\left(\nu_r w'' + \frac{1}{r} w'\right),\\
N_r &= -D_r\left(w''' + \frac{1}{r} w''\right) + D_\theta \frac{1}{r^2} w',\\
H_{r\theta} &= 0, \quad N_\theta = 0;
\end{aligned}\right\} \tag{81.1}$$

$$w^{IV} + \frac{2}{r} w''' + k^2\left(-\frac{1}{r^2} w'' + \frac{1}{r^3} w'\right) = \frac{q(r)}{D_r} \tag{81.2}$$

$$\left(k = \sqrt{\frac{D_\theta}{D_r}} = \sqrt{\frac{E_\theta}{E_r}}\right).$$

With $k = 1$ Eq. (81.2) coincides with the deflection equation of the isotropic plate with the rigidity D_r.* A general expression for the deflection, i.e., a general integral of Eq. (81.2), for $k \neq 1$ has the form

$$w = A + Br^2 + Cr^{1+k} + Er^{1-k} + w_0(r), \tag{81.3}$$

where $w_0(r)$ is a particular solution to Eq. (81.2) which depends on the law according to which the load $q(r)$ is distributed in the radial direction; A, B, C, E are arbitrary constants which are determined from the boundary conditions on the edge of the plate and from the conditions in the center. The boundary conditions will have the form:

for a fixed edge

$$\text{with } r = a \quad w = 0, \quad w' = 0; \tag{81.4}$$

for a supported edge

$$\text{with } r = a \quad w = 0, \quad w'' + \frac{\nu_\theta}{r} w' = 0; \tag{81.5}$$

for a free edge

$$r = a \quad w'' + \frac{\nu_\theta}{r} w' = 0, \quad w''' + \frac{1}{r} w'' - \frac{k^2}{r^2} w' = 0. \tag{81.6}$$

The conditions in the center ($r = 0$) are reduced to the requirement of limitedness of deflection and absence of a corner point on the bent surface $[w'(0) = 0]$ or, in other cases, according to the load, to the requirement of limitedness of the moments and crosscut forces.

By way of example we consider the bending caused by the load q distributed over the whole area of the plate (Fig. 160). This problem was already solved independently by Carrier.**

In this case q = const. Assuming k not to be equal to 1 or 3 we obtain

$$w = A + Cr^{1+k} + \frac{qr^4}{8(9-k^2)D_r}; \tag{81.7}$$

the constants B and E are assumed to be equal to zero since otherwise the deflection and the crosscut force in the center would be infinitely high. Having determined the constants A and C from the

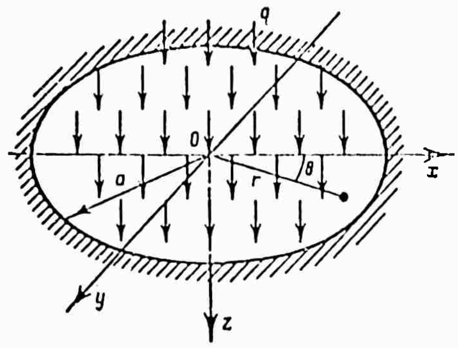

Fig. 160

conditions on the edge we obtain the following results.

When the edge of the plate is fixed,

$$w = \frac{qa^4}{8(9-k^2)(1+k)D_r}\left[3 - k - 4\left(\frac{r}{a}\right)^{1+k} + (1+k)\left(\frac{r}{a}\right)^4\right]; \tag{81.8}$$

$$\left.\begin{aligned} M_r &= \frac{qa^2}{2(9-k^2)}\left[(k+\nu_\theta)\left(\frac{r}{a}\right)^{k-1} - (3+\nu_\theta)\left(\frac{r}{a}\right)^2\right],\\ M_\theta &= \frac{qa^2k^2}{2(9-k^2)}\left[(k\nu_r+1)\left(\frac{r}{a}\right)^{k-1} - (3\nu_r+1)\left(\frac{r}{a}\right)^2\right],\\ N_r &= -\frac{qr}{2}. \end{aligned}\right\} \tag{81.9}$$

The maximum deflection (in the center)

$$w_{max} = \frac{qa^4}{8(3+k)(1+k)D_r}. \tag{81.10}$$

The bending moments at the plate's edge are equal to

$$M_r = -\frac{qa^2}{2(3+k)}, \quad M_\theta = \nu_\theta M_r. \tag{81.11}$$

The value of the bending moments in the center depend essentially on the rigidity ratio D_θ/D_r or, which is the same, on that of Young's moduli E_θ/E_r, i.e., on the quantity k.

With $k > 1$, $E_\theta > E_r$ in the center $M_r = M_\theta = 0$; the bending moment will then reach its maximum value on the edge:

$$M_{max} = |M_r|_{r=a} = \frac{qa^2}{2(3+k)}. \tag{81.12}$$

With $k < 1$, $E_\theta < E_r$ the moments grow unlimitedly as we approach the center; in this case in the center a stress concentration will arise and, theoretically, at this place $M_r = M_\theta = \infty$.

With k = 1 we obtain for the deflection of an isotropic plate

$$w = \frac{qa^4}{64D}\left[1 - \left(\frac{r}{a}\right)^2\right]^2. \qquad (81.13)$$

In the case of the supported edge

$$w = \frac{qa^4}{8(9-k^2)D_r}\left[\frac{(3-k)(4+k+\nu_\theta)}{(1+k)(k+\nu_\theta)} - \frac{4(3+\nu_\theta)}{(1+k)(k+\nu_\theta)}\left(\frac{r}{a}\right)^{k+1} + \left(\frac{r}{a}\right)^4\right]; \qquad (81.14)$$

$$\left.\begin{aligned} M_r &= \frac{q(3+\nu_\theta)a^2}{2(9-k^2)}\left[\left(\frac{r}{a}\right)^{k-1} - \left(\frac{r}{a}\right)^2\right], \\ M_\theta &= \frac{qa^2k^2}{2(9-k^2)}\left[\frac{(3+\nu_\theta)(1+k\nu_r)}{k+\nu_\theta}\left(\frac{r}{a}\right)^{k-1} - (1+3\nu_r)\left(\frac{r}{a}\right)^2\right], \\ N_r &= -\frac{qr}{2}. \end{aligned}\right\} \qquad (81.15)$$

The maximum deflection (in the center) is equal to

$$w_{max} = \frac{qa^4}{8(3+k)D_r}\cdot\frac{4+k+\nu_\theta}{(1+k)(k+\nu_\theta)}. \qquad (81.16)$$

On the edge ($r = a$)

$$M_r = 0, \quad M_\theta = \frac{qa^2k^2}{2(3+k)}\cdot\frac{1-\nu_r\nu_\theta}{k+\nu_\theta}. \qquad (81.17)$$

Also in the case where the edge is resting on a support, with $k > 1$ $M_r = M_\theta = 0$ in the center and $k < 1$ the moments grow unlimitedly as we approach the center.

For an isotropic plate

$$w = \frac{qa^4}{64D}\left[\frac{5+\nu}{1+\nu} - \frac{2(3+\nu)}{1+\nu}\left(\frac{r}{a}\right)^2 + \left(\frac{r}{a}\right)^4\right]. \qquad (81.18)$$

The formulas for the case of $k = 3$ is obtained from those given above by means of a limiting transition.

It is quite simple to obtain a solution of the problem on the bending of a curvilinear-anisotropic plate with respect to the surface of revolution in the case where a round hole is located in the center. In addition to the two conditions on the outer edge we have two conditions for the edge of the aperture. From these four conditions we can determine all constants, A, B, C, E.

§82. BENDING OF A ROUND CURVILINEAR-ANISOTROPIC PLATE BY A CONCENTRATED FORCE

If a circular plate possessing cylindrical anisotropy such as considered in the preceding section is bent by a normal concentrated force P applied to the center (Fig. 161), the deflection equation for it has the form (81.3) where $w_0 = 0$. The constant E must be set equal to zero since otherwise the bent surface would have a corner point at $r = 0$, whereas we should have $w'(0) = 0$. From the conditions on the contour we obtain two equations for the three constants A, B, C. The lacking equation is obtained when we cut out of the plate a disk with arbitrary radius r and consider its equilibrium. It is obvious that the crosscut forces distributed on the edge of this disk must be in equi-

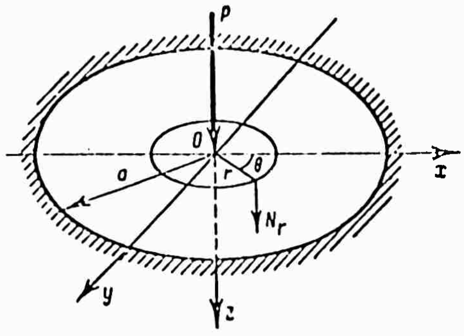

Fig. 161

librium with force P and therefore

$$N_r \cdot 2\pi r + P = 0, \tag{82.1}$$

from which we obtain

$$B = \frac{P}{4\pi(1-k^2)D_r}. \tag{82.2}$$

The expression for the deflection takes the form

$$w = \frac{P}{4\pi(1-k^2)D_r}r^2 + A + Cr^{1+k}. \tag{82.3}$$

For a plate with fixed edge and $k \neq 1$

$$w = \frac{Pa^2}{4\pi(1-k^2)(1+k)D_r}\left[1-k+(1+k)\left(\frac{r}{a}\right)^2 - 2\left(\frac{r}{a}\right)^{1+k}\right]; \tag{82.4}$$

$$\left.\begin{aligned} M_r &= \frac{P}{2\pi(1-k^2)}\left[(k+\nu_\theta)\left(\frac{r}{a}\right)^{k-1} - (1+\nu_\theta)\right],\\ M_\theta &= \frac{P}{2\pi(1-k^2)}\left[(1+k\nu_r)\left(\frac{r}{a}\right)^{k-1} - (1+\nu_r)\right],\\ N_r &= -\frac{P}{2\pi r}. \end{aligned}\right\} \tag{82.5}$$

The maximum deflection (in the center) is equal to

$$w_{max} = \frac{Pa^2}{4\pi(1+k)^2 D_r}. \tag{82.6}$$

It must be remarked that with $k > 1$ $(E_\theta > E_r)$ the bending moments in the center are finite, but with $k \leqslant 1$ $(E_\theta \leqslant E_r)$ they grow unlimitedly toward the center. Approaching the center, the crosscut force grows unlimitedly, independent of the ratio of moduli E_θ/E_r.

For an isotropic plate with fixed edge

$$w = \frac{Pa^2}{16\pi D}\left[1-\left(\frac{r}{a}\right)^2 + 2\left(\frac{r}{a}\right)^2 \ln\frac{r}{a}\right]. \tag{82.7}$$

In the case of a supported edge

$$w = \frac{Pa^3}{4\pi(1-k^2)D_r}\left[\frac{(2+k+\nu_\theta)(1-k)}{(1+k)(k+\nu_\theta)} + \left(\frac{r}{a}\right)^3 - \frac{2(1+\nu_\theta)}{(1+k)(k+\nu_\theta)}\left(\frac{r}{a}\right)^{1+k}\right]; \tag{82.8}$$

$$\left.\begin{aligned} M_r &= \frac{P(1+\nu_\theta)}{2\pi(1-k^2)}\left[\left(\frac{r}{a}\right)^{k-1} - 1\right], \\ M_\theta &= \frac{Pk^3}{2\pi(1-k^2)}\left[\frac{(1+\nu_\theta)(k\nu_r+1)}{k+\nu_\theta}\left(\frac{r}{a}\right)^{k-1} - (1+\nu_r)\right], \\ N_r &= -\frac{P}{2\pi r}; \end{aligned}\right\} \tag{82.9}$$

$$w_{max} = \frac{Pa^3}{4\pi(1+k)^3 D_r}\cdot\frac{2+k+\nu_\theta}{k+\nu_\theta}. \tag{82.10}$$

As regards the moments the same holds true as in the case of the plate fixed at the edge.

For an isotropic plate

$$w = \frac{Pa^2}{16\pi D}\left\{\frac{3+\nu}{1+\nu}\left[1-\left(\frac{r}{a}\right)^2\right] + 2\left(\frac{r}{a}\right)^2 \ln\frac{r}{a}\right\}. \tag{82.11}$$

Note that the deflection, moments and crosscut forces in a curvilinear-anisotropic plate bent by a force [or also by an arbitrary symmetric load $q(r)$], only depends on Young's moduli E_r, E_θ and Poisson's coefficients ν_r, ν_θ and is independent of the modulus of shear $G_{r\theta}$. For a curvilinear-anisotropic plate for which $E_r = E_\theta$ and $\nu_r = \nu_\theta$, the deflection, moments and crosscut forces obtained are precisely the same as for an isotropic plate under the same conditions.

A solution of the problem on the bending of a circular plate with cylindrical anisotropy by a force attacking at the center was also obtained independently by Carrier.*

The problem on the bending of a circular plate possessing cylindrical anisotropy by a normal force applied at a certain distance b from the center (which agrees with the pole of anisotropy) is more complex. For the case of a fixed edge this problem was solved by Sen Gupta.** We shall discuss the method of solution and give the fundamental results obtained by this author.

We place the origin of coordinates in the center of the plate and direct the x-axis from which the angle θ is measured along the line connecting the center with the point of attack of the force (Fig. 162). Let us denote by w_1 the deflection of a part of the plate which is bounded by a circle of $r = b$ and the outer edge $r = a$, and by w_2 the deflection of the part of the plate which is inside the circle $r = b$. These deflections satisfy Eq. (70.7) while the moments and crosscut forces for the outer and inner parts are connected with the deflection by Eqs. (70.4) and (70.5) where w must be replaced by w_1 and w_2 respectively.

The concentrated force P may be considered as a limiting case of a load distributed on a small arc of the circle $r = b$ whose resultant is equal to P. Having expanded this load as a function of the variable θ in a Fourier series, in the limiting case we obtain a divergent series

$$\frac{P}{\pi b}\left(\frac{1}{2}+\sum_{n=1}^{\infty}\cos n\theta\right). \tag{82.12}$$

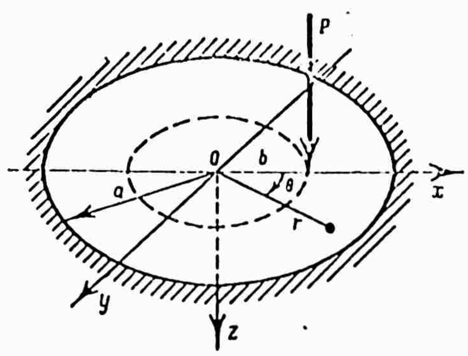

Fig. 162

In the center of the plate the deflection w_2 and its derivative with respect to r, and also the crosscut forces, must be finite quantities. On the outer contour $r = a$ two conditions must be satisfied which depend on the way of fixing of the edge. Inside the plate the deflection, its derivative with respect to r and the moments M_r are steady functions whereas the force N_r has a discontinuity in the outline of the circle $r = b$. Consequently, on the contour $r = b$ the following conditions must be satisfied:

$$\left.\begin{aligned} &w_1 = w_2, \qquad \frac{\partial w_1}{\partial r} = \frac{\partial w_2}{\partial r}, \\ &M_r^{(1)} = M_r^{(2)}, \qquad N_r^{(1)} - N_r^{(2)} = \frac{P}{\pi b}\left(\frac{1}{2}+\sum_{n=1}^{\infty}\cos n\theta\right), \end{aligned}\right\} \tag{82.13}$$

where $M_r^{(1)}, N_r^{(1)}$ denote the moment and crosscut force for the outer part of the plate and $M_r^{(2)}, N_r^{(2)}$ these quantities for the inner part of it.

Seeking the solutions to Eqs. (70.7) in the form of series

$$\left.\begin{aligned} w_1 &= R_0^{(1)}(r) + R_1^{(1)}(r) + \sum_{n=2}^{\infty} R_n^{(1)}(r)\cos n\theta, \\ w_2 &= R_0^{(2)}(r) + R_1^{(2)}(r) + \sum_{n=2}^{\infty} R_n^{(2)}(r)\cos n\theta, \end{aligned}\right\} \tag{82.14}$$

we obtain the following expressions for the functions $R_n^{(i)}$:

$$\left.\begin{aligned} R_0^{(i)} &= A_0^{(i)} + B_0^{(i)} r^2 + C_0^{(i)} r^{1+k} + E_0^{(i)} r^{1-k}, \\ R_1^{(i)} &= A_1^{(i)} r^{1+\beta_1} + B_1^{(i)} r^{1-\beta_1} + C_1^{(i)} r + E_1^{(i)} r \ln r, \\ R_n^{(i)} &= A_n^{(i)} r^{1+\beta_n} + B_n^{(i)} r^{1-\beta_n} + C_n^{(i)} r^{1+\gamma_n} + E_n^{(i)} r^{1-\gamma_n} \end{aligned}\right\} \tag{82.15}$$

($i = 1,\ 2,\ n = 2,\ 3,\ 4,\ \ldots,\ A_k^{(i)},\ B_k^{(i)},\ C_k^{(i)},\ E_k^{(i)}$ are arbitrary constants).

We used the denotations

$$
\left.
\begin{aligned}
k &= \sqrt{\frac{D_\theta}{D_r}} = \sqrt{\frac{E_\theta}{E_r}}, \qquad \beta_1 = \sqrt{1 + \frac{D_\theta + 2D_{r\theta}}{D_r}},\\
\beta_n &= \sqrt{\frac{1}{2}\left(1 + \frac{D_\theta + 2n^2 D_{r\theta}}{D_r}\right) + \sqrt{\frac{1}{4}\left(1 + \frac{D_\theta + 2n^2 D_{r\theta}}{D_r}\right)^2 - \frac{D_\theta}{D_r}(n^2-1)^2}},\\
\gamma_n &= \sqrt{\frac{1}{2}\left(1 + \frac{D_\theta + 2n^2 D_{r\theta}}{D_r}\right) - \sqrt{\frac{1}{4}\left(1 + \frac{D_\theta + 2n^2 D_{r\theta}}{D_r}\right)^2 - \frac{D_\theta}{D_r}(n^2-1)^2}}.
\end{aligned}
\right\}
\tag{82.16}
$$

For the inner part ($r < b$) we obtain from the conditions for the center

$$
\left.
\begin{aligned}
B_0^{(2)} &= E_0^{(2)} = 0,\\
B_1^{(2)} &= E_1^{(2)} = 0,\\
B_n^{(2)} &= E_n^{(2)} = 0.
\end{aligned}
\right\}
\tag{82.17}
$$

The other constants are all obtained from the conditions on the outer edge and on the line $r = b$.

For a plate with fixed edges we have

with $r = a$

$$
w_1 = 0, \qquad \frac{\partial w_1}{\partial r} = 0. \tag{82.18}
$$

The final expressions for the variable coefficients of the series (82.14) take the form

$$
\left.
\begin{aligned}
R_0^{(1)}(r) &= \frac{Pa^3}{4\pi(k^2-1)D_r}\left[\frac{1}{1+k}\left(2 + \frac{1-k}{k}d^{1+k}\right)\left(\frac{r}{a}\right)^{1+k} + \right.\\
&\qquad \left. + \frac{k-1+2d^{1+k}}{1+k}\left(\frac{r}{a}\right)^3 - \frac{d^{1+k}}{k}\left(\frac{r}{a}\right)^{1-k}\right],\\
R_1^{(1)}(r) &= -\frac{Pb^3}{2\pi\beta_1^3 dD_r}\left[d^{\beta_1}\left(\frac{r}{a}\right)^{1-\beta_1} + (d^{\beta_1} - 2)\left(\frac{r}{a}\right)^{1+\beta_1} + \right.\\
&\qquad \left. + 2(1 - d^{\beta_1})\frac{r}{a} + 2\beta_1\frac{r}{a}\ln\frac{r}{a}\right],\\
R_n^{(1)}(r) &= \frac{Pb^3}{2\pi(\beta_n^2 - \gamma_n^2)\gamma_n dD_r}\left[\frac{2\gamma_n}{\beta_n - \gamma_n}\left(d^{\gamma_n} - \frac{\beta_n + \gamma_n}{2\beta_n}d^{\beta_n}\right)\left(\frac{r}{a}\right)^{1+\beta_n} + \right.\\
&\qquad + \left(\frac{2\gamma_n}{\beta_n - \gamma_n}d^{\beta_n} - \frac{\beta_n + \gamma_n}{\beta_n - \gamma_n}d^{\gamma_n}\right)\left(\frac{r}{a}\right)^{1+\gamma_n} + \\
&\qquad \left. + \frac{\gamma_n d^{\beta_n}}{\beta_n}\left(\frac{r}{a}\right)^{1-\beta_n} + d^{\gamma_n}\left(\frac{r}{a}\right)^{1-\gamma_n}\right];
\end{aligned}
\right\}
\tag{82.19}
$$

$$
\left.
\begin{aligned}
R_0^{(2)}(r) &= \frac{Pa^2}{4\pi (k^2-1) D_r}\left[\frac{d^{1+k}}{1+k}\left(2+\frac{1-k}{k}d^{1+k}\right)\left(\frac{r}{b}\right)^{1+k}+\right.\\
&\qquad\left.+\frac{k-1+2d^{1+k}}{1+k}-d^2-\frac{d^2}{k}\left(\frac{r}{b}\right)^{1+k}\right],\\
R_1^{(2)}(r) &= -\frac{Pb^2}{2\pi\beta_1^3 D_r}\left[(1-d^{\beta_1})^2\left(\frac{r}{b}\right)^{1+\beta_1}+2(1-d^{\beta_1})\frac{r}{b}+\right.\\
&\qquad\left.+2\beta_1 \ln d \cdot \frac{r}{b}\right],\\
R_n^{(2)}(r) &= \frac{Pb^2}{2\pi(\beta_n^2-\gamma_n^2)\gamma_n D_r}\left[\frac{2\gamma_n d^{\beta_n}}{\beta_n-\gamma_n}\left(d^{\gamma_n}-\frac{\beta_n+\gamma_n}{2\beta_n}d^{\beta_n}\right)\left(\frac{r}{b}\right)^{1+\beta_n}+\right.\\
&\qquad+\left(\frac{2\gamma_n}{\beta_n-\gamma_n}d^{\beta_n}-\frac{\beta_n+\gamma_n}{\beta_n-\gamma_n}d^{\gamma_n}\right)d^{\gamma_n}\left(\frac{r}{b}\right)^{1+\gamma_n}-\\
&\qquad\left.-\frac{\gamma_n}{\beta_n}\left(\frac{r}{b}\right)^{1+\beta_n}+\left(\frac{r}{b}\right)^{1+\gamma_n}\right].
\end{aligned}
\right\}
\tag{82.20}
$$

Here $d = b/a$.

The deflection at the point of attack of the force is equal to

$$
(w)_{\substack{r=b\\ \theta=0}} = R_0^{(1)}(b) + \sum_{n=1}^{\infty} R_n^{(1)}(b) + R_0^{(2)}(b) + \sum_{n=1}^{\infty} R_n^{(2)}(b). \tag{82.21}
$$

For a plate of orthotropic material

$$
k = 1, \qquad \beta_1 = 2, \qquad \beta_n = 1 + n, \qquad \gamma_n = 1 - n.
$$

The expressions for the functions $R_0^{(i)}$, $R_n^{(i)}$, which will not be given here are obtained from Eqs. (82.19) and (82.20) by means of a limiting transition.*

Example. An anisotropic plate is given for which

$$
\frac{E_r}{E_\theta} = \frac{1}{4}, \qquad \nu_r = \frac{1}{8}, \qquad \nu_\theta = \frac{1}{2}, \qquad G_{r\theta} = \frac{2}{15}.
$$

When the force is applied at a distance equal to half the radius ($b = 0.5a$) the deflection caused by the force is, according to Sen Gupta, determined by the formula

$$
(w)_{\substack{r=b\\ \theta=0}} = 0{,}2044\,\frac{Pa^2}{16\pi D_r}. \tag{82.22}
$$

§83. APPROXIMATION METHODS OF DETERMINING THE DEFLECTIONS OF AN ANISOTROPIC PLATE

In practice one often encounters such cases of bending for which we have no exact solution. An exact solution has not been obtained so far for a homogeneous, anisotropic circular plate with supported edge, or for a rectangular orthotropic plate fixed on its whole contour, not to mention plates of more complex outlines. For anisotropic plates approximation methods for the determination of deflections can be given, which are analogous to the methods applied successfully in order to solve problems on the bending of isotropic plates. We restrict ourselves to two methods which are based on the theorem of the minimum energy of an elas-

tic body formulated previously in §4.

Let us consider a given anisotropic plate bent by a normal load q. Denote by w the possible deflection of the plate, i.e., a deflection for which the plate remains unbroken and the conditions on the fixed parts of the edge are satisfied. In this case the theorem mentioned above, applied to an anisotropic plate, may be formulated as follows: the real deflection of a plate, which is in agreement with the given conditions of fixing of the edge and the load q, differs from all possible deflections by the fact that for it the following expression is minimum:

$$Э = \frac{1}{2}\int\int\left[D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{12}\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} + D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \right.$$
$$\left. + 4D_{66}\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 + 4\left(D_{16}\frac{\partial^2 w}{\partial x^2} + D_{26}\frac{\partial^2 w}{\partial y^2}\right)\frac{\partial^2 w}{\partial x\,\partial y} - 2qw\right]dx\,dy. \tag{83.1}$$

Integration extends over the area of the plate. When the plate is orthotropic and the directions of the axes x and y coincide with the principal directions of elasticity, Eq. (83.1) is simplified a little and takes the form

$$Э = \frac{1}{2}\int\int\left[D_1\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_1\nu_2\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} + D_2\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \right.$$
$$\left. + 4D_k\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - 2qw\right]dx\,dy. \tag{83.2}$$

When we use Ritz's method an approximate solution of the bending problem can be obtained in the following way. When we give the expression for the possible deflection in the form of a sum

$$w = \sum_m \sum_n A_{mn} w_{mn}(x, y), \tag{83.3}$$

where w_{mn} are functions, which satisfies the boundary conditions [at least the kinematic ones],* yielding a smooth surface; they depend in the two integral parameters m and n, while A_{mn} are indefinite coefficients. Substituting this expression in (83.1) or, correspondingly, in (83.2), we obtain after all integrations the quantity $Э$ in the form of a quadratic function (a polynomial of degree two) of the coefficients A_{mn}. We then minimize the function for which the equation

$$\frac{\partial Э}{\partial A_{mn}} = 0. \tag{83.4}$$

is established and solved.

Note that with a successful choice of the functions w_{mn} we can obtain the deflection sufficiently accurate for practice when we retain a few terms of sum (83.3): two, three and sometimes only the first single term (first approximation).**

The errors of the bending and torsional moments and, to a

yet higher degree of the crosscut forces obtained in a first approximation are usually much higher than the deflection error.

Another method of deriving approximate solutions is also based on the theorem of the minimum of energy of the elastic body; it consists of the following. The expression for the deflection is sought in the form of a sum of products of functions of a single variable:

$$w = \sum_{k=1}^{n} \varphi_k(x)\psi_k(y). \tag{83.5}$$

Here $\varphi_k(x)$ $(k = 1, 2, \ldots, n)$ is a system of linearly independent functions; they are chosen in such a way that the expression for w satisfies part of the boundary conditions while $\psi_k(y)$ are unknown functions which must be determined.

Substituting w in Eq. (83.1) and (83.2) and integrating with respect to x, we obtain an integral which contains a function of a single variable, $\psi_k(y)$. We then have to solve a variational problem: we have to find the function $\psi_k(y)$, which minimizes the integral $Э$. Solving this problem according to the rules of the variational calculus, after a series of transformations we obtain a system of ordinary differential equations for the determination of the functions $\psi_k(y)$; the number of these equations is equal to the number of unknown functions. In an abbreviated form this system may be written as follows:*

$$\int \Big[D_{11}\frac{\partial^4 w}{\partial x^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26}\frac{\partial^4 w}{\partial x \partial y^3} + D_{22}\frac{\partial^4 w}{\partial y^4} - q\Big]\varphi_k(x)\,dx = 0 \quad (k = 1, 2, \ldots, n). \tag{83.6}$$

w is here understood to be the sum (83.5).

If we restrict ourselves to the first approximation, i.e., with

$$w = \varphi(x)\psi(y), \tag{83.7}$$

where $\varphi(x)$ is a given function we obtain a single equation instead of the system (83.6). For an orthotropic rectangular plate with sides of the length a and b it reads

$$D_2\psi^{IV}(y)\int_0^a \varphi^2\,dx + 2D_3\psi''(y)\int_0^a \varphi\varphi''\,dx + D_1\psi(y)\int_0^a \varphi\varphi^{IV}\,dx = \int_0^a q\varphi\,dx \tag{83.8}$$

(the origin of coordinates lies at an arbitrary point of side b, the x-axis is parallel to the side whose length is given by a).

Other approximation methods will not be considered here.

§84. APPROXIMATE SOLUTIONS FOR RECTANGULAR PLATES

Let us give approximate solutions for two cases of rectangular orthotropic plates loaded by normal forces q which are distributed uniformly over the entire area.

1. Rectangular plate with supported sides. A rectangular plate whose principal directions of elasticity is parallel to the directions of the sides a and b is supported on all sides and bent uniformly by a distributed load.

With the directions of the axes of coordinates as shown in Fig. 140 (see §72) we may assume

$$w_{mn}=\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}; \tag{84.1}$$

$$w=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \tag{84.2}$$

These functions w_{mn} and w satisfy *all* conditions on the four sides (and not only the kinematic ones). A substitution of the function w in Eq. (83.2) and subsequent integration results in

$$Э=\frac{ab}{8}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{A_{mn}^2\pi^4\left[D_1\left(\frac{m}{a}\right)^4+2D_3\left(\frac{mn}{ab}\right)^2+D_2\left(\frac{n}{b}\right)^4\right]-2A_{mn}a_{mn}\right\}, \tag{84.3}$$

where

$$a_{mn}=\frac{4}{ab}\int_0^a\int_0^b q\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy. \tag{84.4}$$

When we set equal to zero the derivatives with respect to all A_{mn} of (84.3) we obtain

$$A_{mn}=\frac{b^4}{\pi^4}\cdot\frac{a_{mn}}{D_1\left(\frac{m}{c}\right)^4+2D_3n^2\left(\frac{m}{c}\right)^2+D_2n^4}\quad\left(c=\frac{a}{b}\right). \tag{84.5}$$

These coefficients agree with the coefficients of the double series (72.5) which represents an accurate solution for a given plate such that Eq. (84.2) with an infinite number of terms of the sum proves to be identical with the exact solution.

When we want to solve this problem by another method in a first approximation, we lay the x-y frame as shown in Fig. 163 and have

$$w=\frac{q}{24D_1}(x^4-2ax^3+a^3x)\varphi(y). \tag{84.6}$$

The first factor, which depends on x, represents the deflection of a beam of length a supported on its ends, which is under the action of a uniform load. This expression satisfies the necessary conditions on the supported sides of the plate, namely

with $x = 0$ and $x = a$ $w = M_x = 0$. (84.7)

For the unknown function $\psi(y)$ we obtain the differential equation

$$\frac{31a^4}{3024}\cdot\frac{D_3}{D_1}\psi^{IV} - \frac{17a^3}{84}\cdot\frac{D_3}{D_1}\psi'' + \psi = 1. \qquad (84.8)$$

Solving this equation and determining the arbitrary constants from the conditions on the sides $y = \pm b/2$ where $w = M_y = 0$, we obtain the following expression for the deflection:

$$w = \frac{q}{24D_1}(x^4 - 2ax^3 + a^3x)\left[1 + \frac{k_2^2 \operatorname{ch} k_1 y}{(k_1^2 - k_2^2)\operatorname{ch}\frac{k_1 b}{2}} - \frac{k_1^2 \operatorname{ch} k_2 y}{(k_1^2 - k_2^2)\operatorname{ch}\frac{k_2 b}{2}}\right]. \qquad (84.9)$$

Here we used the denotation

$$k_{1,2} = \frac{1}{a}\sqrt{9{,}871\frac{D_3}{D_2} \pm \sqrt{97{,}436\left(\frac{D_3}{D_2}\right)^2 - 97{,}548}}. \qquad (84.10)$$

The deflection in the center $(x = a/2,\ y = 0)$ is determined from the formula

$$w_{max} = \frac{5}{384}\cdot\frac{qa^4}{D_1}\left[1 + \frac{1}{k_1^2 - k_2^2}\left(\frac{k_2^2}{\operatorname{ch}\frac{k_1 b}{2}} - \frac{k_1^2}{\operatorname{ch}\frac{k_2 b}{2}}\right)\right]. \qquad (84.11)$$

A calculation for an isotropic plate with Poisson's coefficient $\nu = 0.3$ according to a formula obtained from (84.10) by means of a limiting transition shows that the error of the approximate solution is small. Thus, for example, the difference between the amount of deflection w_{max} obtained according to the approximate formula and the value obtained from a well-known exact formula amounts to: 2.1% for a square plate, 0.4% for a plate with a sides ratio of 1:1.5 and only 0.2% for a plate with a sides ratio of 1:2.*

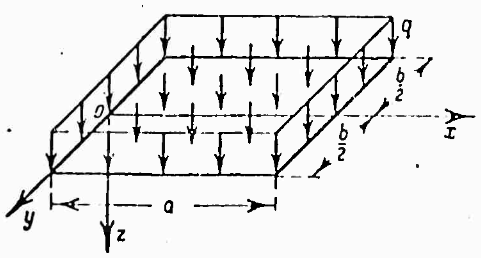

Fig. 163

2. Rectangular plate with four sides fixed. With an orientation of the x-y frame as shown in Fig. 164 we have the following boundary conditions:

$$\left.\begin{aligned} &\text{with } x = \pm\frac{a}{2} \quad w = \frac{\partial w}{\partial x} = 0; \\ &\text{with } y = \pm\frac{b}{2} \quad w = \frac{\partial w}{\partial y} = 0. \end{aligned}\right\} \tag{84.12}$$

We can give a whole series of expressions for w_{mn} which are steady functions and satisfy Conditions (84.12).

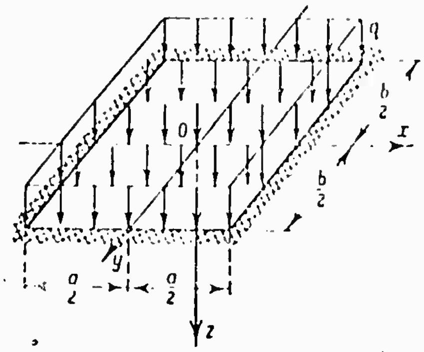

Fig. 164

For example, we can choose for w_{mn} an integral polynomial

$$w_{mn} = \left(x^2 - \frac{a^2}{4}\right)^2 \left(y^2 - \frac{b^2}{4}\right)^2 x^m y^n \tag{84.13}$$

$$(m = 0,\ 1,\ 2,\ \ldots;\ n = 0,\ 1,\ 2,\ \ldots).$$

We obtain in a first approximation

$$w = A\left(x^2 - \frac{a^2}{4}\right)^2 \left(y^2 - \frac{b^2}{4}\right)^2. \tag{84.14}$$

Substituting w in Eq. (83.2) we obtain

$$Э = \frac{a^5b^5}{225}\left[\frac{A^2}{49}\left(7D_1b^4 + 4D_3a^2b^2 + 7D_2a^4\right) - A\frac{q}{4}\right]. \tag{84.15}$$

The minimum value is obtained with

$$A = \frac{49}{8} \cdot \frac{q}{7D_1b^4 + 4D_3a^2b^2 + 7D_2a^4}. \tag{84.16}$$

From this we can derive an approximate expression for the deflection

$$w = \frac{49q}{8} \cdot \frac{\left(x^2 - \frac{a^2}{4}\right)^2\left(y^2 - \frac{b^2}{4}\right)^2}{7D_1b^4 + 4D_3a^2b^2 + 7D_2a^4}. \tag{84.17}$$

The deflection in the center is

$$w_{max} = 0{,}003118\,\frac{qa^4}{D_1 + 0{,}5714D_3c^2 + D_2c^4}. \tag{84.18}$$

where $c = a/b$.

In particular, for an isotropic plate with the rigidity D

$$w_{max} = 0{,}003418 \frac{qa^4}{D(1 + 0{,}5714c^2 + c^4)}. \tag{84.19}$$

We can also derive an approximate expression for w with the help of trigonometrical functions which satisfy all Conditions (84.12). Without giving the intermediate calculations we only give the expression for the deflection at an arbitrary point and in the center as obtained in a first approximation:

$$w_{max} = \frac{qa^4}{4\pi^4} \cdot \frac{\left(1 + \cos\frac{2\pi x}{a}\right)\left(1 + \cos\frac{2\pi y}{b}\right)}{3D_1 + 2D_3c^2 + 3D_2c^4}; \tag{84.20}$$

$$w_{max} = 0{,}003422 \frac{qa^4}{D_1 + 0{,}6667D_3c^2 + D_2c^4}. \tag{84.21}$$

For an isotropic plate

$$w_{max} = 0{,}003422 \frac{qa^4}{D(1 + 0{,}6667c^2 + c^4)}. \tag{84.22}$$

Without knowing the exact solution it is difficult to judge which of the two approximate solutions is more accurate, (84.18) or (84.22).

It is easy to show that, as to their values, the expressions obtained for w_{max} differ only slightly from one another. Consider, for example, a quadratic plate ($c = 1$) of isotropic material. From Eq. (84.19) we obtain for it

$$w_{max} = 0{,}00133 \frac{qa^4}{D}, \tag{84.23}$$

and from Eq. (84.22)

$$w_{max} = 0{,}00128 \frac{qa^4}{D}. \tag{84.24}$$

The results coincide precisely in the first two decimal places: when we retain in both cases two significant ciphers we obtain one and the same coefficient: 0.0013.

Treating the problem considered from the point of view of the second method, we can put

$$w = \frac{q}{24D_1}(x - a)^2 x^2 \psi(y). \tag{84.25}$$

This expression satisfies the conditions on the two sides $x = 0$ and $x = a$ (see Fig. 163). Determining the unknown function $\psi(y)$ analogously as in the case of four supported sides we arrive at the following final result:*

$$w = \frac{q}{24D_1}(x - a)^2 x^2 \left(1 + \frac{k_2 \operatorname{sh} \frac{k_2 b}{2} \operatorname{ch} k_1 y - k_1 \operatorname{sh} \frac{k_1 b}{2} \operatorname{ch} k_2 y}{k_1 \operatorname{sh} \frac{k_1 b}{2} \operatorname{ch} \frac{k_2 b}{2} - k_2 \operatorname{sh} \frac{k_2 b}{2} \operatorname{ch} \frac{k_1 b}{2}}\right). \qquad (84.26)$$

We used the denotation

$$k_{1,2} = \frac{1}{a}\sqrt{12\frac{D_3}{D_2} \pm \sqrt{144\left(\frac{D_3}{D_2}\right)^2 - 504}}. \qquad (84.27)$$

An approximate formula for the maximum deflection has the form

$$w_{max} = \frac{qa^4}{384D_1}\left(1 + \frac{k_2 \operatorname{sh} \frac{k_2 b}{2} - k_1 \operatorname{sh} \frac{k_1 b}{2}}{k_1 \operatorname{sh} \frac{k_1 b}{2} \operatorname{ch} \frac{k_2 b}{2} - k_2 \operatorname{sh} \frac{k_2 b}{2} \operatorname{ch} \frac{k_1 b}{2}}\right). \qquad (84.28)$$

§85. APPROXIMATE SOLUTIONS FOR TRIANGULAR PLATES

With the help of Ritz's method it is easy to derive an approximate solution for a triangular orthotropic plate with fixed or supported sides bent by a load q distributed uniformly. Let us give the approximate solutions for several cases of triangular plates; in all cases the expression for the deflection is given in the form of an integral polynomial with one or three indefinite coefficients which are obtained from the minimum condition of the integral Э [see (83.2)].

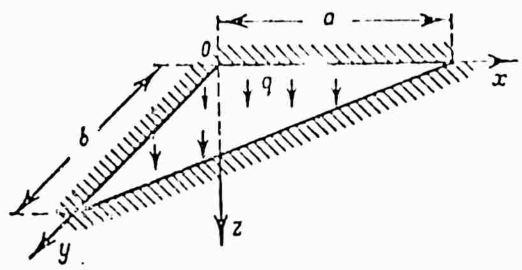

Fig. 165

1. Plate in the form of a rectangular triangle fixed on all sides. A plate is given which (in the ground plan) is of the form of a rectangular triangle cut out of an orthotropic sheet such that the principal directions of elasticity in it are parallel to the legs a and b. All three sides are fixed. The axes x and y are directed along the legs as shown in Fig. 165.

An expression for the deflection obtained in a first approximation has the form*

$$w = \frac{0.3125qc^2}{D_1 + D_3c^2 + D_2c^4} x^2y^2\left(1 - \frac{x}{a} - \frac{y}{b}\right)^2. \qquad (85.1)$$

Here and in the following $c = a/b$.

The maximum deflection is obtained for the center of gravity of the triangle, i.e., for the point $x = a/3$, $y = b/3$:

$$w_{max} = \frac{0{,}000129}{D_1 + D_3c^2 + D_2c^4} qa^4. \tag{85.2}$$

In the case of an isotropic material we obtain

$$w = \frac{0{,}3125qc^3}{D(1+c^2+c^4)} x^2y^2\left(1 - \frac{x}{a} - \frac{y}{b}\right)^2; \tag{85.3}$$

$$w_{max} = \frac{0{,}000129}{D(1+c^2+c^4)} qa^4. \tag{85.4}$$

2. Plate in the form of a right triangle with supported sides. Consider a plate as represented in Fig. 165 where all three sides are resting on a support. As a first approximation we can use an integral polynomial of degree three which vanishes on all three sides and contains an indefinite coefficient. Substituting the expression for w in Eq. (83.2) and requiring that Э must become minimum we determine the coefficient and at the same time the deflection in a first approximation:

$$w = \frac{qa^2c}{40(D_1 + 0{,}33D_3c^2 + D_2c^4)} xy\left(1 - \frac{x}{a} - \frac{y}{b}\right); \tag{85.5}$$

$$w_{max} = w\left(\frac{a}{3}, \frac{b}{3}\right) = \frac{0{,}000926}{D_1 + 0{,}33D_3c^2 + D_2c^4} qa^4. \tag{85.6}$$

To obtain the solution of the bending problem in a second approximation we suppose

$$w = (A_{00} + A_{10}x + A_{01}y) xy\left(1 - \frac{x}{a} - \frac{y}{b}\right). \tag{85.7}$$

The expression for Э is represented in the form of a square function of the three coefficients A_{00}, A_{10}, A_{01}. Minimizing this function we obtain the values of the coefficients of the polynomials (85.7). As a result in second approximation we obtain for the deflection of the plate considered the formula

$$w = \frac{qac}{21} xy\left(1 - \frac{x}{a} - \frac{y}{b}\right)\left(\frac{0{,}6a}{D_1 + 0{,}33D_3c^2 + D_2c^4} + \right.$$
$$\left. + \frac{x}{D_1 + 1{,}6D_3c^2 + 2D_2c^4} + \frac{y}{2D_1 + 1{,}6D_3c^2 + D_2c^4}\right). \tag{85.8}$$

The maximum deflection is equal to

$$w_{max} = qa^4\left(\frac{0{,}000926}{D_1 + 0{,}33D_3c^2 + D_2c^4} + \frac{0{,}000515}{D_1 + 1{,}6D_3c^2 + 2D_2c^4} + \right.$$
$$\left. + \frac{1}{c} \cdot \frac{0{,}000515}{2D_1 + 1{,}6D_3c^2 + D_2c^4}\right). \tag{85.9}$$

In the case of an isotropic plate

$$w_{max} = \frac{qa^4}{D}\left(\frac{0{,}000926}{1 + 0{,}33c^2 + c^4} + \frac{0{,}000515}{1 + 1{,}6c^2 + 2c^4} + \frac{1}{c} \cdot \frac{0{,}000515}{2 + 1{,}6c^2 + c^4}\right). \tag{85.10}$$

In particular, for an isotropic plate in the form of an isosceles right triangle (c = 1) Eq. (85.10) reads

$$w_{max} = 0{,}000619 \frac{qa^4}{D}. \tag{85.11}$$

The approximate solutions given for the triangular plate with supported sides were obtained by V.G. Rusin.*

3. Plate in the form of an isosceles or equilateral triangle with fixed sides. An orthotropic plate having the form of an isosceles triangle with the angles $\pi/2 - \alpha$, 2α, $\pi/2 - \alpha$, which is fixed on all sides and bent by a load distributed uniformly. The principal directions of elasticity are assumed parallel and perpendicular to the base of the triangle.

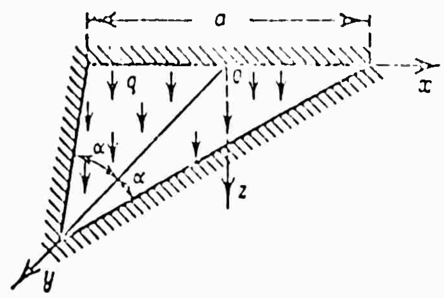

Fig. 166

We direct the x-y frame as shown in Fig. 166.

The final formula according to which we can determine (in a first approximation) the deflection of a plate with an arbitrary angle $\alpha < 90°$ can be written in the following form:

$$w = \frac{5}{16} q a^2 \frac{\cos^2\alpha}{D_1 \operatorname{ctg}^4\alpha + 2D_3 \operatorname{ctg}^2\alpha + 9D_2} y^2 \times \\ \times \left[\left(\frac{y}{a\cos\alpha} - 1\right)^2 - \frac{x^2}{a^2 \sin^2\alpha}\right]^2 \tag{85.12}$$

(a is the length of the lateral sides).

The maximum deflection at a point coinciding with the center of gravity of the triangle is given by

$$w_{max} = 0{,}00686 q a^4 \frac{\cos^4\alpha}{D_1 \operatorname{ctg}^4\alpha + 2D_3 \operatorname{ctg}^2\alpha + 9D_2}. \tag{85.13}$$

For an isotropic plate

$$w_{max} = 0{,}00686 \frac{q a^4}{D} \cdot \frac{\sin^4\alpha \cos^4\alpha}{1 + 8\sin^4\alpha}. \tag{85.14}$$

When the plate has the form of an equilateral triangle, for it $\alpha = 30°$ and on the basis of Eqs. (85.12) and (85.13) we obtain the following expressions for the deflections at an arbitrary point and in the center of gravity:

$$w = \frac{5}{192} \cdot \frac{q a^2}{D_1 + 0{,}667 D_3 + D_2} y^2 \left[\left(\frac{2y}{a\sqrt{3}} - 1\right)^2 - \frac{4x^2}{a^2}\right]^2; \tag{85.15}$$

$$w_{max} = \frac{0{,}000127}{D_1 + 0{,}667 D_3 + D_2} q a^4. \qquad (85.16)$$

In particular, the maximum deflection of an isotropic plate in the form of an equilateral triangle is in a first approximation determined according to the formula

$$w_{max} = 0{,}000160 \frac{q a^4}{D}. \qquad (85.17)$$

An approximate solution of the problem for a plate with arbitrary angle α was obtained by R.V. Feodos'yev,* and for a plate in the form of an equilateral triangle by Ye.F. Burmistrov.**

Manuscript Page No.	[Footnotes]
301	Timoshenko, S.P., Plastinki i obolochki [Plates and Shells] Gostekhizdat, Moscow, 1948, page 119.
302	As to the solution for an isotropic plate, see, e.g., the book by S.P. Timoshenko "Plastinki i obolochki" [Plates and Shells] Gostekhizdat, Moscow, 1948, pages 127-134.
309	See paper by M.T. Huber, Probleme der Statik technisch sichtiger orthotroper Platten [Problems of the Statics of Technically Important Orthotropic Plates] Warsaw, 1929, pages 74-75.
311	See pages 38-56 of M.T. Huber's paper mentioned in §74.
313	See pages 44 and 49 of Huber's paper mentioned.
314	See pages 42-45 of Huber's paper.
315	See paper by W. Nowacki: 1) Pasmo plytowe ortotropowe [A Strip of an Orthotropic Plates] Archiwum Mechaniki stosowanej [Archives of Applied Mechanics] Vol. III, No. 3-4, Gdansk 1951; 2) Beitrag zur Theorie der orthotropen Platten [Contribution to the Theory of Orthotropic Plates] Acta technika Academiae scientiarum hungaricae [Technical Acta of the Hungarian Academy of Sciences] Vol. VIII, fasc. 1-2, Budapest, 1954.
316*	See Huber's paper, page 55.
316**	See Huber's paper mentioned in the preceding sections, pages 56-58.
319*	See pages 59-60 of Huber's paper mentioned.
319**	See his paper mentioned in the preceding section.

319*** Cywinska, Z., Mossakowski, J., Powierzchnie wplywowe ortotropowego polpasma plytowego [Surfaces of Influence of an Orthotropic Semistrip of Plate] Archiwum mechaniki stosowanej, Vol. VI, No. 1, 1964.

320* Kaczkowski, Z., 1) Obliczanie plyt anizotropowich metoda nakladania ugiec faldowych [Investigation of Anisotropic Plates by the Method of Superposition of Crippled Buckling] Archiwum mechaniki stosowanej, Vol. V, No. 3, 1953; 2) raschet anizotropnykh plastinok metodom nalozheniya volnoobraznykh progibov [Calculation of Anisotropic Plates by the Method of Superposition of Crippled Buckling] Byull. Pol'skoy Akademii nauk [Bull. of the Polish Academy of Sciences] Part IV, Vol. II, No. 2, 1954.

320** Lokshin, A.S., K raschetu plastinok, podkreplennykh zhestkimi rebrami [To the Calculation of Plates Reinforced by Stiffening Ribs] Prikladnaya matematika i mekhanika, Vol. II, No. 2, 1935.

320*** Filippov, A.P., Pryamougol'nyye plastinki, podkreplennyye uprugimi rebrami i tochechnymi uprugimi oporami [Rectangular Plates Reinforced by Elastic Ribs and Elastic Point Supports] Prikladnaya matematika i mekhanika, New Series, Vol. I, No. 2, 1937.

320**** Nowacki, W., 1) Z zagadnien teorii rusztow plaskich [Problems of the Theory of Flat Grids] (I). Archiwum mechaniki stosowanej, Vol. VI, No. 1, 1954; 2) Zagadnienia statyki i dynamiki plyt, wzmocnionych zebrami [Problems of the Statics and Dynamics of Plates Reinforced by Ribs] Ibid., Vol. VI, No. 4, 1954.

320***** Lekhnitskiy, S.G., Izgib pryamougol'noy ortotropnoy plastinki s parallel'nymi rebrami zhestkosti [The Bending of an Orthotropic Plate with Parallel Stiffening Ribs] Prikladnaya matematika i mekhanika, Vol. XII, No. 3, 1948.

327 Lekhnitskiy, S.G., Ustoychivost' anisotropnykh plastinok [Stability of Anisotropic Plates] Gostekhizdat, Moscow-Leningrad, 1943, page 66.

337 See, for example, S.P. Timoshenko, Plastiki i obolochki [Plates and Shells] Gostekhizdat, 1948, page 279.

338 Okubo, H., Bending of a Thin Circular Plate of an Aeolotropic Material Under Uniform Lateral Load (Supported Edge), Journal Appl. Phys., Vol. 20, No. 12, 1949.

339* See S.P. Timoshenko, Plastinki i oboloch i [Plates and Shells] Gostekhizdat, Moscow, 1949, pages 63, 249. Particular cases of bending of a circular plate, cf. pages 64-82.

339** Carrier, G.F., The Bending of the Cylindrically Aeolo-

tropic Plate, Journ. of Appl. Mech., Vol. 11, 1944, No. 3.

343* See his paper referred to in §81.

343** Sen Gupta, A.M., Bending of a Cylindrically Aeolotropic Circular Plate with Eccentric Load, Journ. of Appl. Mech., Vol. 19, 1952, No. 1.

346 See S.P. Timoshenko, Plastinki i obolochki [Plates and Shells] Gostekhizdat, Moscow, 1948, pages 256-259.

347* These conditions read as follows: in the fixed part of the edge the deflection and its derivative with respect to the normal are vanishing; in the supported part of the edge the deflection is equal to zero.

347** Many approximate solutions for an isotropic plate can be found in the book by A. Föppl, Sila i deformatsiya [Force and Strain] Part 1, ONTI, 1933 (see §§19-21, pages 153-170). The detailed analysis of the solutions given in this book enables us to estimate the accuracy of the first approximations for an isotropic plate.

348 Uspenskiy, M.M., Ob izgibe pryamouglo'noy plastinki s ortogonal'noy anizotropiyey [On the Bending of a Rectangular Plate with Orthogonal Anisotropy] Trudy Voronezhskogo inzhenernostroit. in-ta [Transactions of the Voronezh Institute of Engineering and Construction] 1950, Coll. 2.

350 See pages 174-177 of the paper by M.M. Uspenskiy mentioned in §83.

352 See pages 178-179 of the paper by M.M. Uspenskiy mentioned.

353 Burmistrov, Ye.F., Nekotoryye sluchai izgiba treugol'noy ortotropnoy plastinki [Certain Cases of Bending of a Triangular Orthotropic Plate] Vestnik inzhenerov i tekhnikov, 1947, No. 2.

355 Rusin, V.G., Nekotoryye sluchai izgiba i svobodnykh poperechnykh kolebaniy ortotropnykh treugol'nykh plastinok [Certain Cases of Bending and Free Transverse Vibrations of Orthotropic Triangular Plates] Diploma Thesis, Saratov State University, Saratov 1950.

356* Feodos'yev, R.V., Izgib i kolebaniya anizotropnoy plastinki v forme ravnobedrennogo treugol'nika [Bending and Vibrations of an Anisotropic Plate in the Form of an Isosceles Triangle] Diploma Thesis, Saratov State University, Saratov, 1951.

356** See his paper mentioned.

Manuscript Page No.	Transliterated Symbols
314	пр = pr = privedennyy = reduced

Chapter 11

THE BENDING OF A PLATE BY A LOAD DISTRIBUTED ALONG THE EDGE

§86. THE LOCAL STRESSES AROUND A RECTILINEAR EDGE OF A PLATE

In the present chapter we consider some cases of bending of orthotropic anisotropic plates by a load in the form of bending moments and forces distributed along the edge.

Let us consider an orthotropic anisotropic plate whose contour has a rectilinear section. Let us assume the plate bent by a load distributed on a short part of the rectilinear section of the boundary. As regards the elastic properties, we shall suppose that the plate is orthotropic, with the principal directions parallel and perpendicular to the rectilinear section of the edge.

We can obtain approximate formulas for the determination of the moments and crosscut forces near the loaded part of the rectilinear boundary when the plate is considered to be infinitely large, as an "elastic semiplane." The method with which we solve this problem has much in common with the method used in order to investigate the plane state of stress of an "elastic semiplane" described in Chapter 4.*

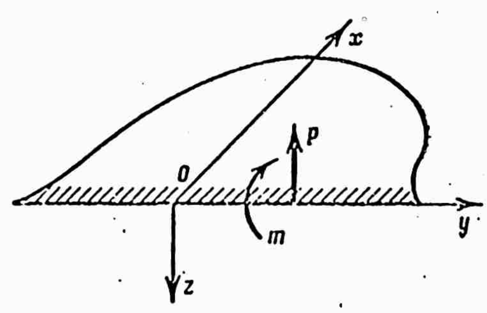

Fig. 167

Restricting ourselves to cases where a finite section of a rectilinear boundary loaded by bending moments and normal forces distributed symmetrically relative to the midpoint of this section, we assume the midpoint of the loaded section coincident with the origin of coordinates, the y-axis agrees with the direction of the boundary and the x-axis lies inside the semiplane (Fig. 167). The deflection equation will be homogeneous ($q = 0$):

$$D_1\frac{\partial^4 w}{\partial x^4}+2D_3\frac{\partial^4 w}{\partial x^2\partial y^2}+D_2\frac{\partial^4 w}{\partial y^4}=0. \qquad (86.1)$$

Let us denote by $m(y)$ the bending moments and by $p(y)$ the bending normal forces per unit length (given functions of y which are even by virtue of the symmetrical distribution). The boundary conditions read:

$$M_x=m(y), \quad N_x+\frac{\partial H_{xy}}{\partial y}=p(y). \qquad (86.2)$$

We represent the given moments and forces in the form of Fourier integrals:

$$m(y)=\frac{2}{\pi}\int_0^\infty \psi(\alpha)\cos\alpha y\,d\alpha, \quad p(y)=\frac{2}{\pi}\int_0^\infty \chi(\alpha)\cos\alpha y\,d\alpha, \qquad (86.3)$$

where

$$\psi(\alpha)=\int_0^\infty m(\eta)\cos\alpha\eta\,d\eta, \quad \chi(\alpha)=\int_0^\infty p(\eta)\cos\alpha\eta\,d\eta, \qquad (86.4)$$

and seek the solution of Eq. (86.1) in the form

$$w=\int_0^\infty \Phi(\alpha,x)\cos\alpha y\,d\alpha. \qquad (86.5)$$

The form of the function Φ will depend on the roots of the equation

$$D_1 s^4-2D_3 s^2+D_2=0, \qquad (86.6)$$

which was already considered in §73.

Taking into account that with increasing distance from the edge the corresponding stresses $\sigma_x, \sigma_y, \tau_{xy}$ must tend to zero, we obtain:

Case 1

$$\Phi=Ae^{-s_1\alpha x}+Be^{-s_2\alpha x}. \qquad (86.7)$$

Case 2

$$\Phi=(A+Bx)e^{-s\alpha x}. \qquad (86.8)$$

Case 3

$$\Phi=(A\cos t\alpha x+B\sin t\alpha x)e^{-s\alpha x}. \qquad (86.9)$$

Satisfying the boundary conditions we obtain the following expressions for the moments and crosscut forces in the case of unequal roots ($s_1 \neq s_2$):

$$M_x = D_1 \int_0^\infty [A(\nu_2 - s_1^2) e^{-s_1 \alpha x} + B(\nu_2 - s_2^2) e^{-s_2 \alpha x}] \cos \alpha y \, d\alpha,$$

$$M_y = D_2 \int_0^\infty [A(1 - \nu_1 s_1^2) e^{-s_1 \alpha x} + B(1 - \nu_1 s_2^2) e^{-s_2 \alpha x}] \cos \alpha y \, d\alpha, \qquad (86.10)$$

$$H_{xy} = -2D_k \int_0^\infty (A s_1 e^{-s_1 \alpha x} + B s_2 e^{-s_2 \alpha x}) \sin \alpha y \, d\alpha;$$

$$N_x = \int_0^\infty [A(D_1 s_1^2 - D_3) s_1 e^{-s_1 \alpha x} + B(D_1 s_2^2 - D_3) s_2 e^{-s_2 \alpha x}] \alpha \cos \alpha y \, d\alpha,$$

$$N_y = \int_0^\infty [A(D_3 s_1^2 - D_2) e^{-s_1 \alpha x} + B(D_3 s_2^2 - D_2) e^{-s_2 \alpha x}] \alpha \sin \alpha y \, d\alpha, \qquad (86.11)$$

where

$$A = \frac{24}{\pi D_1 h^3 (s_1 - s_2)\left(E_2 + 4G\sqrt{\frac{E_2}{E_1}}\right)} \times \left[\psi(\alpha) s_2 (D_1 s_2^2 - D_3 - 2D_k) + \frac{\chi(\alpha)}{\alpha} D_1 (s_2^2 - \nu_2)\right],$$

$$B = -\frac{24}{\pi D_1 h^3 (s_1 - s_2)\left(E_2 + 4G\sqrt{\frac{E_2}{E_1}}\right)} \times \left[\psi(\alpha) s_1 (D_1 s_1^2 - D_3 - 2D_k) + \frac{\chi(\alpha)}{\alpha} D_1 (s_1^2 - \nu_2)\right]. \qquad (86.12)$$

The determination of the moments and crosscut forces in each concrete case is reduced to a calculation of the integrals $\psi(\alpha)$ and $\chi(\alpha)$ and the integrals of Eqs. (86.10) and (86.11). With a simple law of load distribution all integrals can be calculated without much work and expressions for $M_x, M_y, \ldots, N_y$ are obtained in a finite form.

The solutions for Case 2 can be obtained by means of a limiting transition with $s_1 = s_2 = s$; a solution for the case of complex roots may be obtained from the above solution with $s_1 = s + ti$ and $s_2 = s - ti$ and when the real and the imaginary parts are separated.

A solution of this problem can also be obtained by another method, when we represent the deflections, moments and crosscut forces in terms of functions of the complex variables $w_1(z_1)$ and $w_2(z_2)$, using Cauchy integrals or Schwartz's formula. The course of the calculation is essentially the same as in the case of the plane problem (see §29) and we shall not consider it.

§87. THE ACTION OF A CONCENTRATED MOMENT

Let us suppose that at point O of the rectilinear edge of the plate (which is considered to be infinitely large) a concentrated moment is applied; we have to determine the local stresses caused by this moment (Fig. 168).

Replacing the concentrated moment M by moments which are

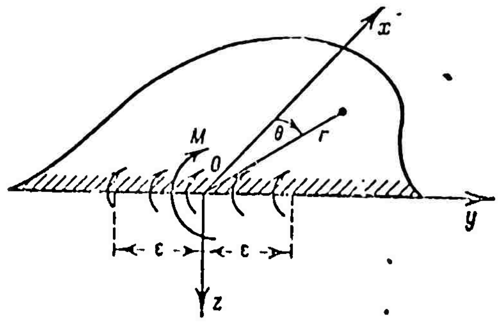

Fig. 168

statically equivalent to it and which are of the intensity $m = M/2\varepsilon$ and distributed uniformly along a small section of a length of 2ε, we obtain

$$\psi(\alpha) = \frac{M}{2} \cdot \frac{\sin 2\varepsilon}{2\varepsilon}, \quad \chi(\alpha) = 0. \tag{87.1}$$

Substituting these expressions in (86.12) and then in (86.10)-(86.11) and carrying out the limiting transition in which we let the length of the section 2ε tend to zero, we obtain integrals which are easy to calculate. As the result we obtain the following distributions of the moments M_x, M_y, H_{xy} caused by the concentrated moment.*

$$\left.\begin{aligned} M_x &= M\beta \frac{(E_2 x^2 + 4Gy^2)\, x}{D_2 x^4 + 2D_3 x^2 y^2 + D_1 y^4}, \\ M_y &= M\beta \frac{(4G\nu_2 - E_2)\, x y^2}{D_2 x^4 + 2D_3 x^2 y^2 + D_1 y^4}, \\ H_{xy} &= M\beta \frac{2G(\nu_2 x^2 + y^2)\, y}{D_2 x^4 + 2D_3 x^2 y^2 + D_1 y^4}. \end{aligned}\right\} \tag{87.2}$$

Here

$$\beta = \frac{\sqrt{D_1 D_2}}{\pi} \cdot \frac{s_1 + s_2}{E_2 + 4G\sqrt{\frac{E_2}{E_1}}}. \tag{87.3}$$

In order to determine the crosscut forces we can use the formulas

$$N_x = \frac{\partial M_x}{\partial x} + \frac{\partial H_{xy}}{\partial y}, \quad N_y = \frac{\partial M_y}{\partial y} + \frac{\partial H_{xy}}{\partial x}. \tag{87.4}$$

The moments M_r, M_θ, $H_{r\theta}$ referred to the polar coordinates (the point of application of the moment M is the pole, the x-axis coincides with the polar axis) are determined by the formulas

$$\left.\begin{aligned} M_r &= \frac{M\beta}{rD_\theta} \cos\theta \{2G(1+\nu_2) + [E_2 - 2G(1+\nu_2)]\cos 2\theta\}, \\ M_\theta &= 0, \\ H_{r\theta} &= -\frac{M\beta}{2rD_\theta} \sin\theta \{2G(1-\nu_2) + E_2 + [E_2 - 2G(1+\nu_2)]\cos 2\theta\}. \end{aligned}\right\} \tag{87.5}$$

Here

$$D_\theta = D_1 \sin^4 \theta + 2D_3 \sin^2 \theta \cos^2 \theta + D_2 \cos^4 \theta \qquad (87.6)$$

is the rigidity of bending about an axis in the radial direction r [cf. Eqs. (69.6)].

The expressions given show that the bending and torsional moments decrease with increasing distance from the point of application of the concentrated moment, its variation is inversely proportional to the distance r; the variations of the stresses $\sigma_x, \sigma_y, \tau_{xy}$ and $\sigma_r, \tau_{r\theta}$ are governed by the same law. The crosscut forces vary inversely proportional to the square of the distance r; the same way of variation is shown by the tangential stresses τ_{xz}, τ_{yz}. At point O itself we obtain (in the theory) infinitely high stresses. At a certain distance r from point O the moment M_r reaches its maximum value on the normal to the edge of the plate and the torsional moment $H_{r\theta}$ is there vanishing; the bending moment vanishes at the edge of the plate.

In an isotropic plate

$$\left.\begin{aligned} M_r &= \frac{2M}{\pi} \cdot \frac{1+\nu}{3+\nu} \cdot \frac{\cos\theta}{r}, \\ M_\theta &= 0, \\ H_{r\theta} &= -\frac{2M}{\pi} \cdot \frac{1}{3+\nu} \cdot \frac{\sin\theta}{r}. \end{aligned}\right\} \qquad (87.7)$$

On the basis of these formulas it is easy to see that the points at which the radial moment M_r has one and the same value lie on a circle with centers on the normal to the boundary (i.e., on the x-axis) passing through 0, the point of application of the moment (we met these circles already in the theory of the plane problem, see §30, Fig. 54). The lines of equal torsional moments $H_{r\theta}$ are circles orthogonal to the first; they pass through the point of application of the concentrated moment and their centers lie on the edge of the plate. In an anisotropic plate the lines of equal moments will be more complex curves of fourth order.

§88. THE BENDING OF A PLATE WITH ELLIPTIC APERTURE WHOSE EDGE IS FIXED ARBITRARILY

Let us consider an anisotropic, homogeneous but generally not orthotropic plate of arbitrary form, with an elliptic aperture and bending strengths and moments which are distributed on the outer edge and along the edges of the aperture.

The question of the influence of an elliptic or circular aperture on the stress distribution is well investigated for the case of a generalized plane state of stress (at least for a small aperture distant from the outer edge, see §§37-40). A complex representation of the deflection, the moments and the crosscut forces with the help of the two functions $w_1(z_1)$ and $w_2(z_2)$ enables us also to study this problem in the case of bending.*

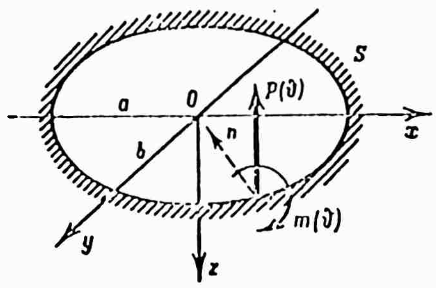

Fig. 169

Assuming the dimensions of the aperture to be small compared with the dimensions of the plate and the aperture itself being remote from the edge, we shall consider the plate to be infinitely large, i.e., to represent an "elastic plane with elliptic cutout." Let us consider the case where the load in the form of bending moments m and the bending normal forces p are only distributed along the edge of the aperture.

We assume the median surface in the xy-plane and the origin of coordinates in the center of the aperture, the x-y frame being coincident with the principal axes of the ellipse (Fig. 169) and all rigidities D_{ij} for the x- and y-axes are assumed to be known. The contour equation of the ellipse, in a parameterical form, can then be written down:

$$x = a\cos\vartheta, \quad y = b\sin\vartheta; \tag{88.1}$$

m and p will be functions of ϑ. In the general case we shall suppose that the load distributed on the edge of the aperture is reduced to the force P_z and the moments m_x and m_y (relative to the axes x and y).

The moments and crosscut forces can be given in terms of the two functions $w_1(z_1)$ and $w_2(z_2)$ of the complex variables $z_1 = x + \mu_1 y$ and $z_2 = x + \mu_2 y$ [see Eqs. (63.4) and (63.5)]; the derivatives w_1' and w_2' satisfy Conditions (63.7) on the edge of the aperture. Having obtained the expressions entering the right-hand sides of the boundary conditions (63.7), in the general case of load distribution we have

$$
\left.
\begin{aligned}
-\int_0^s (m\,dy + f\,dx) &= \frac{P_x a}{2\pi}\,\theta\cos\theta - \frac{m_y}{2\pi}\,\theta + \alpha_0 + \\
&\quad + \sum_{m=1}^{\infty} (\alpha_m \sigma^m + \bar{\alpha}_m \sigma^{-m}), \\
\int_0^s (-m\,dx + f\,dy) &= -\frac{P_x b}{2\pi}\,\theta\sin\theta - \frac{m_x}{2\pi}\,\theta + \beta_0 + \\
&\quad + \sum_{m=1}^{\infty} (\beta_m \sigma^m + \bar{\beta}_m \sigma^{-m}).
\end{aligned}
\right\} \qquad (88.2)
$$

Here

$$
f = \int_0^s p\,ds, \quad \sigma = e^{i\theta};
$$

α_m, β_m are coefficients which depend on the law of load distribution [we obtain them when we expand m and p in Fourier series with respect to the variable   and integrate as provided for in Eqs. (88.2)]; $\bar{\alpha}_m$, $\bar{\beta}_m$ are coefficients conjugated to the former.

The solution is obtained with the help of the functions w_1' and w_2', in the form

$$
\left.
\begin{aligned}
w_1'(z_1) &= (A' z_1 + A)\ln\zeta_1 + A_0 + \\
&\quad + \frac{\mu_1}{d}\left[\mu_2 q_2 \bar{\alpha}_1 - p_2\bar{\beta}_1 - \frac{C}{2}(\mu_2 q_2 a + p_2 b i)\right]\frac{1}{\zeta_1} + \\
&\quad + \frac{\mu_1}{d}\sum_{m=2}^{\infty}(\mu_2 q_2 \bar{\alpha}_m - p_2\bar{\beta}_m)\zeta_1^{-m}, \\
w_2'(z_2) &= (B' z_2 + B)\ln\zeta_2 + B_0 - \\
&\quad - \frac{\mu_2}{d}\left[\mu_1 q_1 \bar{\alpha}_1 - p_1\bar{\beta}_1 - \frac{C}{2}(\mu_1 q_1 a + p_1 b i)\right]\frac{1}{\zeta_2} - \\
&\quad - \frac{\mu_2}{d}\sum_{m=2}^{\infty}(\mu_1 q_1 \bar{\alpha}_m - p_1\bar{\beta}_m)\zeta_2^{-m}.
\end{aligned}
\right\} \qquad (88.3)
$$

ζ_1, ζ_2 are here functions of z_1 and z_2:

$$
\left.
\begin{aligned}
\zeta_1 &= \frac{z_1 + \sqrt{z_1^2 - a^2 - \mu_1^2 b^2}}{a - i\mu_1 b}; \\
\zeta_2 &= \frac{z_2 + \sqrt{z_2^2 - a^2 - \mu_2^2 b^2}}{a - i\mu_2 b};
\end{aligned}
\right\} \qquad (88.4)
$$

$$
d = p_1 q_2 \mu_2 - p_2 q_1 \mu_1; \qquad (88.5)
$$

A, B, A', B' are constants determined from the following system of equations [see §63, Eqs. (63.9)-(63.10)]:

$$\left.\begin{aligned} &A' + B' - \bar{A}' - \bar{B}' = 0, \\ &\mu_1 A' + \mu_2 B' - \bar{\mu}_1 \bar{A}' - \bar{\mu}_2 \bar{B}' = 0, \\ &\mu_1^2 A' + \mu_2^2 B' - \bar{\mu}_1^2 \bar{A}' - \bar{\mu}_2^2 \bar{B}' = 0, \\ &\frac{1}{\mu_1} A' + \frac{1}{\mu_2} B' - \frac{1}{\bar{\mu}_1} \bar{A}' - \frac{1}{\bar{\mu}_2} \bar{B}' = \frac{P_z}{2\pi i D_{11}}; \end{aligned}\right\} \tag{88.6}$$

$$\left.\begin{aligned} &A + B - \bar{A} - \bar{B} = 0, \\ &\mu_1 A + \mu_2 B - \bar{\mu}_1 \bar{A} - \bar{\mu}_2 \bar{B} = 0, \\ &\mu_1^2 A + \mu_2^2 B - \bar{\mu}_1^2 \bar{A} - \bar{\mu}_2^2 \bar{B} = -\frac{m_x}{2\pi i D_{22}}, \\ &\frac{1}{\mu_1} A + \frac{1}{\mu_2} B - \frac{1}{\bar{\mu}_1} \bar{A} - \frac{1}{\bar{\mu}_2} \bar{B} = -\frac{m_y}{2\pi i D_{11}}. \end{aligned}\right\} \tag{88.7}$$

The constant C is determined on the basis of the requirement that the deflection w is an unambiguous function.

When the contour of the hole is subject to the action of a load in equilibrium, i.e., if $P_z = m_x = m_y = 0$, we have $A' = B' = A = B = 0$ and the solution becomes simpler.

Let us also give the solution of the second fundamental problem for an infinite plate with an elliptic aperture.

The deflection and the angle of slope of the curved surface with respect to the initial xy-plane of the plate are assumed given on the contour of the aperture. Let us also know that with increasing distance from the aperture the deflection and the slope decrease and can be considered to be vanishing at a sufficiently large distance from the edge (theoretically: at infinity). In this case the boundary conditions read:

$$w = F(\vartheta), \quad \frac{dw}{dn} = \alpha(\vartheta), \tag{88.8}$$

where $F(\vartheta)$, $\alpha(\vartheta)$ are the deflection and angle of slope, both given as functions of the parameter ϑ. These conditions can be written in the form

$$\left.\begin{aligned} &\frac{\partial w}{\partial x} = \frac{dF}{ds}\frac{dx}{ds} - \alpha\frac{dy}{ds} = \alpha_0 + \sum_{m=1}^{\infty}(\alpha_m \sigma^m + \bar{\alpha}_m \sigma^{-m}), \\ &\frac{\partial w}{\partial y} = \frac{dF}{ds}\frac{dy}{ds} + \alpha\frac{dx}{ds} = \beta_0 + \sum_{m=1}^{\infty}(\beta_m \sigma^m + \bar{\beta}_m \sigma^{-m}). \end{aligned}\right\} \tag{88.9}$$

Here ds is the differential arc of the ellipse, a function of the parameter ϑ, α_m, β_m are given coefficients whose conjugated quantities are denoted by $\bar{\alpha}_m$, $\bar{\beta}_m$

The deflection and slope of the curved surface given for the contour must be periodic functions of ϑ. Hence it follows for the relation between the coefficients of the first terms of the series on the right-hand sides of Conditions (88.9)

$$(\alpha_1 - \bar{\alpha}_1)a + (\beta_1 + \bar{\beta}_1)bi = 0. \tag{88.10}$$

For definiteness of the problems it is necessary to know the vector sum and the resultant moment of the forces acting on the edge of the aperture, P_z, m_x, m_y.

The functions of the complex variables determining the solution of the problem have the form

$$\left.\begin{aligned} w_1'(z_1) &= (A'z_1 + A)\ln\zeta_1 + A_0 + \sum_{m=1}^{\infty} \frac{\bar{\beta}_m - \mu_2\bar{\alpha}_m}{\mu_1 - \mu_2}\zeta_1^{-m}, \\ w_2'(z_2) &= (B'z_2 + B)\ln\zeta_2 + B_0 - \sum_{m=1}^{\infty} \frac{\bar{\beta}_m - \mu_1\bar{\alpha}_m}{\mu_1 - \mu_2}\zeta_2^{-m}. \end{aligned}\right\} \quad (88.11)$$

The coefficients A', B' and A, B are obtained from Eqs. (88.6 and (88.7) and A_0 and B_0 and the conjugate quantities satisfy the two equations

$$\left.\begin{aligned} A_0 + B_0 + \bar{A}_0 + \bar{B}_0 &= \alpha_0, \\ \mu_1 A_0 + \mu_2 B_0 + \bar{\mu}_1\bar{A}_0 + \bar{\mu}_2\bar{B}_0 &= \beta_0. \end{aligned}\right\} \quad (88.12)$$

Some simplifications are obtained for the case where the plate is orthotropic.

When a load is also applied to the outer edge of the plate, the stress distribution is obtained when we add the stresses in a massive plate and the stresses in a plate with aperture loaded along its contour. An approximate solution for some of these cases are given in §§89-90.

§89. THE PURE BENDING OF A PLATE WEAKENED BY A CIRCULAR APERTURE

A rectangular orthotropic plate weakened by a round hole in its center is bent by the moments M distributed uniformly on two sides; the edge of the hole is assumed free from loads. The diameter of the aperture is considered to be small with respect to the dimensions of the sides.

To begin with we consider the general case where one of the principal directions of elasticity makes an arbitrary angle φ with the direction of the axis of symmetry of the plate denoted by x' (Fig. 170).

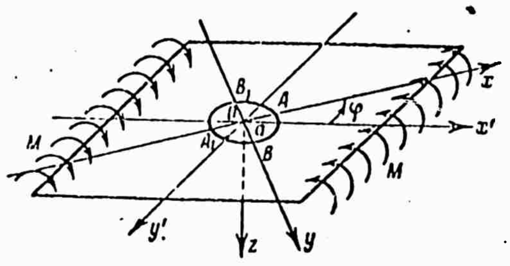

Fig. 170

The directions of the coordinates axes x and y coincide with the principal directions of elasticity. An approximate solution to the problem is obtained when we add the moments and crosscut forces in the massive plate

$$\left.\begin{gathered} M_x = M\cos^2\varphi, \quad M_y = M\sin^2\varphi, \quad H_{xy} = M\sin\varphi\cos\varphi, \\ N_x = N_y = 0 \end{gathered}\right\} \tag{89.1}$$

to the moments and crosscut forces in the infinite plate with an aperture whose edge is loaded by forces and moments. The latter must be chosen in such a way that on the edge of the aperture to following conditions are satisfied:

$$M_r = 0, \quad N_r + \frac{1}{r}\frac{\partial H_{r\theta}}{\partial\theta} = 0. \tag{89.2}$$

In the case given the additional moments and crosscut forces are determined by means of functions of the form

$$\left.\begin{aligned} w_1'(z_1) &= \left\{\frac{\mu_1(i\mu_2 q_2\cos^2\varphi - p_2\sin^2\varphi)}{2d} + \right. \\ &\left. + \frac{(\mu_2 i - 1)\sin\varphi\cos\varphi}{D_2(\mu_1 - \mu_2)[n(1 - \mu_1\mu_2) + 2(\nu_1 - \mu_1\mu_2)]}\right\}\frac{Ma}{\zeta_1}, \\ w_2'(z_2) &= -\left\{\frac{\mu_2(i\mu_1 q_1\cos^2\varphi - p_1\sin^2\varphi)}{2d} + \right. \\ &\left. + \frac{(\mu_1 i - 1)\sin\varphi\cos\varphi}{D_2(\mu_1 - \mu_2)[n(1 - \mu_1\mu_2) + 2(\nu_1 - \mu_1\mu_2)]}\right\}\frac{Ma}{\zeta_2} \\ n &= -i(\mu_1 + \mu_2) \end{aligned}\right\} \tag{89.3}$$

(for the denotations see §§63 and 88).

After a series of rather cumbersome transformations we obtain the following expressions for the moments M_θ and $H_{r\theta}$, referred to polar coordinates, on the edge of the aperture:

$$\left.\begin{aligned} M_\theta &= M + \\ &+ M\frac{\sqrt{D_1D_2}}{D_r}\left\{\frac{1}{k + ig}(a_0\sin^4\theta + a_2\sin^2\theta\cos^2\theta + a_4\cos^4\theta) - \right. \\ &- \frac{2n(k + n + 1)}{n(k + 1) + 2(\nu_1 + k)}\left[\frac{1 + \nu_1}{k}\sin^2\theta + (1 + \nu_2)k\cos^2\theta\right]\times \\ &\left.\times\sin\varphi\cos\varphi\sin\theta\cos\theta\right\}, \\ H_{r\theta} &= M\frac{\sqrt{D_1D_2}}{D_r}\left\{\frac{1}{k + ig}(a_1\sin^2\theta + a_3\cos^2\theta)\sin\theta\cos\theta + \right. \\ &\left. + \frac{n(k + n + 1)}{n(k + 1) + 2(\nu_1 + k)}\left(k\cos^4\theta - \frac{1}{k}\sin^4\theta\right)\sin\varphi\cos\varphi\right\}. \end{aligned}\right\} \tag{89.4}$$

Here we used the new denotations

$$\left.\begin{gathered} k = -\mu_1\mu_2 = \sqrt{\frac{D_1}{D_2}}, \quad g = \frac{G}{E_2}, \quad n = -i(\mu_1 + \mu_2); \\ D_r = D_1\cos^4\theta + 2D_3\sin^2\theta\cos^2\theta + D_2\sin^4\theta \end{gathered}\right\} \tag{89.5}$$

(the rigidity of bending about an axis tangent to the contour of the hole):

$$
\left.
\begin{aligned}
a_0 &= n\cos^2\varphi + \left(1 - k - 4g\frac{1+\nu_1}{k}\right)\sin^2\varphi,\\
a_2 &= [k - n - n^2 - 1 + 4g(1+\nu_2)(1+n)]\cos^2\varphi +\\
&\quad + \left[k(1-k-n) - n^2 + 4g(1+\nu_1)\left(\frac{n}{k}+1\right)\right]\sin^2\varphi,\\
a_4 &= k[1 - k - 4g(1+\nu_2)]\cos^2\varphi + kn\sin^2\varphi,\\
a_1 &= -\left[n + 1 + 2g\left(\frac{1}{k} - \nu_2 - \nu_2 n\right)\right]\cos^2\varphi -\\
&\quad - \left[k - 2g\left(\frac{n+n^2}{k} + \nu_2 k - 1\right)\right]\sin^2\varphi,\\
a_3 &= \left[k + 2g\left(1 - n - \frac{\nu_1 + n^3}{k}\right)\right]\cos^2\varphi +\\
&\quad + \left[k(k+n) + 2g\left(k - \nu_1 - \frac{\nu_1 n}{k}\right)\right]\sin^2\varphi.
\end{aligned}
\right\}
\qquad (89.6)
$$

For an isotropic plate we obtain:

$$
\left.
\begin{aligned}
M_\theta &= M\left[1 - \frac{2(1+\nu)}{3+\nu}\cos 2(\theta - \varphi)\right],\\
H_{r\theta} &= -M\frac{2}{3+\nu}\sin 2(\theta - \varphi).
\end{aligned}
\right\}
\qquad (89.7)
$$

Figure 171 shows the distributions of the moments M_θ and $H_{r\theta}$ on the edge of a hole in a veneer plate cut out such that the principal directions (the direction of the sheet fibers and the directions perpendicular to it) make an angle of intersection with the sides of 45°. The numerical values of the complex parameters are taken equal to $1.04 \pm 1.55i$ (the direction of the x-axis agrees with the direction of maximum rigidity, see §67).

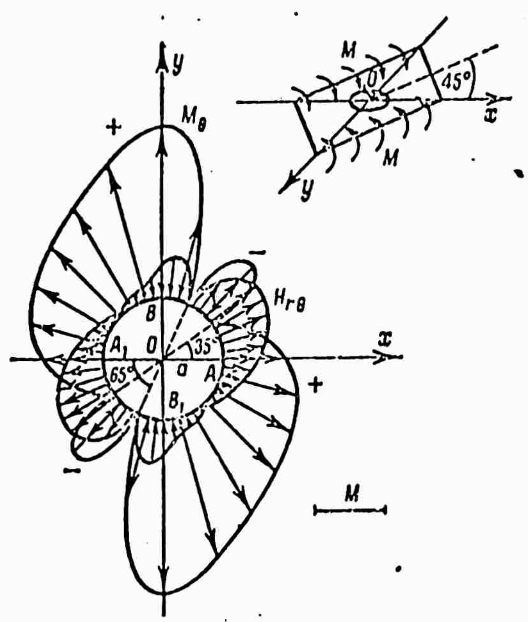

Fig. 171

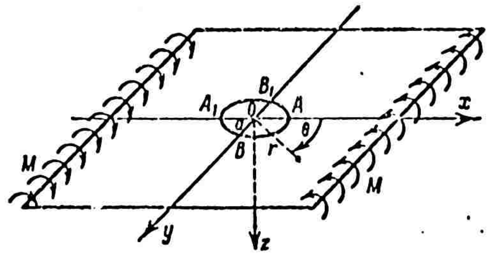

Fig. 172

The maximum values of the bending moments are obtained near the point $\theta = 90°$ and the point symmetrical with respect to the center; they are equal to

$$(M_\theta)_{\max} = 2{,}55M. \tag{89.8}$$

The torsional moment reaches its highest value with

$$(H_{r\theta})_{\max} = 0{,}82M. \tag{89.9}$$

When the plate is cut in such a way that its principal directions are parallel to the sides ($\varphi = 0$, Fig. 172) all formulas become simpler. Instead of (89.4) we obtain:*

$$\left.\begin{aligned} M_\theta &= M\left[1 + \frac{\sqrt{D_1 D_2}}{D_r} \times \right. \\ &\quad \left. \times \frac{1}{k+4g}\left(a_0 \sin^4\theta + a_2 \sin^2\theta \cos^2\theta + a_4 \cos^4\theta\right)\right], \\ H_{r\theta} &= M \frac{\sqrt{D_1 D_2}}{D_r} \cdot \frac{1}{k+4g}\left(a_1 \sin^2\theta + a_3 \cos^2\theta\right) \sin\theta \cos\theta, \end{aligned}\right\} \tag{89.10}$$

where

$$\left.\begin{aligned} a_0 &= n, \\ a_2 &= k - n - n^2 - 1 + 4g(1+\nu_2)(1+n), \\ a_4 &= k\left[1 - k - 4g(1+\nu_2)\right], \\ a_1 &= -n - 1 + 2g\left(\nu_2 + \nu_2 n - \frac{1}{k}\right), \\ a_3 &= k + 2g\left(1 - n - \frac{\nu_1 + n^2}{k}\right). \end{aligned}\right\} \tag{89.11}$$

The bending moment M_θ reaches its maximum value either at points A, A_1 or at points B, B_1 (Fig. 172) where the principal directions of elasticity intersect with the contour of the hole.

At the points A, A_1 (Fig. 172)

$$M_\theta = M\frac{1 - 4g\nu_2}{k+4g}, \quad H_{r\theta} = 0, \tag{89.12}$$

at the points B and B_1

$$M_\theta = M\left(1 + \frac{kn}{k+4g}\right), \quad H_{r\theta} = 0. \tag{89.13}$$

For a plate of an isotropic material at the points A and A_1

$$M_\theta = M\frac{1-\nu}{3+\nu}; \quad (89.14)$$

and at the points B and B_1

$$M_\theta = M\frac{5+3\nu}{3+\nu}. \quad (89.15)$$

Figure 173 shows the moment distribution on the edge of a hole in a veneer plate. The direction of the fibers of the sheet agree with the direction of the x-axis; the outer sides are perpendicular to the fibers.

At the points A and A_1

$$M_\theta = 0{,}17M; \quad (89.16)$$

at the points B and B_1

$$M_\theta = 2{,}97M. \quad (89.17)$$

The maximum torsional moment is

$$(H_{r\theta})_{max} = 0{,}6M. \quad (89.18)$$

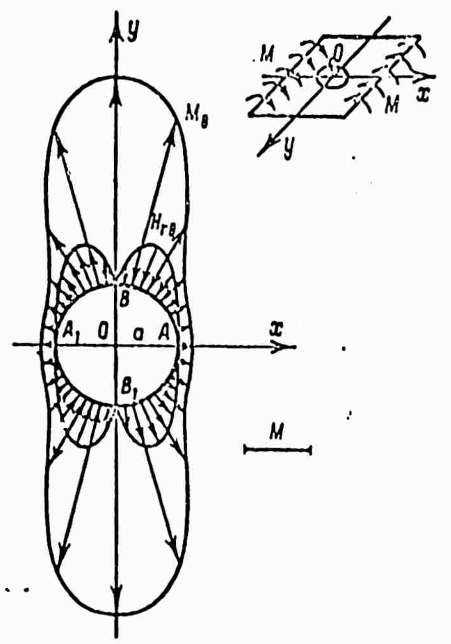

Fig. 173

Figure 174 gives the moment distribution on the edge of a hole in a veneer plate where the sides parallel to the fibers of the sheet are loaded.

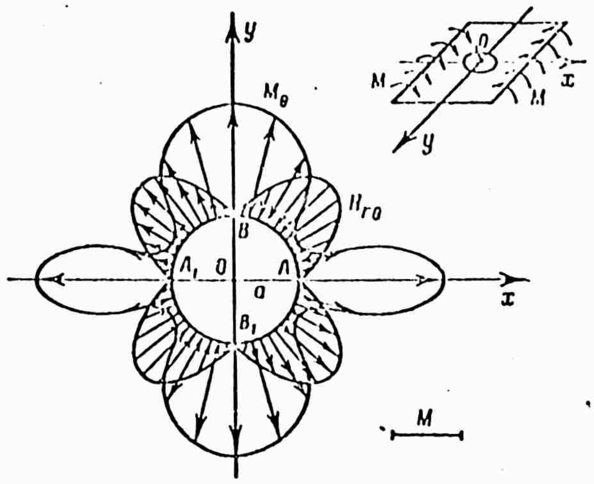

Fig. 174

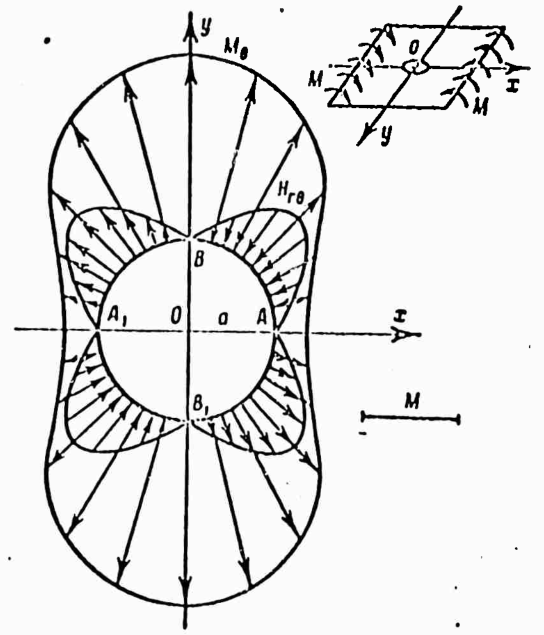

Fig. 175

At the points A and A_1

$$M_\theta = 2{,}09M; \tag{89.19}$$

at the points B and B_1

$$M_\theta = 1{,}56M. \tag{89.20}$$

The maximum torsional moment is given by

$$(H_{r\theta})_{\max} = M. \tag{89.21}$$

Comparing the graphs for the three cases of orientation of the principal axes relative to the loaded sides of the veneer plate which we considered, we see that the highest concentration of the moments M_θ and the directions σ_θ is obtained for the case where the loaded sides are perpendicular to the fibers of the sheet. Inversely, the minimum concentration occurs in the case where the sides parallel to the fibers of the sheet are loaded. The case where the fibers of the sheet make an angle of 45° with respect to the sides of the plate takes an intermediate position.

For comparison we show in Fig. 175 the moment distribution on the edge of an aperture in an isotropic plate with a Poisson coefficient equal to 0.3. Here, at the points A and A_1 (Fig. 172)

$$M_\theta = 0{,}21M; \tag{89.22}$$

at the points B and B_1

$$M_\theta = 1{,}79M. \tag{89.23}$$

The maximum torsional moment is

$$(H_{r\theta})_{max} = 0{,}61M. \qquad (89.24)$$

A comparison of all graphs shown gives us a clear idea on the influence of the anisotropy of the material on the stress distribution in pure bending in a plate weakened by a round hole.

§90. OTHER CASES OF DEFORMATION OF AN ORTHOTROPIC PLATE WITH A CIRCULAR APERTURE

Let us still consider three cases of bending of an orthotropic plate with round hole and give the formulas and graphs of moment distribution for them.

1. Bending on all sides. A rectangular orthotropic plate with a circular central hole is bent by the moments M distributed uniformly on all four sides (Fig. 176). It is supposed that the principal directions of elasticity are parallel to the sides.

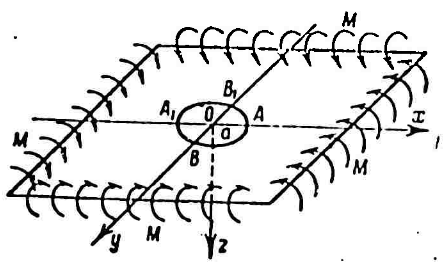

Fig. 176

Knowing the solution for the case of unilateral pure bending considered in the last sections, it is easy to obtain a solution also for the case of omnilateral bending by means of a superposition. The results are the following.

The formulas for the bending and torsional moments M_θ and $H_{r\theta}$ on the contour of the aperture have the form

$$\left.\begin{aligned} M_\theta &= 2M + \\ &+ M\frac{\sqrt{D_1D_2}}{D_r}\cdot\frac{1}{k+4g}\left(b_0\sin^4\theta + b_2\sin^2\theta\cos^2\theta + b_4\cos^4\theta\right), \\ H_{r\theta} &= M\frac{\sqrt{D_1D_2}}{D_r}\cdot\frac{1}{k+4g}\left(b_1\sin^2\theta + b_3\cos^2\theta\right)\sin\theta\cos\theta. \end{aligned}\right\} \qquad (90.1)$$

Here

$$
\left.
\begin{aligned}
b_0 &= k + n - 1 - 4g\frac{1+\nu_1}{k},\\
b_2 &= -(1-k)^2 - n(1+k) - 2n^2 + 4g(1+\nu_2)(1+n) +\\
&\quad + 4g(1+\nu_1)\frac{k+n}{k},\\
b_4 &= k[n - k + 1 - 4g(1+\nu_2)],\\
b_1 &= (k+n+1)(2g\nu_2 - 1) + 2g\left(\frac{n^2+n-1}{k} - 1\right),\\
b_3 &= (k+n+1)\left(k - \frac{2g\nu_1}{k}\right) + 2g\left(k - n + 1 - \frac{n^3}{k}\right).
\end{aligned}
\right\} \qquad (90.2)
$$

The other denotations are the same as in the previous section. In an isotropic plate

$$M_\theta = 2M, \quad H_{r\theta} = 0. \qquad (90.3)$$

At the points A and A_1 of an orthotropic plate (Fig. 176)

$$M_\theta = M\left(1 + \frac{1+n-4g\nu_2}{k+4g}\right), \quad H_{r\theta} = 0; \qquad (90.4)$$

at the points B and B_1

$$M_\theta = M\left(1 + \frac{k^2+kn-4g\nu_1}{k+4g}\right), \quad H_{r\theta} = 0. \qquad (90.5)$$

In Fig. 177 we show the distribution of the moments M_θ and $H_{r\theta}$ along the edge of the hole in the veneer plate; the direction of the x-axis agrees with the direction of the fibers of the sheet; the dashed circle shows the moment distribution in an isotropic plate.

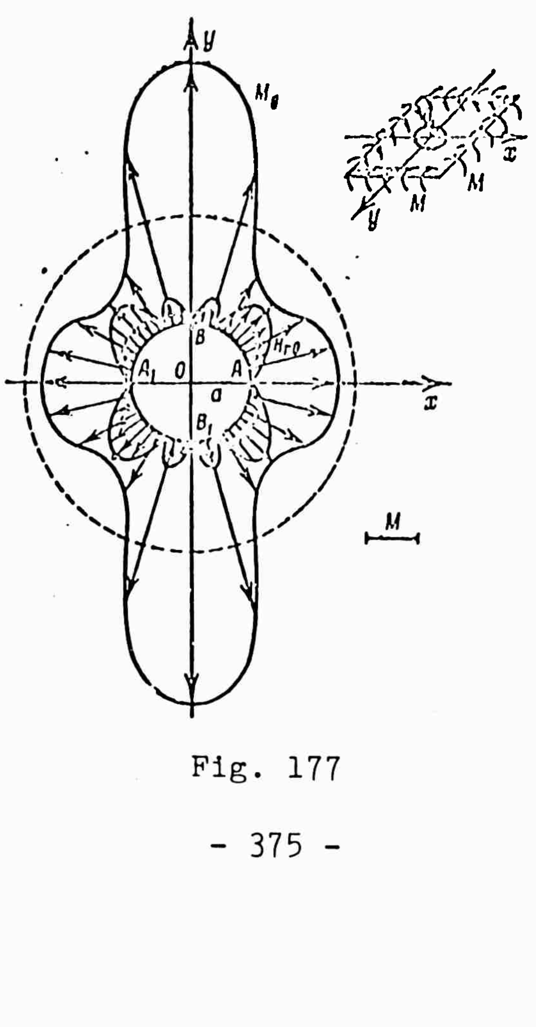

Fig. 177

At the points A and A_1 (Fig. 176)

$$M_\theta = 1.74M; \qquad (90.6)$$

At the points B and B_1

$$M_\theta = 5.06M. \qquad (90.7)$$

The maximum torsional moment is equal to

$$(H_{r\theta})_{max} = 0.8M. \qquad (90.8)$$

As we see from these values, the concentrations of the stresses σ_θ in a veneer plate, which may be estimated on the basis of the value of the moment M_θ, proves to be more than 2.5 times higher than in an isotropic plate. This case of bending was investigated for the first time by M.V. Nikulin.*

2. Torsion. An orthotropic rectangular plate with round aperture in its center is deformed by torsional moments H, which are distributed uniformly on all four sides (Fig. 178).

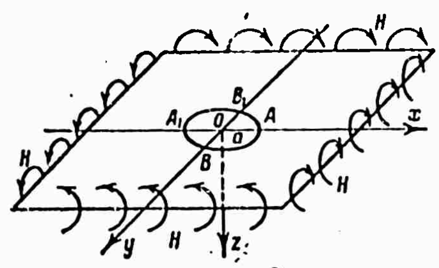

Fig. 178

Let us only consider the simplest case where the principal directions in the plate are parallel to the sides. The solution of the problem is obtained by a superposition of the distributions of the moments and crosscut forces in the form

$$M_x = M_y = 0, \quad H_{xy} = H, \quad N_x = N_y = 0 \qquad (90.9)$$

and the distribution corresponding to the functions

$$\left.\begin{aligned} w_1'(z_1) &= -\frac{Ha(1-i\mu_2)}{D_2(\mu_1-\mu_2)[n(k+1)+2(\nu_1+k)]}\cdot\frac{1}{\zeta_1},\\ w_2'(z_2) &= \frac{Ha(1-i\mu_1)}{D_2(\mu_1-\mu_2)[n(k+1)+2(\nu_1+k)]}\cdot\frac{1}{\zeta_2}.\end{aligned}\right\} \qquad (90.10)$$

After the transformations we obtain the following distributions for the bending and torsional moments M_θ and $H_{r\theta}$ on the contour of the hole:

$$
\left.
\begin{aligned}
M_\theta &= -2H\frac{\sqrt{D_1D_2}}{D_r}\cdot\frac{n(k+n+1)}{n(k+1)+2(\nu_1+k)}\times\\
&\quad\times\left[\frac{1+\nu_1}{k}\sin^2\theta+(1+\nu_2)\,k\cos^2\theta\right]\sin\theta\cos\theta,\\
H_{r\theta} &= H\frac{\sqrt{D_1D_2}}{D_r}\cdot\frac{n(k+n+1)}{n(k+1)+2(\nu_1+k)}\times\\
&\quad\times\left(k\cos^4\theta-\frac{1}{k}\sin^4\theta\right).
\end{aligned}
\right\}
\qquad (90.11)
$$

The moment distribution near the aperture in an isotropic plate is characterized by the formulas:

$$
\left.
\begin{aligned}
M_\theta &= -H\frac{4(1+\nu)}{3+\nu}\sin 2\theta,\\
H_{r\theta} &= H\frac{4}{3+\nu}\cos 2\theta.
\end{aligned}
\right\}
\qquad (90.12)
$$

Figure 179 shows the graphs of distribution of the moments M_θ and $H_{r\theta}$ on the contour of the aperture in a veneer plate; the x-axis is parallel to the fibers of the sheet. The bending moment reaches its maximum value near the point $\theta = 60°$ and the points symmetrical with respect to it; the torsional moment reaches its maximum absolute value at the four points A, A_1, B, B_1:

$$|M_\theta|_{max} = 2{,}15H; \qquad (90.13)$$

$$|H_{r\theta}|_{max} = 1{,}09H. \qquad (90.14)$$

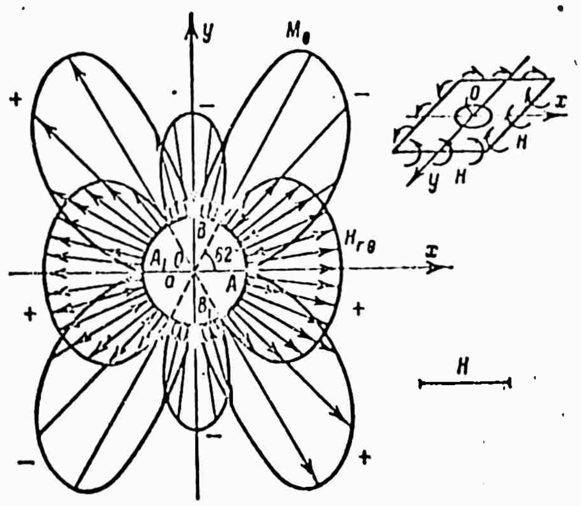

Fig. 179

In such a plate of isotropic material, with Poisson's coefficient equal to 0.3, we obtain

$$|M_\theta|_{max} = 1{,}58H; \qquad (90.15)$$

$$|H_{r\theta}|_{max} = 1{,}21H. \qquad (90.16)$$

The moment distribution graphs for the contour of the hole in an isotropic plate are in their nature the same as the graphs

for the veneer plate and we shall not give them here.

This problem was solved by V.N. Al'pert.*

3. Pure bending of an orthotropic plate with a rigid circular core. Consider an orthotropic rectangular plate with a round hole in its middle in which a core is soldered or glued in, which was not prestressed and which consists of a perfectly rigid non-deformed material. Let us consider the pure bending of such a plate for the case where the principal directions of elasticity are parallel to the sides. The position of the axes and the load distribution is shown in Fig. 180.

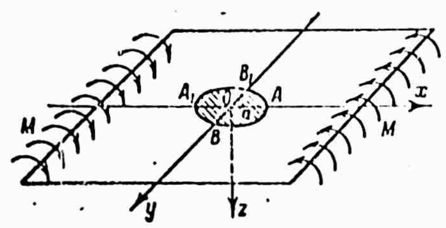

Fig. 180

In this case we are concerned with the second fundamental problem of the theory of elasticity since on the contour of the aperture the strains are given: owing to the rigidity of the core the edge of the aperture cannot be deformed, it can only be displaced in space and rotated. Using the results of §88 we arrive at the following results. The bending and torsional moments and the crosscut forces are constituted from the moments and forces corresponding to pure bending of the massive plate

$$M_x = M, \quad M_y = H_{xy} = N_x = N_y = 0 \tag{90.17}$$

and the additional moments and forces determined by functions of complex variables in the form

$$\left.\begin{aligned} w_1'(z_1) &= A_0 - \frac{Ma}{2D_1(1-\nu_1\nu_2)} \cdot \frac{i\nu_1+\mu_2}{\mu_1-\mu_2} \cdot \frac{1}{\zeta_1}, \\ w_2'(z_2) &= B_0 + \frac{Ma}{2D_1(1-\nu_1\nu_2)} \cdot \frac{i\nu_1+\mu_1}{\mu_1-\mu_2} \cdot \frac{1}{\zeta_2}. \end{aligned}\right\} \tag{90.18}$$

Here A_0 and B_0 are constants which do not influence the stress distribution in the plate; we can set them equal to zero.

The formula for the bending moment M_r acting on the edge of the core from the side of the plate and on the edge of the aperture from the side of the core has the form

$$M_r = \frac{M}{1-\nu_1\nu_2}\left[\left(1+\frac{\nu_1+n}{k}\right)\cos^2\theta - \frac{D_2}{D_1}(\nu_1+\nu_1 n+k)\sin^2\theta\right]. \tag{90.19}$$

The moments M_θ and $H_{r\theta}$ near the core are determined by much more complex formulas. We cannot give these formulas in their

full length, we restrict ourselves to the values of the moments at significant points.

At the points A and A_1 (Fig. 180)

$$M_r = \frac{M}{1-\nu_1\nu_2}\left(1 + \frac{\nu_1 + n}{k}\right), \quad M_\theta = \nu_2 M_r, \quad H_{r\theta} = 0, \tag{90.20}$$

at the points B and B_1

$$\left.\begin{aligned} M_r &= -\frac{M}{1-\nu_1\nu_2}\cdot\frac{D_2}{D_1}(\nu_1 + \nu_1 n + k), \\ M_\theta &= \nu_1 M_r, \quad H_{r\theta} = 0. \end{aligned}\right\} \tag{90.21}$$

In an isotropic plate the moments near the core are equal to

$$\left.\begin{aligned} M_r &= M\left(\frac{1}{1+\nu} + \frac{2}{1-\nu}\cos 2\theta\right), \\ M_\theta &= \nu M_r, \quad H_{r\theta} = 0 \end{aligned}\right\} \tag{90.22}$$

(ν is Poisson's coefficient).

At the points A and A_1 of an isotropic plate

$$M_r = M\frac{3+\nu}{1-\nu^2}, \quad M_\theta = \nu M_r; \tag{90.23}$$

at the points B and B_1

$$M_r = -M\frac{1+3\nu}{1-\nu^2}, \quad M_\theta = \nu M_r. \tag{90.24}$$

In Fig. 181 we show the distribution of the moments M_r along the core for a veneer plate loaded on the sides perpendicular to the fibers of the sheet, in Fig. 182 the same is shown for the case where the sides parallel to the fibers are loaded. The dashed curves represent the moment distribution in an isotropic plate for which ν = 0.3.

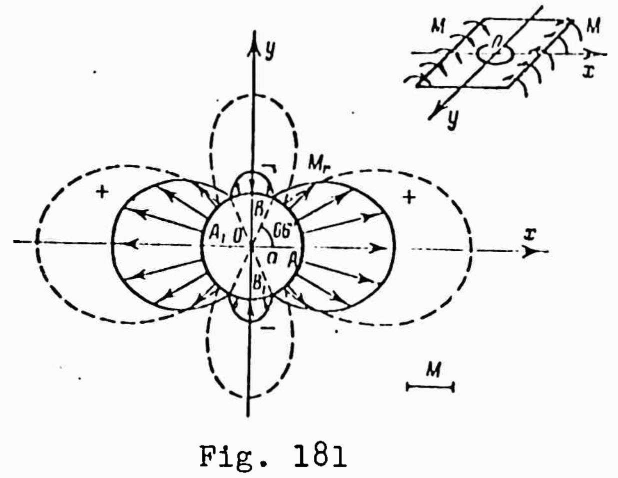

Fig. 181

In the first case of a veneer plate the moment M_r is equal to

$$(M_r)_{max} = 1{,}996M, \tag{90.25}$$

and in the second case

$$(M_r)_{max} = 4{,}213M. \tag{90.26}$$

In the case of an isotropic plate

$$(M_r)_{max} = 1{,}088M. \tag{90.27}$$

Comparing the graphs of Fig. 181 and Fig. 182, we can see that the case where the sides parallel to the sheet fibers are loaded is less favorable. We see from these two graphs that the difference between the distributions of the moments M_r in these two cases is quite significant. In both cases the moment M_r reaches a maximum which is much higher than in an isotropic plate.

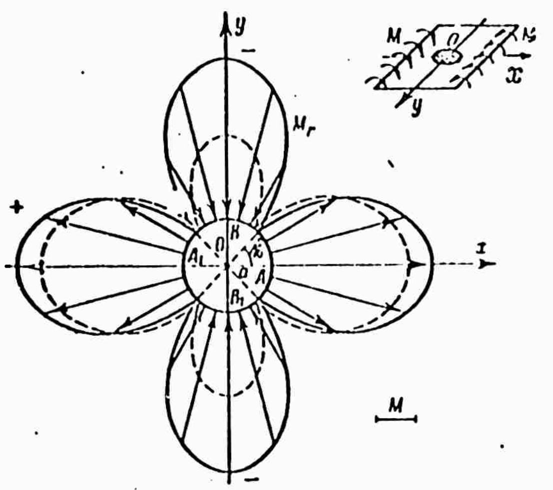

Fig. 182

An approximate solution of the problem of bending of an anisotropic plate with rigid core, circular or elliptic, was obtained by B.Ya. Rodin.*

M.F. Sheremet'yev derived a solution of the problem on the bending of an infinite anisotropic plate with a circular aperture whose edge is reinforced by a thin elastic ring (in the case of constant bending and torsional moments applied "at infinity."**

Manuscript Page No.	[Footnotes]
359	A solution of this problem is to be found in our paper: "O nekotorykh voprosakh, svyazannykh s teoriyey izgiba tonkikh plit" [On Some Problems Connected with the Theory of Bending of Thin Plates] Prikladnaya matematika i mekhanika [Applied Mathematics and Mechanics] Nov. Seriya [New Series], Vol. II, No. 2, 1938.
362	See our paper mentioned in the last section. In this paper we used somewhat different denotations.
364	This paper has been dealt with carefully in our paper mentioned in §86.
371	Lekhnitskiy, S.G., O nekotorykh voprosakh svyazannykh s teoriyey izgiba tonkikh plit [On Some Problems Connected with the Theory of Bending of Thin Plates] Prikladnaya matematika i mekhanika, Nov. Seriya, Vol. II, No. 2, 1938. In this paper other notations were used.
376	Nikulin, M.V., Izgib pryamougol'noy tonkoy plity s ellipticheskim i krugovym otverstiyem izgibayushchim momentami, raspredelennymi ravnomerno po storonam [The Bending of a Rectangular Thin Plate with Elliptic or Circular Holes Bent by Moments Distributed Uniformly on the Sides]. Diploma thesis, Saratov State University, Saratov 1954.
378	Al'pert, V.N., Izgib pryamougol'noy ortotropnoy plity s ellipticheskim otverstiyem pod deystviyem krutyashchikh momentov, ravnomerno raspredelennykh po krayu plity [The Bending of a Rectangular Orthotropic Plate with Elliptic Aperture Under the Action of Critical Moments Distributed Uniformly on the Edge of the Plate] Diploma paper, Saratov State University, Saratov, 1954.
380*	Rodin, B.Ya., Izgib pryamougol'noy anizotropnoy plastinki s zhestkim ellipticheskim yadrom momentami, ravnomerno raspredelennymi po dvum storonam [The Bending of a Rectangular Anisotropic Plate with a Rigid Elliptic Core by Moments Uniformly Distributed on Two Sides], Diploma thesis, Saratov State University, Saratov, 1954.
380**	Sheremet'yev, M.P., 1) Izgib tonkikh plit s podkreplennym krayem [Bending of Thin Plates with Reinforced Edge] Ukrainskiy matematicheskiy zhurnal [Ukrainian Mathematical Journal] Vol. 5, No. 1, 1953; 2) Izgib anizotropnykh i izotropnykh plit oslablennykh otverstiyem, kray kotorogo podkreplen uprugim tonkom kol'tsom [Bending of Anisotropic and Isotropic Plates Weakened by an Aperture, the Edge Being Reinforced by an Elastic Thin Ring] DAN UkrSSR, 1950, No. 6.

Chapter 12

TRANSVERSE VIBRATIONS OF ANISOTROPIC PLATES

§91. FREE VIBRATIONS OF A PLATE

In all cases dealt with previously the plates were assumed to be deformed by static loads. The problem of the investigation of strains and stresses in an anisotropic plate becomes much more complicate when the load is assumed to vary with time or to be applied suddenly, i.e., in the case of dynamic loads. The dynamic problems are closely related with the problems of vibrations of plates. In the present chapter we shall consider briefly some problems of transverse vibrations of anisotropic plates characterized by curvatures of the median surface.

Let us consider a plate of arbitrary form which is anisotropic and homogeneous but generally not orthotropic, with arbitrarily fixed or free edges. Let us suppose that certain forces distributed on the surface impart deflections and velocities in a direction perpendicular to the initial (nondeformed) median surface, to the particles arranged on the median surface; at the moment of time following the initial, the plate is suddenly discharged from all external loads. Having receive the initial deformation and velocity the plate, when unloaded, begins to vibrate and the particles in the median surface move perpendicularly to it; the result is that the plate is bent at any moment of time. This type of oscillation is called free transverse vibration.

The differential equation of vibration is obtained when we set up the equation of motion of an element of the form of a rectangular parallelepiped of height h and base $dxdy$.

Let γ be the specific weight of the material, $w(x, y, t)$ is the deflection of the median surface; the other denotations for rigidities, moments and crosscut forces are the same as previously. Let us return to Fig. 126 (load q must be assumed absent).

Instead of the first equilibrium equation (62.1) we obtain the equation of motion

$$\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y}\right) dx\,dy = \frac{h\gamma}{g}\, dx\,dy\, \frac{\partial^2 w}{\partial t^2}, \tag{91.1}$$

or

$$\frac{\partial^2 w}{\partial t^2} + \frac{g}{h\gamma}\left[D_{11}\frac{\partial^4 w}{\partial x^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3\,\partial y} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2\,\partial y^2} + 4D_{26}\frac{\partial^4 w}{\partial x\,\partial y^3} + D_{22}\frac{\partial^4 w}{\partial y^4}\right] = 0 \tag{91.2}$$

where the force of inertia is taken into account while the resistance of the surrounding medium and internal friction are neglected.

In particular, for an orthotropic plate we obtain the equation

$$\frac{\partial^2 w}{\partial t^2}+\frac{g}{h\gamma}\left(D_1\frac{\partial^4 w}{\partial x^4}+2D_3\frac{\partial^4 w}{\partial x^2\partial y^2}+D_2\frac{\partial^4 w}{\partial y^4}\right)=0, \qquad (91.3)$$

and for an isotropic plate with rigidity D, the equation

$$\frac{\partial^2 w}{\partial t^2}+\frac{gD}{h\gamma}\nabla^2\nabla^2 w=0. \qquad (91.4)$$

The deflection w satisfies boundary conditions which depend on the method of fixing of the plate's edge (with fixed, supported or free edges these conditions do not differ from the conditions in the case of equilibrium), and the initial conditions

$$\text{with } t=0 \quad w=w_0(x, y), \; \frac{\partial w}{\partial t}=v_0(x, y), \qquad (91.5)$$

where w_0, v_0 are given quantities of initial deflection and initial velocity of point (x, y).

A comprehensive investigation into the problem of free vibrations must be reduced to the determination of the deflection at an arbitrary point and at an arbitrary instant of time, but the most important part of the problem is the determination of the frequency of natural vibrations and the eigenfunctions. In problems of the dynamics of plates the frequencies of the natural vibrations play an important part; they must be known in order to determine the dynamic stresses caused by a variable load.

Let us briefly describe the course of solution of the problem on free transverse vibrations by the Fourier method.

Let us introduce (for the sake of brevity) the operator L:

$$L=D_{11}\frac{\partial^4}{\partial x^4}+4D_{16}\frac{\partial^4}{\partial x^3\partial y}+2(D_{12}+2D_{66})\frac{\partial^4}{\partial x^2\partial y^2}+ \\ +4D_{26}\frac{\partial^4}{\partial x\,\partial y^3}+D_{22}\frac{\partial^4}{\partial y^4}. \qquad (91.6)$$

Equation (91.2) can then be written in the abbreviated form

$$\frac{\partial^2 w}{\partial t^2}+\frac{g}{h\gamma}Lw=0. \qquad (91.7)$$

We seek a solution of this equation in the form of the product

$$w=(A\cos pt+B\sin pt)W(x, y), \qquad (91.8)$$

where p is the proper frequency of the plate which is to be determined.*

Substituting (91.8) in (91.7) we obtain for W the equation

$$LW - \frac{p^2 h \gamma}{g} W = 0. \qquad (91.9)$$

We then determine the solution of this equation which satisfies the boundary conditions (the conditions for W in the case of a fixed, supported or free edge will not differ from the conditions for w). In the case of simple contours, for example, a rectangular plate, an expression for W can be chosen beforehand which makes it possible to satisfy the boundary conditions or rather satisfy them; the expression for them will contain arbitrary constants. It is required that the function W satisfies the boundary conditions and is a solution to Eq. (91.7) so that we obtain a system of homogeneous equations for the unknown constants; this system has solutions which are nonvanishing only in the case where its determinant $\Delta(p)$ vanishes. From it we obtain the frequency equation

$$\Delta(p) = 0. \qquad (91.10)$$

This equation will have an infinite set of solutions which represent the frequency spectrum for the plate given. The frequencies will in general depend on the two parameters: m and n ($m = 1, 2, 3, \ldots$; $n = 1, 2, 3, \ldots$). The lowest frequency is called the frequency of the first harmonic, the other frequencies are the frequencies of higher order or the higher harmonics.

A function $W_{mn}(x, y)$ corresponds to each frequency p_{mn}, which, on the basis of the homogeneous system of equations is determined to within a constant factor which can be taken equal to unity. The functions W_{mn} which are called eigenfunctions determine the form of the vibrations (i.e., the form of the bent surface corresponding to vibrations with frequencies p_{mn}).

When the problem is to be solved finally, i.e., when we have to determine $w(x, y, t)$ at an arbitrary point and an arbitrary instant of time the following is to be done. The given initial deflection and initial velocity are expanded in series with respect to the eigenfunctions W_{mn}, i.e., we represent them in the form of

$$w_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \alpha_{mn} W_{mn}, \quad v_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \beta_{mn} W_{mn} \qquad (91.11)$$

and find the solution to Eq. (91.7) in the form of a sum of all solutions of the form (91.8).

When the coefficients α_{mn}, β_{mn} have been determined, the determination of the constants A and B is quite simple. The result obtained reads

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\alpha_{mn} \cos p_{mn} t + \frac{\beta_{mn}}{p_{mn}} \sin p_{mn} t \right) W_{mn}. \qquad (91.12)$$

The total deflection at an arbitrary point is obtained as the result of a superposition of an infinite series of deflections whose variation with time is governed by the law of the

law of the simple harmonic vibrations with the frequencies p_{mn}.

The equation of free transverse vibrations in the more general case where the plate rests on a massive elastic base with an elastic coefficient k and is subject to the action of longitudinal forces T_x, T_y, S_{xy} which are time-independent, has the form:

$$\frac{\partial^2 w}{\partial t^2} + \frac{g}{h\gamma}\left(Lw + kw - T_x \frac{\partial^2 w}{\partial x^2} - 2S_{xy}\frac{\partial^2 w}{\partial x\,\partial y} - T_y \frac{\partial^2 w}{\partial y^2}\right) = 0. \qquad (91.13)$$

In order to determine the natural frequencies and eigenfunctions for this case we can use the same Fourier method which results in an equation for the function of W which, of course, is more complicate compared with (91.9).

§92. DETERMINATION OF FREQUENCIES OF A RECTANGULAR ORTHOTROPIC PLATE

Let us consider a given rectangular homogeneous orthotropic plate whose principal directions of elasticity are parallel to the sides; we have to determine the frequencies of the natural vibrations. This problem can be solved exactly only in the case of an orthotropic plate with four supported sides.

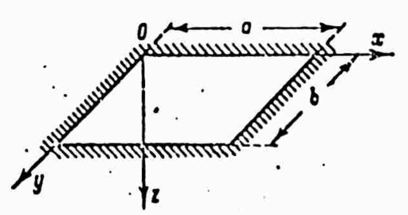

Fig. 183

We let the axes coincide with the sides of the plate (Fig. 183) whose lengths are denoted by a and b. The function W [see Eq. (91.8)] must satisfy the conditions

$$\text{with } x = 0,\ x = a \qquad W = 0,\quad \frac{\partial^2 W}{\partial x^2} + \nu_2 \frac{\partial^2 W}{\partial y^2} = 0; \qquad (92.1)$$

$$\text{with } y = 0,\ y = b \qquad W = 0,\quad \frac{\partial^2 W}{\partial y^2} + \nu_1 \frac{\partial^2 W}{\partial x^2} = 0. \qquad (92.2)$$

These conditions are satisfied by the expression

$$W_{mn} = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \qquad (92.3)$$

where m and n are integral numbers. This expression is required to be a solution to Eq. (91.9) which, in the case of an orthotropic plate assumes the form

$$D_1 \frac{\partial^4 W}{\partial x^4} + 2D_3 \frac{\partial^4 W}{\partial x^2\,\partial y^2} + D_2 \frac{\partial^4 W}{\partial y^4} - \frac{p^2 h\gamma}{g} W = 0. \qquad (92.4)$$

Substituting Eq. (92.3) in the left-hand side and setting the result equal to zero we obtain

$$D_1\left(\frac{m\pi}{a}\right)^4+2D_3\left(\frac{mn\pi^2}{ab}\right)^2+D_2\left(\frac{n\pi}{b}\right)^4-\frac{p^2h\gamma}{g}=0. \qquad (92.5)$$

From this we obtain the frequency $p = p_{mn}$:

$$p_{mn}=\frac{\pi^2}{b^2}\sqrt{\frac{g}{h\gamma}}\sqrt{D_1\left(\frac{m}{c}\right)^4+2D_3n^2\left(\frac{m}{c}\right)^2+D_2n^4}, \qquad (92.6)$$

where $c = a/b$.

The fundamental frequency is equal to

$$p_{11}=\frac{\pi^2}{a^2}\sqrt{\frac{g}{h\gamma}}\sqrt{D_1+2D_3c^2+D_2c^4}. \qquad (92.7)$$

In particular, for a quadratic plate of side a

$$p_{mn}=\frac{\pi^2}{a^2}\sqrt{\frac{g}{h\gamma}}\sqrt{D_1m^4+2D_3n^2m^2+D_2n^4}, \qquad (92.8)$$

$$p_{11}=\frac{\pi^2}{a^2}\sqrt{\frac{g}{h\gamma}}\sqrt{D_1+2D_3+D_2}. \qquad (92.9)$$

When the plate is isotropic, we have*

$$p_{mn}=\frac{\pi^2}{b^2}\sqrt{\frac{gD}{h\gamma}}\left[\left(\frac{m}{c}\right)^2+n^2\right]. \qquad (92.10)$$

The given frequency p_{mn} corresponds to the deflection

$$w_{mn}=(A_{mn}\cos p_{mn}t+B_{mn}\sin p_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \qquad (92.11)$$

The eigenfunctions (92.3) determine the form of the vibrations, i.e., the form of the bent surface of the plate executing the vibrations with the given frequencies p_{mn}. In Fig. 184a we see the shape of the bent surface at an arbitrary moment of time for a plate vibrating at the first harmonic frequency p_{11}; the point of maximum deviation or antinode is in the middle. Figure 184b and c show the bent surfaces of a plate vibrating at the frequencies p_{21} and p_{12}; in each case there are two antinodes and one of the axes of symmetry remains immobile, i.e., it represents a line of node.

The total deflection of the plate whose arbitrary initial deviation w_0 and initial velocity v_0 are given is obtained as the result of a superposition of an infinite series of deflections of the form (92.11)

$$w=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\alpha_{mn}\cos p_{mn}t+\frac{\beta_{mn}}{p_{mn}}\sin p_{mn}t\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad (92.12)$$

Here α_{mn}, β_{mn} are the expansion coefficients of initial deflection and initial velocity in series of the eigenfunctions; in the given case they are double Fourier series whose coefficients

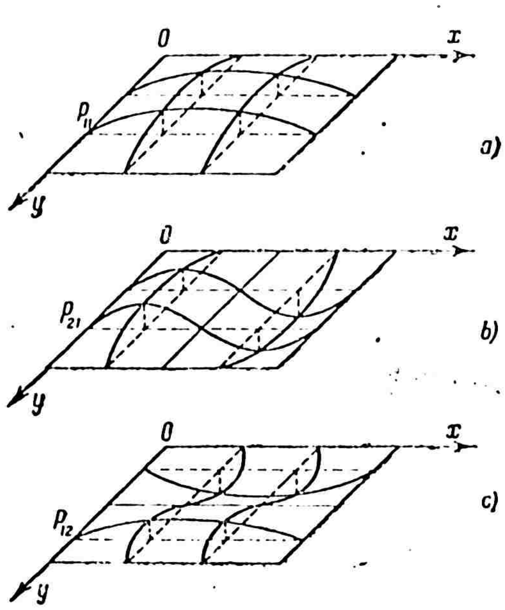

Fig. 184

are determined from the well-known formulas

$$
\left.\begin{aligned}
\alpha_{mn} &= \frac{4}{ab}\int_0^a\int_0^b w_0 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy,\\
\beta_{mn} &= \frac{4}{ab}\int_0^a\int_0^b v_0 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy.
\end{aligned}\right\} \qquad (92.13)
$$

For a plate where the two sides $x = 0$, $x = a$ are supported and the others are fixed arbitrarily or free, the expression of the function of W has the form

$$
\begin{aligned}
W = \Big(C_{1m}\sin\frac{m\pi k_1 y}{a} + C_{2m}\cos\frac{m\pi k_1 y}{a} &+\\
+ C_{3m}\,\mathrm{sh}\frac{m\pi k_2 y}{a} + C_{4m}\,\mathrm{ch}\frac{m\pi k_2 y}{a}\Big)&\sin\frac{m\pi x}{a},
\end{aligned} \qquad (92.14)
$$

where

$$
\left.\begin{aligned}
k_1 &= \sqrt{\frac{1}{D_2}\sqrt{D_3^2 - D_1D_2 + D_2\frac{p^2h\gamma a^4}{g\pi^4m^4}} - \frac{D_3}{D_2}},\\
k_2 &= \sqrt{\frac{1}{D_2}\sqrt{D_3^2 - D_1D_2 + D_2\frac{p^2h\gamma a^4}{g\pi^4m^4}} + \frac{D_3}{D_2}}.
\end{aligned}\right\} \qquad (92.15)
$$

This function satisfies the conditions for supported sides. From the conditions on the other sides a system of four homogeneous equations are obtained with four unknowns: $C_{1m}, C_{2m}, C_{3m}, C_{4m}$ the frequencies are determined from equations obtained when the determinant of the homogeneous system is set equal to zero. To each value of $m = 1, 2, 3, \ldots$ corresponds an infinite series of frequencies p_{mn} since the equation $\Delta(p) = 0$ is transcendental and has an infinite number of solutions. In each case where all sides of the plate are supported we obtain Eq. (92.6) for the frequencies.

§93. APPROXIMATION METHODS FOR THE DETERMINATION OF FREQUENCIES

An accurate determination of the frequencies of a plate (except for a rectangular plate with two or four supported sides) entails considerable difficulties which are connected with the integration of the fourth-order equation (91.9) or (92.4)

In practice it may prove to be valuable to have approximation methods for the determination of the fundamental frequency, which are analogous to the methods applied in the investigation of the natural vibrations of beams. There exists a series of approximation methods for the determination of the frequencies of transverse vibrations of beams which permit a quick determination of the frequency without an integration of differential equations* (mainly the frequencies of the fundamental harmonics which are of greatest interest in practice). These methods can easily be generalized for the case of a plate.

We shall here consider one of them, the method by Rayleigh-Ritz, and we shall use it in order to determine approximately the fundamental frequency of rectangular, round and triangular plates.

Using this method, we consider an elastic body executing free vibrations with the fundamental frequency (in our case a plate) as a system with one degree of freedom whose state in an arbitrary moment of time is determined by a single generalized coordinate $q(t)$.

In the case of a plate the deflection at an arbitrary instant of time is assumed to have the form of

$$w = q(t)\,W(x, y), \tag{93.1}$$

where W is a given steady function satisfying the boundary conditions (depending on the method of fastening of the edge) and representing approximately the form of the bent surface of the vibrating plate. We then set up the equation of motion of the system with the help of the well-known Lagrange equations. In the given case we obtain a single equation corresponding to the number of degrees of freedom, which has the form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q'}\right) - \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} = 0, \tag{93.2}$$

where T is the kinetic and V the potential energy of the system. For an orthotropic and homogeneous plate

$$\left.\begin{aligned} T &= \frac{q'^2}{2}\cdot\frac{h\gamma}{g}\int\int W^2 dx\,dy, \\ V &= \frac{q^2}{2}\int\int\left[D_1\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + 2D_1\nu_2\frac{\partial^2 W}{\partial x^2}\cdot\frac{\partial^2 W}{\partial y^2} + \right. \\ &\left. + D_2\left(\frac{\partial^2 W}{\partial y^2}\right)^2 + 4D_k\left(\frac{\partial^2 W}{\partial x\,\partial y}\right)^2\right] dx\,dy \end{aligned}\right\} \tag{93.3}$$

(γ is the specific weight and h the thickness).

We denote the expression for V in the abbreviated form

$$V = q^2 \int\int \bar{V}(W)\,dx\,dy. \qquad (93.4)$$

After having substituted the expressions for T and V the Lagrange equations takes the form

$$q'' + p^2 q = 0, \qquad (93.5)$$

where p is the frequency determined by the equation

$$p^2 = \frac{2g}{h\gamma} \cdot \frac{\int\int \bar{V}(W)\,dx\,dy}{\int\int W^2\,dx\,dy}. \qquad (93.6)$$

The expression for the deflection is obtained in the form

$$w = (A\cos pt + B\sin pt)\,W. \qquad (93.7)$$

The accuracy of the determination of the frequencies according to Eq. (93.6) depends essentially on the suitable choice of the expression of W. For simple contours the choice of these functions is not too difficult; sometimes one can give at once several different expressions for them. We can, for example, take an expression proportional to the static deflection of the plate as a first approximation for W, with the same conditions for the fixing of the edge under the influence of a uniformly distributed load. This is equivalent to the supposition that the surface of the plate executing the vibration of the lowest harmonic has the same form as the surface of a plate bent by uniform pressure.

Equation (93.6) determines the frequency p in a first approximation. More accurate values of the frequency can be obtained when we determine the minimum of the expression $\text{Э}'$ dealt with in §4 [see Eq. (4.5)]. In the case of a plate executing a simple harmonic vibration with the frequency p, the expression for $\text{Э}'$ takes the form

$$\text{Э}' = \frac{h\gamma}{2g} q^2 S, \qquad (93.8)$$

where

$$S = \int\int \left[\frac{2g}{h\gamma}\bar{V}(W) - p^2 W^2\right] dx\,dy \qquad (93.9)$$

(integration over the area of the plate). The problem is reduced to the determination of the function W satisfying the boundary conditions and minimizing the integral (93.9).

An approximate solution of this variational problem can be obtained by means of, e.g., Ritz's method, applying it in the same order as in the case of static bending of a plate, namely, by choosing an expression for W in the form of a sum with indefinite coefficients

$$W = \sum_m \sum_n A_{mn} W_{mn}(x, y), \qquad (93.10)$$

where W_{mn} are continuous functions depending on two parameters

and satisfying the conditions on the edge of the plate (at least the kinematic ones). After having substituted this expression in Eq. (93.9) and integration, the quantity S can be represented in the form of a homogeneous square function of the coefficients A_{mn}. This function is then minimized and a system of homogeneous equations of first degree is obtained for the coefficients A_{mn}. It is required that at least one of the coefficients is nonvanishing, the determinant of this system is set equal to zero and the following frequency equation is obtained:

$$\Delta(p) = 0. \qquad (93.11)$$

The lowest nonzero solution will also be the approximate value of the fundamental natural frequency.

§94. EXAMPLES OF FREQUENCY DETERMINATION IN A FIRST APPROXIMATION

Let us consider some examples.

1. Rectangular plate with fixed sides. Let us determine the fundamental natural frequency of a rectangular orthotropic plate whose four sides are all fixed (Fig. 185).

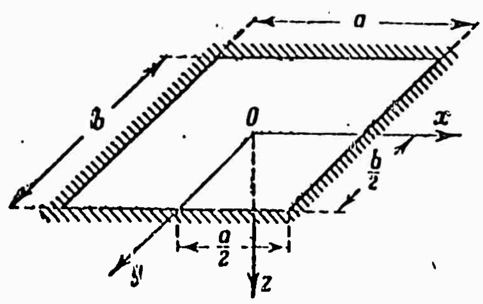

Fig. 185

An exact solution to this problem is unknown as yet, but we can obtain an approximate one assuming that, say,

$$W_{mn} = \left(x^2 - \frac{a^2}{4}\right)^2 \left(y^2 - \frac{b^2}{4}\right)^2 x^m y^n, \qquad (94.1)$$

where m, n are integral numbers.

It is obvious that all these functions (and the function W as a whole) will satisfy the boundary conditions

$$\text{with } x = \pm\frac{a}{2} \quad W_{mn} = \frac{\partial W_{mn}}{\partial x} = 0; \qquad (94.2)$$

$$\text{with } y = \pm\frac{b}{2} \quad W_{mn} = \frac{\partial W_{mn}}{\partial y} = 0. \qquad (94.3)$$

As a first approximation we take only the first term of the sum (93.10):

$$W = A\left(x^2 - \frac{a^2}{4}\right)^2\left(y^2 - \frac{b^2}{4}\right)^2. \tag{94.4}$$

Substituting w in Eq. (93.6) and integrating, we arrive at

$$p_{11} = \frac{22,45}{a^2}\sqrt{\frac{g}{h\gamma}}\sqrt{D_1 + 0,571 D_3 c^2 + D_2 c^4}. \tag{94.5}$$

In particular, for a quadratic isotropic plate with the rigidity D

$$p_{11} = \frac{36,0}{a^2}\sqrt{\frac{gD}{h\gamma}}. \tag{94.6}$$

We can also give another expression for W_{mn}

$$W_{mn} = \left[1 - (-1)^m \cos\frac{2m\pi x}{a}\right]\cdot\left[1 - (-1)^n \cos\frac{2n\pi y}{b}\right], \tag{94.7}$$

which, obviously, also satisfies all boundary conditions. The first approximation for the fundamental natural frequency is obtained on the assumption that

$$W = A\left(1 + \cos\frac{2\pi x}{a}\right)\left(1 + \cos\frac{2\pi y}{b}\right). \tag{94.8}$$

The result of all calculations reads

$$p_{11} = \frac{22,79}{a^2}\sqrt{\frac{g}{h\gamma}}\sqrt{D_1 + 0,667 D_3 c^2 + D_2 c^4}. \tag{94.9}$$

For an isotropic quadratic plate we obtain from this formula

$$p_{11} = \frac{37,2}{a^2}\sqrt{\frac{gD}{h\gamma}}. \tag{94.10}$$

The difference between Eqs. (94.6) and (94.10) is small: it amounts to 3.5% referred to the lower of the two values.

2. Circular plate with fixed rim. Let us determine the fundamental natural frequency of an orthotropic homogeneous plate whose rim is fixed (Fig. 186, the x-y frame coincides with the principal directions of elasticity).

In order to obtain a first-order solution we use the expression for the static deflection under the action of a uniformly distributed load derived in §80. Assuming

$$W = A(a^2 - x^2 - y^2)^2, \tag{94.11}$$

we obtain

$$p_{11} = \frac{6,33}{a^2}\sqrt{\frac{g}{h\gamma}}\sqrt{D_1 + 0,667 D_3 + D_2}. \tag{94.12}$$

In particular, for such a plate of isotropic material*

$$p_{11} = \frac{10,32}{a^2}\sqrt{\frac{gD}{h\gamma}}. \tag{94.13}$$

3. Plate in the form of a rectangular triangle. Let us consider an orthotropic plate in the form of a rectangular triangle with the legs a and b, the principal directions of elasticity being parallel to the legs (Fig. 187).

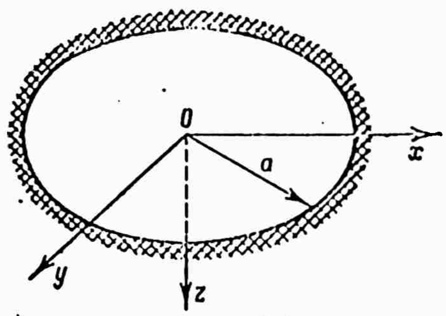

Fig. 186

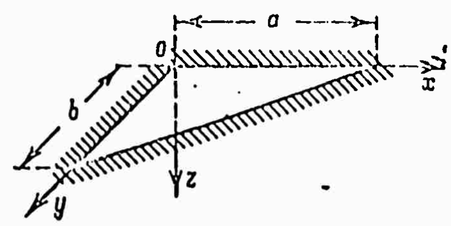

Fig. 187

When all three sides are fixed we can assume (see §85)

$$W = Ax^2y^2\left(1 - \frac{x}{a} - \frac{y}{b}\right)^2 \tag{94.14}$$

and we then obtain from Eq. (93.6) the fundamental natural frequency

$$p_{11} = \frac{63,28}{a^2}\sqrt{\frac{g}{h\gamma}}\sqrt{D_1 + D_3c^2 + D_2c^4}. \tag{94.15}$$

Here $c = a/b$.

The formula for the frequency of the first harmonic of an isotropic plate is obtained from (94.15), assuming that $D_1 = D_2 = D_3 = D$. In particular, for an isotropic plate with equal legs (c = 1)

$$p_{11} = \frac{109,6}{a^2}\sqrt{\frac{gD}{h\gamma}}. \tag{94.16}$$

In the case of three sides supported we have in a first approximation

$$W = Axy\left(1 - \frac{x}{a} - \frac{y}{b}\right). \tag{94.17}$$

For the frequency of the first harmonic we obtain

$$p_{11} = \frac{40,98}{a^2}\sqrt{\frac{g}{h\gamma}}\sqrt{D_1 + 0,33D_3c^2 + D_2c^4} \tag{94.18}$$

For an isotropic plate in the form of an isosceles rectangular triangle

$$p_{11} = \frac{62,61}{a^2}\sqrt{\frac{gD}{h\gamma}}. \tag{94.19}$$

Equations (94.15)-(94.19) have been obtained by V.G. Rusin (in his diploma thesis, see reference in §85).

4. Plate in the form of an isosceles and an equally-sided triangle. The next example we consider is a plate in the form of an isosceles triangle with a vertex angle of 2α fixed on all three sides. It is supposed that the principal directions of elasticity are parallel and perpendicular to the axes of symmetry of the triangle.

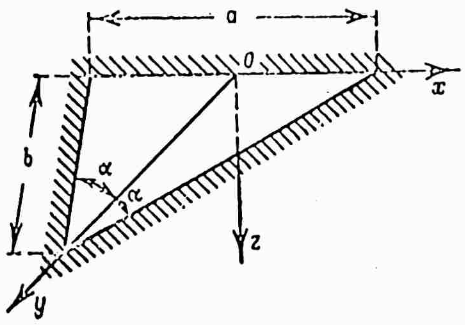

Fig. 188

Using the approximate expression of static deflection [see Eq. (85.12)] we obtain

$$W = Ay^2\left[\left(\frac{y}{a\cos\alpha} - 1\right)^2 - \frac{x^2}{a^2\sin^2\alpha}\right]^2. \qquad (94.20)$$

Substituting (94.20) in Eq. (93.6) we obtain after a transformation

$$p_{11} = \frac{15{,}82}{a^2}\sqrt{\frac{g}{h\gamma}} \cdot \frac{\sqrt{D_1 \operatorname{ctg}^4\alpha + 2D_3 \operatorname{ctg}^2\alpha + 9D_3}}{\cos^3\alpha}. \qquad (94.21)$$

In particular, for a plate in the form of an equilateral triangle with fixed sides Eq. (94.12) takes the form

$$p_{11} = \frac{63{,}28}{a^3}\sqrt{\frac{g}{h\gamma}}\sqrt{D_1 + 0{,}667D_3 + D_2}. \qquad (94.22)$$

The frequency of the first harmonic for such an isotropic plate is determined in a first approximation from the formula

$$p_{11} = \frac{103{,}15}{a^2}\sqrt{\frac{gD}{h\gamma}}. \qquad (94.23)$$

These results were obtained by R.V. Feodos'yev.*

§95. FORCED OSCILLATIONS OF A PLATE

The equation of forced oscillations, i.e., the equation of motion of a plate under the action of a variable load $q(x, y, t)$, is derived analogously as in the case of free vibrations, with

the only exception that the normal load q must be taken into account in the derivation (see Fig. 126).

For a nonorthotropic plate we obtain the equation

$$\frac{\partial^2 w}{\partial t^2} + \frac{g}{h\gamma} Lw = \frac{g}{h\gamma} q(x, y, t). \tag{95.1}$$

and for an orthotropic one we have

$$\frac{\partial^2 w}{\partial t^2} + \frac{g}{h\gamma}\left(D_1 \frac{\partial^4 w}{\partial x^4} + 2D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4}\right) = \frac{g}{h\gamma} q(x, y, t). \tag{95.2}$$

The problem is reduced to the determination of a solution to Eq. (95.1) or (95.2), which satisfies the boundary conditions and the initial conditions. An exact solution can be sought in the following way.

At first we solve the problem of free oscillations and determine the frequencies of the natural oscillations, p_{mn}, and the eigenfunctions W_{mn}. We then represent the load q, as a function of x and y, in the form of a series expanded with respect to the eigenfunctions,

$$q = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn}(t) W_{mn}(x, y) \tag{95.3}$$

and seek the solution in the form

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} T_{mn}(t) W_{mn}(x, y). \tag{95.4}$$

For the functions T_{mn} we obtain the equation

$$T''_{mn} + p^2_{mn} T_{mn} = \frac{g}{h\gamma} a_{mn}(t), \tag{95.5}$$

from which we arrive at

$$T_{mn} = A_{mn} \cos p_{mn}t + B_{mn} \sin p_{mn}t + \tau_{mn}(t), \tag{95.6}$$

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [A_{mn} \cos p_{mn}t + B_{mn} \sin p_{mn}t + \tau_{mn}(t)] W_{mn}. \tag{95.7}$$

τ_{mn} is here a particular solution to the nonhomogeneous equation (95.5); its form depends on a_{mn}, i.e., on the law of variation of the load with time. The constants A_{mn} and B_{mn} are determined from the initial conditions just as in the case of the free vibrations.

Let us give the solution for a particular case.

Consider the load acting on a rectangular supported plate given in the form of the function

$$q = q_0(x, y) \cos pt, \tag{95.8}$$

i.e., its distribution on the surface of the plate remains unchanged and the magnitude of it varies at each point according to the law of a simple harmonic oscillation with the frequency p. When we suppose that at the initial instant of time the plate is at rest, we obtain

$$w = \frac{g}{h\gamma} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{c_{mn}}{p_{mn}^2 - p^2} (\cos pt - \cos p_{mn} t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \quad (95.9)$$

where

$$c_{mn} = \frac{4}{ab} \int_0^a \int_0^b q_0 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy. \quad (95.10)$$

If the frequency of the load p coincides with any of the frequencies of the plate, the plate will oscillate in resonance. Thus, if for any m and n $p = p_{mn}$, the corresponding term of the series (95.9) w_{mn} takes the indefinite form 0/0; disclosing the indeterminacy we obtain

$$w_{mn} = \frac{g}{h\gamma} \cdot \frac{c_{mn}}{2p_{mn}} t \sin p_{mn} t \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \quad (95.11)$$

This motion appears as a free oscillation with an amplitude increasing unlimitedly with time (proportional to the time). But this result was obtained when the resistance was ignored completely, i.e., both the resistance of the surrounding medium and internal friction were neglected in spite of the cast that both factors influence essentially the process of oscillation; owing to them the amplitude of the oscillation remains finite.

In practice resonance in the fundamental natural frequency may be dangerous if $p = p_{11}$; the fundamental harmonic oscillations characterized by the antinode in the middle of the plate are amplified in the case of resonance and this may result in effects which can be dangerous for the strength of the plate.

Manuscript Page No.	[Footnotes]
383	More accurately, p is the circular frequency, a quantity connected with the oscillation period T by the relation $p = 2\pi/T$.
386	See, e.g., Timoshenko, S.P., Teoriya kolebaniy v inzhenernom dele [The Theory of Oscillations in Engineering] GTTI, 1931, §56, page 308.
388	In this connection see, e.g., the book by Timoshenko, S.P., mentioned in the preceding section and the book by S.A. Bernshteyn "Osnovy dinamiki sooruzheniy" [Fundamentals of the Dynamics of Buildings] Gosstroyizdat,

1938, Chapter 5, where various methods of determining the frequencies of beams are considered in detail and illustrated by many examples.

391 See, e.g., the book by S.P. Timoshenko, mentioned in §92, page 313. Equation (94.13) yields a first approximation; in the second approximation, for an isotropic plate, $p_{11} = \frac{10,21}{a^2}\sqrt{\frac{gD}{h\gamma}}$ (cf. this book).

393 See footnote at the end of §85.

Chapter 13

THE FUNDAMENTALS OF THE THEORY OF STABILITY OF PLATES

§96. GENERAL STATEMENT OF THE PROBLEM OF THE STABILITY OF PLATES

In the present chapter we shall formulate the problem of stability of plates and give the basic methods used to determine the critical loads, together with some general formulas which are in connection with the stability problem.

A plate which is deformed by forces acting in the median surface possesses a single state of equilibrium which is stable as long as the forces are small in magnitude. The single and stable form of equilibrium corresponding to sufficiently small loads is characterized by the fact that the median surface of the plate remains plane; in the following we shall call this form of equilibrium fundamental. When, without altering the law of force distribution on the edge of the plate, the forces are increased, the moment may be reached at which the fundamental form of equilibrium is no longer unique and stable and other forms become possible which are characterized by a curved median surface. The plane form becomes unstable and the plate deformed by the forces may, under the action of even an insignificant transverse force, pass over from the unstable form to a stable, with a curved median surface. In such cases the plate is said to lose stability. In most cases the loss of stability of a construction element in the form of a plate is an undesirable effect; it may disturb the correctness of constructional work and even destroy it. In strength calculations of thin-walled construction elements in the form of slabs it is therefore necessary to pay great attention to that part of the calculation which deals with the choice of the dimensions or the magnitude of the effective forces such that in the construction work no effects of stability loss may appear.

The problem of the stability of a plane plate in its general statement may be formulated in the following way.

Consider a plate of given form on which external forces are applied such that they act in the median surface. In the general case the load is supposed to consist of two components: a load whose magnitude and law of distribution on the edge remain unchanged and a load which is given to within the factor λ; this means that the law of force distribution on the edge is given, but the magnitude of the forces may vary between zero and arbitrarily high values. It is assumed that with $\lambda = 0$ the plate is in stable equilibrium (general state of plane stresses), but as λ increases a moment may be reached when the uniqueness of equilibrium becomes disturbed. It is required to determine those

values of $\lambda = \lambda_1, \lambda_2, \lambda_3, \ldots$ which describe the branching points of equilibrium, i.e., where besides the plane form other possible forms of equilibrium may exist (in the general case there is an infinite series of such values). The lowest value of λ with which the fundamental form of equilibrium ceases to be the only stable form is called the critical value and the load corresponding to it is the critical load.

Though an investigation of all possible forms of equilibrium of a plate is of great theoretical interest, in practice one restricts oneself usually to the determination of the critical load and the form of equilibrium of the plate which becomes possible besides the plane one, when the load is equal to the critical load. The obtaining of an exact solution of the problem of the various forms of equilibrium and the stability is connected with great mathematical difficulties; one has to solve nonlinear equations of the theory of elasticity taking high strains into account.

In practice we restrict ourselves therefore almost always to approximate solutions.

§97. THE BASIC METHODS OF DETERMINING THE CRITICAL LOAD*

Among the various approximation methods used in the determination of critical loads, which at present are at our disposal, we shall consider the following three: 1) the static method; 2) the energy method and 3) the dynamic method.

1. The static method. Considering the equilibrium of a plate loaded by forces acting in the median surface (part of these forces remains constant, the other part is given to within a factor λ) it is supposed that with a certain value of λ a small curvature of the median surface becomes possible. A differential equation is set up to describe the curved surface taking the longitudinal forces into account which are caused by the external load. Some of the coefficients of this equation will contain the factor λ as a parameter. We then seek a solution to the equation obtained which satisfies all boundary conditions (depending on the way of fixing of the edge of the plate), which is not identically equal to zero. Such a solution exists not with all values of λ, but only with certain definite values $\lambda_1, \lambda_2, \lambda_3 \ldots$ (the characteristic numbers); the lowest nonvanishing value of them is the critical, denoted λ_{kr}.

The equation of the curved surface is obtained from Eq. (64.6) or (64.7) where we must set $q = 0$ and substitute the values of the longitudinal forces T_x, T_y, S_{xy}. In order to determine the longitudinal forces we must solve the plane problem; we arrive at the result:

$$T_x = T_1 + \lambda T_1', \quad T_y = T_2 + \lambda T_2', \quad S_{xy} = S + \lambda S' \qquad (97.1)$$

and the equation of the curved surface of an orthotropic plate has the form

$$D_1 \frac{\partial^4 w}{\partial x^4} + 2D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} - (T_1 + \lambda T_1') \frac{\partial^2 w}{\partial x^2} - 2(S + \lambda S') \frac{\partial^2 w}{\partial x \partial y} - (T_2 + \lambda T_2') \frac{\partial^2 w}{\partial y^2} = 0. \quad (97.2)$$

According to the form of the plate's contour and the fixing of its edges we seek a solution to the deflection equation in such a form which makes it possible to satisfy the boundary conditions or even satisfies them. The solution, a function $w(x, y)$, will contain arbitrary constants. Satisfying the boundary conditions, one obtains a system of homogeneous equations for the constants which has a nonzero solution only if its determinant is vanishing. When we set the determinant of the system equal to zero we obtain an equation for λ. From the infinite series of solutions (the equation is usually obtained in a transcendental form) we must take the lowest nonzero solution. When the critical value λ_{kr} has been obtained the expression for the deflection can be determined with an accuracy to within an arbitrary constant factor.

An arbitrary constant factor indicates that the state of equilibrium of the plate will be indifferent, which is also the case when the load is precisely equal to the critical. In particular, when the factor is equal to zero, the form of equilibrium will be plane.

2. The energy method. This method is based on the general theorems of equilibrium of a mechanical system.

As we know from mechanics, the potential energy of a system in the state of equilibrium has an extremum. The equilibrium will be stable when the potential energy in the position of equilibrium has its minimum value (relative to the values corresponding to arbitrary possible small deviations from the position of equilibrium) and unstable when the energy has its maximum value; it is indefinite when in the position of equilibrium the energy has neither a maximum nor a minimum. Applying this criterion to a plate one proceeds as follows.

We consider two states of the plate: the state of equilibrium under the action of a given load with which the median surface remains plane, and the neighboring state, in which the median surface is slightly curved as the result of small possible displacements imparted to the plate. Let V_0 be the potential energy in the equilibrium position and V the potential energy in the neighboring state. The equilibrium will be stable, when for all possible small deviations (i.e., deflections w) $V_0 < V$, unstable with $V_0 > V$ and indefinite with $V_0 = V$. If small possible deflections are imparted to the plate the potential energy will grow at the expense of the bending energy V_{izg} and decrease owing to the work A done by the external forces on the bending of the median surface. For the indefinite equilibrium

$$V_0 = V_0 + V_{izg} - A, \quad (97.3)$$

or

$$V_{izg} = A(\lambda). \quad (97.4)$$

Thus, for the determination of the critical value of λ we must set equal to one another the potential energy of bending corresponding to small curvatures of the median surface, and the work of the external forces.

Applying the energy method in practice one usually proceeds in the following way. According to the form of the plate and the fixing of its edge we choose a suitable expression for the deflection in the form of a sum with indefinite coefficients

$$w = \sum_m \sum_n A_{mn} w_{mn}, \qquad (97.5)$$

where w_{mn} are continuous functions satisfying all boundary conditions. Substituting this expression in Eq. (97.4) we obtain a fraction for λ, whose numerator and denominator are both functions of the coefficients A_{mn}

$$\lambda = \frac{M(A_{mn})}{N(A_{mn})}. \qquad (97.6)$$

When in the expression for w only a single term of the sum is retained, in Eq. (97.6) the coefficient can be reduced and an approximate value is obtained for λ which depends on m and n and corresponds to the indefinite equilibrium of the plate. When Expression (97.5) is taken in the form of a sum of two, three or more terms, the coefficient in Eq. (97.6) cannot be reduced. When we want to obtain the smallest of all possible values of λ we must find the minimum of the fraction (97.6), i.e., the derivatives of λ with respect to all A_{mn} are set equal to zero. But as

$$\frac{\partial\lambda}{\partial A_{mn}} = \frac{1}{N}\left(\frac{\partial M}{\partial A_{mn}} - \frac{M}{N}\cdot\frac{\partial N}{\partial A_{mn}}\right) = \frac{1}{N}\cdot\frac{\partial}{\partial A_{mn}}(M - \lambda N), \qquad (97.7)$$

the problem of determining the minimum of the fraction (97.6) is equivalent to the problem of determining the minimum of the expression

$$U = M - \lambda N. \qquad (97.8)$$

As a result we obtain a system of homogeneous equations with respect to the coefficients A_{mn} and setting its determinant equal to zero we obtain an equation for λ; the smallest nonvanishing solution will be λ_{kr}.

3. The dynamic method. When a plate in the state of equilibrium under the action of external forces (where the median surface is a plane) is led out of the state of equilibrium by imparting its particles small initial displacements and velocities in a transverse direction, the plate will move and the nature of the motion will depend on the kind of equilibrium, stable, unstable or indifferent. A plate which, by an initial perturbation, is brought out of its position of stable equilibrium will tend to return to this position, i.e., it recovers; if resistance is ignored, a perfectly elastic plate will perform undamped oscillations about this position of equilibrium. If, however, the equilibrium was unstable, the plate will not tend to return to the

position of equilibrium but will remove from it and its curvature increases.

Using the dynamic method we assume that the plate, which is in a state of equilibrium, receives an initial arbitrary deviation and an initial velocity in a transverse direction. We set up a differential equation of motion, i.e., an equation of the transverse oscillations taking the longitudinal forces into account [see Eq. (91.13)]; this equation will contain λ as a parameter. We then determine the frequencies of the natural oscillations of the plate, p_{mn}, which will depend on the dimensions and elastic constants of the plate and on λ. Considering the expressions for the frequencies (or the equation from which they were derived) we notice the following. As long as λ is small all frequencies obtained are real and the deflection of the plate is the result of superpositions of the deflections w_{mn} in the form

$$w_{mn} = (A_{mn} \cos p_{mn}t + B_{mn} \sin p_{mn}t) W_{mn}. \tag{97.9}$$

This means that the plate will oscillate about a position of equilibrium and equilibrium will be stable. As λ increases, for some frequencies zero or imaginary values are obtained: $p_{mn} = 0$ or $p_{mn} = ip'_{mn}$; the corresponding expressions of the type of (97.9) which constitute the deflection must be replaced by terms which grow unlimitedly with time:

$$w_{mn} = (A_{mn} + B_{mn}t) W_{mn} \tag{97.10}$$

or

$$w_{mn} = (A_{mn}e^{-p'_{mn}t} + B_{mn}e^{p'_{mn}t}) W_{mn}. \tag{97.11}$$

Owing to the presence of terms of this type the deflection will tend to grow unlimitedly with time and, consequently, with such values of λ the plane form of the plate will be unstable (or indifferent).

The smallest nonvanishing value of λ corresponding to the transition from the undamped oscillations (97.7) to a motion, which is characterized by deviations from the plane form which grow unlimitedly, is the critical value.

In the following we shall only use the static method and the energy method.

§98. THE WORK OF THE EXTERNAL FORCES

When solving the problems of the stability of plates by means of the energy method, we encountered an expression for the work A of the external forces, which results in small deviations from the plane form. Let us derive this expression.

The derivation is based on the assumption that the median surface of the plate is bent without suffering tensions or compressions; owing to this, bending is accompanied by a mutual approach of the plate's edges and the longitudinal forces perform work.

Let us consider elements of the median surface perpendicular to the axes x and y which are subject to the action of the compressive forces p_x, p_y and the tangential forces t (per unit length)

$$T_x = -p_x, \quad T_y = -p_y, \quad S_{xy} = -t. \tag{98.1}$$

We cut out of the plate a strip parallel to the x-axis of the width dy, and from it an element of the length dx. The approach of the ends of this element (Fig. 189) is equal to

$$d\lambda = dx - dx\cos\alpha = \frac{1}{2}dx\sin^2\alpha = \frac{1}{2}dx\left(\frac{\partial w}{\partial x}\right)^2. \tag{98.2}$$

The work of the forces is equal to

$$dA_1 = p_x dy \cdot \frac{1}{2}dx\left(\frac{\partial w}{\partial x}\right)^2. \tag{98.3}$$

The work A_1 of the forces for the whole plate is obtained by integrating Eq. (98.3) over the area of the plate:

$$A_1 = \frac{1}{2}\int\int p_x\left(\frac{\partial w}{\partial x}\right)^2 dx\,dy. \tag{98.4}$$

Analogously, we have for the work of the forces p_y

$$A_2 = \frac{1}{2}\int\int p_y\left(\frac{\partial w}{\partial y}\right)^2 dx\,dy. \tag{98.5}$$

In order to calculate the work of the tangential forces we consider the same element $dxdy$ shown in Fig. 189. Under the action of the tangential forces the element is distorted and its projection on the xy-plane has the form of a parallelogram with

$$d\delta = \gamma_{xy}\,dy. \tag{98.6}$$

The displacement γ_{xy} is determined from the sixth formula of (5.1) assuming in it $\sin\gamma_{xy} = \gamma_{xy}$ and $u = v = 0$. Retaining only the terms which are small in second order we have

$$\gamma_{xy} = \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}. \tag{98.7}$$

Consequently,

$$d\delta = \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}dy \tag{98.8}$$

and the work of the tangential forces producing the displacement $d\delta$ is equal to

$$dA_3 = t\,dx\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}dy. \tag{98.9}$$

The work of the tangential forces for the whole plate is

$$A_3 = \int\int t\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}dx\,dy. \tag{98.10}$$

Adding the expressions for A_1, A_2 and A_3 we obtain the work

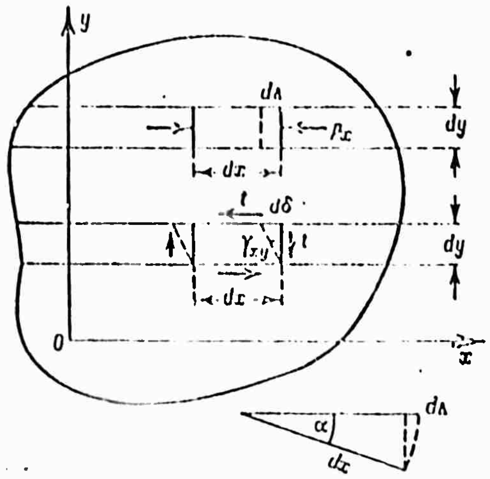

Fig. 189

of the longitudinal forces in the case of small curvature of the plate:

$$A = \frac{1}{2}\int\int\left[p_x\left(\frac{\partial w}{\partial x}\right)^2 + 2t\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y} + p_y\left(\frac{\partial w}{\partial y}\right)^2\right]dx\,dy. \qquad (98.11)$$

Another (stricter) derivation of a formula for the work A may be found in S.P. Timoshenko's book.*

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398	See, e.g., the book by S.P. Timoshenko: 1) Ustoychivost' uprugikh sistem [Stability of Elastic Systems] Gostekhizdat, 1946, Chapter 7, Section 62; 2) Plastinki i obolochki [Plates and Shells] Gostekhizdat, Moscow, 1948, Chapter 8, Section 62.
403	See his book "Ustoychivost' uprugikh sistem" [Stability of Elastic Systems] Gostekhizdat, 1946, Section 58, pages 280-282.

[Transliterated Symbols]

399	изг = izg = izgib = bending
400	кр = kr = kriticheskiy = critical

Chapter 14

THE STABILITY OF PLATES DEFORMED BY A DISTRIBUTED LOAD

§99. THE STABILITY OF A RECTANGULAR ORTHOTROPIC PLATE WITH FOUR SUPPORTED SIDES, WHICH IS COMPRESSED IN THE PRINCIPAL DIRECTION

Among the various problems of stability of anisotropic plates which are interesting for practice, there are relatively few which have been studied in detail and for which numerical results are available. In this connection we must, first of all, mention the stability problems of a rectangular orthotropic plate and a plate in the form of an infinitely long strip, which are deformed by forces distributed along the edges according to a simple law. We only know the solutions for the cases of normal load distributed uniformly and following a linear law, of a uniformly distributed tangential load and of the case of simultaneous action of normal and tangential loads. These solutions will be considered in the present chapter.

Let us begin with the consideration of stability of a rectangular supported plate, which is compressed in a principal direction.

We have a rectangular orthotropic plate, all sides of which are resting on a support; along two sides normal compressive forces are distributed uniformly. The principal directions in the plate are assumed parallel to its edges. We have to determine the critical value of the forces p_{kr}, with which the plane form of equilibrium ceases to be a unique and stable form (the plate loses equilibrium).

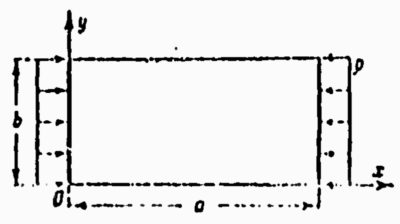

Fig. 190

Let us denote by a and b the length of the sides of the plate, $c = a/b$ is the ratio of the sides and p is the magnitude of the force per unit length; the coordinate frame is allowed to coincide with the sides of the plate (Fig. 190).

Before stability is lost, the state of stress of the plate is plane, with $T_x = -p$, $T_y = S_{xy} = 0$.

The problem is easily solved with the help of any of the three basic methods described in §97. We prefer the static method.*

The equation of the bent surface has the form

$$D_1 \frac{\partial^4 w}{\partial x^4} + 2D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} + p \frac{\partial^2 w}{\partial x^2} = 0. \tag{99.1}$$

We have to find a solution to this equation which is nonzero and satisfies the boundary conditions

$$\text{with } x = 0 \text{ и } x = a \quad w = 0, \ \frac{\partial^2 w}{\partial x^2} + \nu_2 \frac{\partial^2 w}{\partial y^2} = 0; \tag{99.2}$$

$$\text{with } y = 0 \text{ и } y = b \quad w = 0, \ \frac{\partial^2 w}{\partial y^2} + \nu_1 \frac{\partial^2 w}{\partial x^2} = 0. \tag{99.3}$$

These boundary conditions are satisfied by any of the expressions

$$w = A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \tag{99.4}$$

where A_{mn} is a constant coefficient and m and n are integral numbers. Substituting (99.4) in Eq. (99.1) we obtain

$$A_{mn} \left\{ \pi^4 \left[D_1 \left(\frac{m}{a}\right)^4 + 2D_3 \left(\frac{mn}{ab}\right)^2 + D_2 \left(\frac{n}{b}\right)^4 \right] - p\pi^2 \left(\frac{m}{a}\right)^2 \right\} = 0. \tag{99.5}$$

As we are interested in a nonzero solution the expression in the braces must be set equal to zero. From this we obtain

$$p = \frac{\pi^2 \sqrt{D_1 D_2}}{b^2} \left[\sqrt{\frac{D_1}{D_2}} \left(\frac{m}{c}\right)^2 + \frac{2D_3}{\sqrt{D_1 D_2}} n^2 + \sqrt{\frac{D_2}{D_1}} \left(\frac{c}{m}\right)^2 n^4 \right]. \tag{99.6}$$

The constant A_{mn} remains indefinite. Equation (99.6) yields all values of p corresponding to the values of $m = 1, 2, 3, \ldots$, $n = 1, 2, 3, \ldots$, with which a curvature of the form (99.4) becomes possible. Among the set of p-values we must choose the smallest one; it will also be the critical. It is obvious that the smallest value of p is obtained with $n = 1$ corresponding to a curvature in the direction of the side b, in the form of a sinusoidal semiwave. We then have to determine for which m the expression for p, which corresponds to the given side ratio c, has the smallest value, and we have to determine this smallest value.

With $n = 1$ the equation for p assumes the form

$$p = \frac{\pi^2 \sqrt{D_1 D_2}}{b^2} \left[\sqrt{\frac{D_1}{D_2}} \left(\frac{m}{c}\right)^2 + \frac{2D_3}{\sqrt{D_1 D_2}} + \sqrt{\frac{D_2}{D_1}} \left(\frac{c}{m}\right)^2 \right]. \tag{99.7}$$

We shall here not enter into details of elementary investigations into Eq. (99.7) and only give the fundamental results.

1) When the side ratio c satisfies the condition

$$c = m' \sqrt[4]{\frac{D_1}{D_2}}, \qquad (99.8)$$

where m' is an integral number, we must set $m = m'$ in Eq. (99.7) and we obtain the following formula* for the critical load

$$p_{\text{кр}} = \frac{\pi^2 \sqrt{D_1 D_2}}{b^2} \cdot 2\left(1 + \frac{D_3}{\sqrt{D_1 D_2}}\right). \qquad (99.9)$$

This value will be the smallest of all values determined from Eq. (99.7).

2) When

$$c = c_{\text{пр}} = \sqrt{m(m+1)} \sqrt[4]{\frac{D_1}{D_2}}, \qquad (99.10)$$

where m is an arbitrary integer, two forms of equilibrium are possible with one and the same critical load: with m semiwaves in the direction of side a:

$$w = A_{m1} \sin \frac{m\pi x}{a} \sin \frac{\pi y}{b}; \qquad (99.11)$$

and with the m + 1st semiwave

$$w = A_{m+1,1} \sin \frac{(m+1)\pi x}{a} \sin \frac{\pi y}{b}. \qquad (99.12)$$

On the basis of Eq. (99.10) it is easy to establish the number of semiwaves m corresponding to a given ratio c. We have

$$\left.\begin{array}{l} \text{if } 0 < c < 1{,}41 \sqrt[4]{\frac{D_1}{D_2}}, \text{ then } m = 1; \\ \text{if } 1{,}41 \sqrt[4]{\frac{D_1}{D_2}} < c < 2{,}45 \sqrt[4]{\frac{D_1}{D_2}}, \text{ then } m = 2; \\ \text{if } 2{,}45 \sqrt[4]{\frac{D_1}{D_2}} < c < 3{,}46 \sqrt[4]{\frac{D_1}{D_2}}, \text{ then } m = 3 \end{array}\right\} \qquad (99.13)$$

etc.

3) With an arbitrary value given for the side ratio c the critical load is determined in the following way: the number m is established which corresponds to the given value of c [on the basis of the inequalities described above, or Eq. (99.10)]; the value obtained for m is substituted in Eq. (99.7) which also yields the value for the critical load. With high side ratios $c > 3$ the critical load is determined from Eq. (99.9). The formula for the critical load can be represented in the form

$$p_{\text{кр}} = \frac{\pi^2 \sqrt{D_1 D_2}}{b^2} k, \qquad (99.14)$$

where k is a coefficient which depends on the ratio c/m and the rigidity ratio.

Dividing the quantity of the critical load by the thickness of the plate we obtain the load per unit area or the critical

stress.

In the particular case where the plate consists of an isotropic material, $D_1 = D_2 = D_3 = D$ and we obtain from (99.7) the well-known formula*

$$p_{\kappa p} = \frac{\pi^2 D}{b^2}\left(\frac{m}{c} + \frac{c}{m}\right)^2. \qquad (99.15)$$

The limiting ratios c_{pr}, with which the transition is carried out from m semiwaves in the direction of the x-axis to the $m + 1$ semiwave, are equal to

$$c_{np} = \sqrt{m(m+1)}. \qquad (99.16)$$

When c is an integral number, $m = c$ and

$$p_{\kappa p} = \frac{\pi^2 D}{b^2} \cdot 4. \qquad (99.17)$$

Let us give the results of calculations and graphic representations for a veneer plate, using the numerical data of §67.

Let us consider a rectangular plate cut out of a veneer sheet in such a way that the principal directions of elasticity in it are parallel to the sides. When the plate is compressed in the direction of the fibers ($D_1 > D_2$), the limiting side ratios corresponding to the transition from m semiwaves in the direction of the compressive forces to the m + 1st semiwave, are equal to

$$c_{np} = \sqrt{m(m+1)} \cdot 1{,}86. \qquad (99.18)$$

Hence we obtain

$$\left.\begin{array}{l} \text{if } \quad 0 < c < 2{,}63, \text{ then } m = 1; \\ \text{if } 2{,}63 < c < 4{,}56, \text{ then } m = 2; \\ \text{if } 4{,}56 < c < 6{,}45, \text{ then } m = 3 \end{array}\right\} \qquad (99.19)$$

etc. The minimum value of the coefficient k is equal to

$$k = 2{,}76 \qquad (99.20)$$

and is obtained with $c = 1.86\ m'$ where m' is an integer.

In the case of compression across the fibers of the sheet ($D_1 < D_2$) the limiting ratios are obtained from the formula

$$c_{np} = \sqrt{m(m+1)} \cdot 0{,}54. \qquad (99.21)$$

Hence we obtain

$$\left.\begin{array}{l} \text{if } \quad 0 < c < 0{,}76, \text{ then } m = 1; \\ \text{if } 0{,}76 < c < 1{,}31, \text{ then } m = 2; \\ \text{if } 1{,}31 < c < 1{,}86, \text{ then } m = 3; \\ \text{if } 1{,}86 < c < 2{,}41 \text{ then } m = 4 \end{array}\right\} \qquad (99.22)$$

etc. The minimum value of k is also in this case equal to 2.76 but

is obtained with $c = 0.54\, m'$ ($m' = 1, 2, 3, \ldots$).

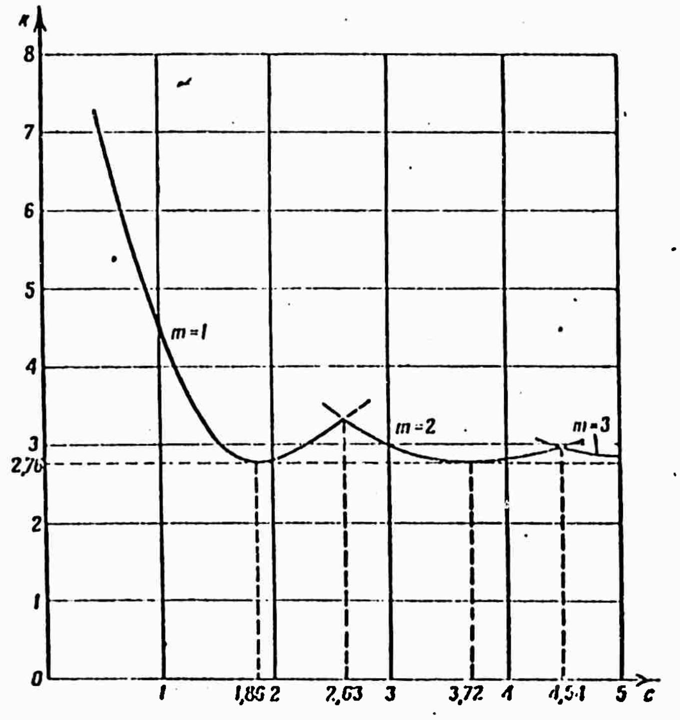

Fig. 191

The values of the coefficient k for certain ratios c are given in Table 20 where also the corresponding number m of the semiwave is given.

Figure 191 shows a graph of k as a function of the side ratio c for a veneer plate compressed along the fibers of the sheet, Fig. 192 shows the same for a plate compressed across the fibers.

TABLE 20

Values of the Coefficient k for a Veneer Plate Compressed in the Principal Direction

1 Сжатие вдоль волокон рубашки ($D_1 > D_2$)

c	0,5	1	1,86	2	2,63	3	∞
k	11,75	4,53	2,76	2,79	3,27	2,96	2,76
m	1	1	1	1	1—2	2	—

2 Сжатие поперек волокон рубашки ($D_1 < D_2$)

c	0,5	0,51	0,76	1	1,31	1,62	1,86	2	∞
k	2,79	2,76	3,27	2,79	2,93	2,76	2,85	2,79	276
m	1	1	1—2	2	2—3	3	3—4	4	—

1) Compression along the fibers of the sheet;
2) compression across the fibers of the sheet.

Each graph consists of sections of the curves $k = f(c, m)$ corresponding to different values of the integer m. When we want to determine k for a given side ratio from the graph we must, from the point on the abscissa corresponding to the given c, drop a normal to the next curve of the set.

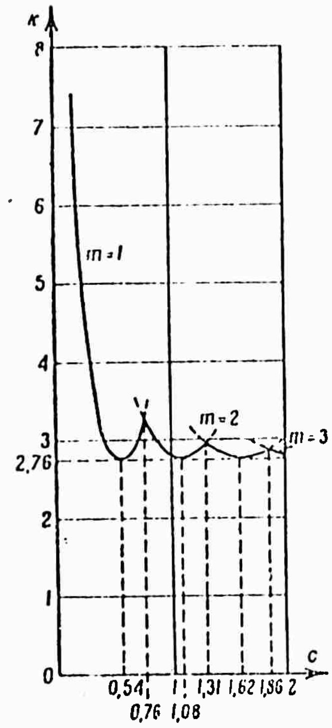

Fig. 192

The value of the coefficient k (in the given scale) is equal to the distance from the abscissa to the next curve, and this curve will show which m corresponds to the critical load. For example, we see from Fig. 191 that the normal dropped from the point $c = 2.63$ on the abscissa, passes through the point of intersection of the curves with the parameters $m = 1$ and $m = 2$; consequently, with $c = 2.63$ two values of m are possible at the same time: $m = 1$ and $m = 2$, corresponding to $k = 3.27$. When we take a ratio of, say, $c = 3$, the next curve will be that with the parameter $m =$ $= 2$. This indicates that a plate with this side ratio, having lost its stability, produces two semiwaves in the direction of the compressive forces. Considering the curve makes us understand the nonuniformity of the variation of k with c.

Table 20 and the graphs show that the number of semiwaves, into which the plate is divided, which is compressed across the fibers, is, as a rule, considerably higher than in the case of compression along the fibers (with the same dimensions). Let us consider, for example, two plates with a side ratio of $c = 2$; the sheet fibers in the one are assumed parallel to the long sides and those in the other parallel to the short sides. In a compression along the long sides the first plate, having lost the stability, will form one semiwave in the direction of the compressive forces, while in the second there will arise four semiwaves.

When for a birch veneer we take the other values for the reduced moduli and coefficients also given in §67 [see Eq. (67.16)], the limiting ratios c_{pr} and the ratios for which k is a minimum remain unchanged while the values of the coefficients k change toward higher values. In particular, the lowest value obtained for k is 3.43. This increase is chiefly due to the higher value of the modulus of shear, G', entering Eqs. (99.7) and (99.9) (through D_3) which is now taken to be not $0.07 \cdot 10^5$ kg/cm^2 but $0.12 \cdot 10^5$ kg/cm^2, which is considerably higher.

§100. STABILITY OF A RECTANGULAR PLATE WITH TWO SUPPORTED SIDES COMPRESSED IN THE PRINCIPAL DIRECTION

When a plate, as shown in Fig. 190, is supported on its loaded sides $x = 0$ and $x = a$, while the sides $y = 0$ and $y = b$ are fixed arbitrarily or are free, the solution of Eq. (99.1) must be sought in the form

$$w = f(y)\sin\frac{m\pi x}{a}. \qquad (100.1)$$

This expression satisfies the conditions on the supported sides. Substituting Eq. (100.1) in (99.1) we obtain the equation used to determine $f(y)$:

$$D_2 f^{IV} - 2\left(\frac{m\pi}{a}\right)^2 D_3 f'' + \left[D_1\left(\frac{m\pi}{a}\right)^4 - p\left(\frac{m\pi}{a}\right)^2\right] f = 0. \qquad (100.2)$$

Let us denote the roots of the characteristic equation

$$D_2 s^4 - 2\left(\frac{m\pi}{a}\right)^2 D_3 s^2 + D_1\left(\frac{m\pi}{a}\right)^4 - p\left(\frac{m\pi}{a}\right)^2 = 0 \qquad (100.3)$$

by $\pm k_1$, $\pm i k_2$, respectively, where

$$\left.\begin{aligned} k_1 &= \sqrt{\frac{m\pi}{a}\sqrt{\left(\frac{D_3}{D_2}\right)^2\left(\frac{m\pi}{a}\right)^2 - \frac{D_1}{D_2}\left(\frac{m\pi}{a}\right)^2 + \frac{p}{D_2}} + \frac{D_3}{D_2}\left(\frac{m\pi}{a}\right)^2},\\ k_2 &= \sqrt{\frac{m\pi}{a}\sqrt{\left(\frac{D_3}{D_2}\right)^2\left(\frac{m\pi}{a}\right)^2 - \frac{D_1}{D_2}\left(\frac{m\pi}{a}\right)^2 + \frac{p}{D_2}} - \frac{D_3}{D_2}\left(\frac{m\pi}{a}\right)^2}. \end{aligned}\right\} \qquad (100.4)$$

The expression for the deflection assumes the form

$$w = (A\,\mathrm{ch}\,k_1 y + B\,\mathrm{sh}\,k_1 y + C\cos k_2 y + D\sin k_2 y)\sin\frac{m\pi x}{a}. \qquad (100.5)$$

The constants A, B, C, D are determined from the conditions on the sides a. For each of the sides we have two conditions, altogether four; the unknown coefficients are equally many. We can therefore satisfy the conditions on the sides a for any loading mode. Satisfying them we obtain a homogeneous system of four equations for A, B, C, D; we set equal to zero the determinant of this system and so obtain an equation for p. Among all the solutions of the latter we must select the lowest nonzero solution which yields the critical load p_{kr}.

For example, with a plate where the sides $y = 0$ and $y = b$ are fixed, the critical load is determined from the equation*

$$\mathrm{th}\,k_1 b\ \mathrm{tg}\,k_2 b = \frac{2k_1 k_2}{k_1^2 - k_2^2}\left(1 - \frac{1}{\mathrm{ch}\,k_1 b\cos k_2 b}\right) \qquad (100.6)$$

(which, of course, cannot be solved with respect to p).

For long plates with a side ratio of $c = a/b > 4$ the critical load may be considered to be independent of c.

In particular, for a given isotropic plate**

$$p_{\kappa p} = \frac{\pi^2 D}{b^2}\cdot 7. \qquad (100.7)$$

An exact determination of the critical load for a plate with two supported sides is connected with the solution of a complicated transcendental equation.*** Approximate formulas for the determination of the critical load can be obtained with the help of the energy method. Assuming that the plate (compressed in the principal direction), having lost its stability, takes a sinusoid-

al curvature in the direction of the compressive load, we shall seek the expression for the deflection in the form of (100.1).

The potential energy for the curved plate has become higher at the expense of the bending energy, which is determined by Eq. (61.22) and in the given case is equal to

$$V_{\text{изг}} = \frac{a\sqrt{D_1 D_2}}{4}\int_0^b \Big[\sqrt{\frac{D_2}{D_1}}f''^2 - 2\left(\frac{m\pi}{a}\right)^2 \nu_2 \sqrt{\frac{D_1}{D_2}}ff'' + \left(\frac{m\pi}{a}\right)^4 \sqrt{\frac{D_1}{D_2}}f^2 + \frac{4D_k}{\sqrt{D_1 D_2}}\left(\frac{m\pi}{a}\right)^2 f'^2\Big]\,dy. \tag{100.8}$$

When the plate is bent by a compressive force, this force performs work which is equal to*

$$A = \frac{pa}{4}\left(\frac{m\pi}{a}\right)^2 \int_0^b f^2\,dy \tag{100.9}$$

Equating Eqs. (100.8) and (100.9) we obtain

$$p = \frac{\sqrt{D_1 D_2}}{\left(\frac{m\pi}{a}\right)^2 \int_0^b f^2\,dy}\int_0^b \Big\{\sqrt{\frac{D_2}{D_1}}f''^2 - 2\left(\frac{m\pi}{a}\right)^2 \nu_2 \sqrt{\frac{D_1}{D_2}}ff'' + \left(\frac{m\pi}{a}\right)^4 \sqrt{\frac{D_1}{D_2}}f^2 + \frac{4D_k}{\sqrt{D_1 D_2}}\left(\frac{m\pi}{a}\right)^2 f'^2\Big\}\,dy. \tag{100.10}$$

We then choose an expression for the function f satisfying the conditions on the sides a and yielding a smooth surface in the form of a sum with indefinite coefficients

$$f = \sum_n A_n f_n(y). \tag{100.11}$$

Substituting this expression in Eq. (100.10) we determine the minimum of p which, as shown in §97, is equivalent to the minimum obtained for the expression

$$U = \int_0^b \Big[\sqrt{\frac{D_2}{D_1}}f''^2 - 2\left(\frac{m\pi}{a}\right)^2 \nu_2 \sqrt{\frac{D_1}{D_2}}ff'' + \left(\frac{m\pi}{a}\right)^4 \sqrt{\frac{D_1}{D_2}}f^2 + \frac{4D_k}{\sqrt{D_1 D_2}}\left(\frac{m\pi}{a}\right)^2 f'^2 - \frac{p}{\sqrt{D_1 D_2}}\left(\frac{m\pi}{a}\right)^2 f^2\Big]\,dy. \tag{100.12}$$

For a plate with fixed sides $y = 0$ and $y = b$ we can obtain

$$f = \sum_n A_n\left(1 - \cos\frac{2n\pi y}{b}\right). \tag{100.13}$$

Let us consider the first approximation in greater detail. In the first approximation, for a plate with two sides supported and two sides fixed, we have

$$f = A_n\left(1 - \cos\frac{2n\pi y}{b}\right). \tag{100.14}$$

where n is an integer to be determined in the following. Substituting f in Eq. (100.10) we obtain

$$p=\frac{\pi^2\sqrt{D_1D_2}}{b^2}\left[\sqrt{\frac{D_1}{D_2}}\left(\frac{m}{c}\right)^2+\frac{8}{3}n^2\frac{D_3}{\sqrt{D_1D_2}}+\frac{16}{3}n^4\sqrt{\frac{D_2}{D_1}}\left(\frac{c}{m}\right)^2\right]. \quad (100.15)$$

It is evident that p will have its smallest value with $n = 1$ and the critical load can therefore be determined by means of the formula

$$p=\frac{\pi^2\sqrt{D_1D_2}}{b^2}\left[\sqrt{\frac{D_1}{D_2}}\left(\frac{m}{c}\right)^2+2{,}67\frac{D_3}{\sqrt{D_1D_2}}+5{,}33\sqrt{\frac{D_2}{D_1}}\left(\frac{c}{m}\right)^2\right]. \quad (100.16)$$

We then have to determine the number m of the semiwaves in the direction of the compressive forces, which are formed by the plate after having lost its stability.

The results of an investigation of Eqs. (100.16) carried out in the same way as in the case of a plate with four supported sides results in the following:

1) If the side ratio satisfies the condition

$$c=0{,}658m'\sqrt[4]{\frac{D_1}{D_2}}, \quad (100.17)$$

where m' is an integer, then $m = m'$ and

$$p_{\text{кр}}=\frac{\pi^2\sqrt{D_1D_2}}{b^2}\cdot 2{,}67\left(1{,}73+\frac{D_3}{\sqrt{D_1D_2}}\right). \quad (100.18)$$

This value will be the lowest of all values determined according to Eq. (100.16).

2) If

$$c=c_{\text{up}}=0{,}658\sqrt{m(m+1)}\sqrt[4]{\frac{D_1}{D_2}}, \quad (100.19)$$

where m is an integer, with one and the same critical load, in the direction of the x-axis there may exist forms of equilibrium with m semiwaves and with $m + 1$ semiwaves:

$$w=A_1\left(1-\cos\frac{2\pi y}{b}\right)\sin\frac{m\pi x}{a}, \quad (100.20)$$

$$w=A_1\left(1-\cos\frac{2\pi y}{b}\right)\sin\frac{(m+1)\pi x}{a}. \quad (100.21)$$

For a ratio c which does not satisfy Conditions (100.17) or (100.19) the number of semiwaves is determined on the basis of the following inequalities:

$$\left.\begin{array}{l}\text{if } 0 < c < 0{,}931\sqrt[4]{\dfrac{D_1}{D_2}}, \text{ then } m=1;\\ \text{if } 0{,}931\sqrt[4]{\dfrac{D_1}{D_2}} < c < 1{,}61\sqrt[4]{\dfrac{D_1}{D_2}}, \text{ then } m=2;\\ \text{if } 1{,}61\sqrt[4]{\dfrac{D_1}{D_2}} < c < 2{,}28\sqrt[4]{\dfrac{D_1}{D_2}}, \text{ then } m=3\end{array}\right\} \quad (100.22)$$

3) In order to determine the critical load with a given side ratio we determine the number m on the basis of Eq. (100.22), i.e., we determine the line in which the given ratio is mentioned and, having obtained m, we determine p_{kr} from Eq. (100.16). With high side ratios the critical load can be determined from Eq. (100.18).

For a long isotropic plate we obtain in a first approximation

$$p_{\text{кр}} = \frac{\pi^2 D}{b^2} \cdot 7{,}29, \quad (100.23)$$

whereas the coefficient of the more exact formula (100.7) is equal to 7. The error of the first approximation amounts to about 4%.

For a given veneer plate we obtain in a first approximation

$$p_{\text{кр}} = \frac{\pi^2 \sqrt{D_1 D_2}}{b^2} \cdot 5{,}63. \quad (100.24)$$

§101. THE STABILITY OF A RECTANGULAR PLATE COMPRESSED IN TWO DIRECTIONS

A rectangular orthotropic plate whose principal directions are parallel to the sides is compressed by the load p_x distributed uniformly on two sides and the load p_y distributed uniformly on the other two sides (Fig. 193). The problem of the stability of such a plate is solved for the case of four supported sides.*

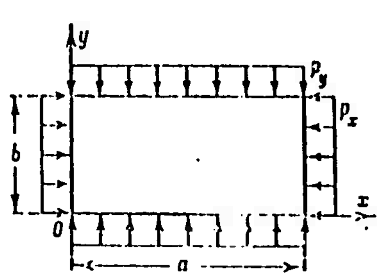

Fig. 193

The deflection equation will have the form

$$D_1 \frac{\partial^4 w}{\partial x^4} + 2D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} + \\ + p_x \frac{\partial^2 w}{\partial x^2} + p_y \frac{\partial^2 w}{\partial y^2} = 0. \quad (101.1)$$

The solution is sought in the form of

$$w = A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \quad (101.2)$$

Requiring that Eq. (101.2) be a solution to Eq. (101.1) we obtain the dependence

$$p_x \left(\frac{m}{a}\right)^2 + p_y \left(\frac{n}{b}\right)^2 = \pi^2 \left[D_1 \left(\frac{m}{a}\right)^4 + 2D_3 \left(\frac{mn}{ab}\right)^2 + D_2 \left(\frac{n}{b}\right)^4\right]. \quad (101.3)$$

For lthe definiteness of the problem it is also necessary to give additional conditions for the forces p_x and p_y which determine the interrelation of these forces. This interrelation may, of course, be of various types; one of the forces may be constant and the other variable, or the ratio of the forces may remain constant, etc. If one of the forces is tensile, it must be provided in Eq. (101.3) with a minus sign.

Let us consider several particular cases of force distribution.

1) The forces p_x and p_y are variable but the ratio of their magnitudes remains constant

$$p_x = \lambda, \quad p_y = \lambda\alpha.$$

The critical value λ is determined on the basis of the formula

$$\lambda = \frac{\pi^2 \sqrt{D_1 D_2}}{b^2} \cdot \frac{\sqrt{\frac{D_1}{D_2}} \left(\frac{m}{c}\right)^2 + \frac{2D_3}{\sqrt{D_1 D_2}} n^2 + \sqrt{\frac{D_2}{D_1}} \left(\frac{c}{m}\right)^2 n^4}{1 + \alpha \left(\frac{c}{m}\right)^2 n^2}. \quad (101.4)$$

Out next task is to determine the values of m and n corresponding to the smallest λ and the critical value λ_{kr} itself. If the force p_y is a tensile force α must be taken with a minus sign.

Let us give some numerical results for a quadratic veneer plate (the x-axis is assumed in the direction of the sheet fibers, i.e., $D_1 > D_2$).

a) A square plate is compressed by forces distributed uniformly on all sides $(p_x = p_y = \lambda, \alpha = 1)$:

$$\lambda_{\text{кр}} = \frac{\pi^2 \sqrt{D_1 D_2}}{b^2} \cdot 2{,}23; \quad m = 1, \; n = 2. \quad (101.5)$$

Having lost its stability, the plate forms one semiwave in the direction of the fibers of the sheet and two semiwaves in a direction perpendicular to the fibers.

b) A square plate is compressed by forces distributed uniformly on two sides and extended by forces of the same magnitude distributed uniformly on the other sides ($p_x = \lambda$, $p_y = -\lambda$, $\alpha = -1$).

In the case of compression along the fibers of the sheet and tension across the fibers

$$\lambda_{\text{кр}} = \frac{\pi^2 \sqrt{D_1 D_2}}{b^2} \cdot 19{,}67; \quad m = 2, \ n = 1. \tag{101.6}$$

With compression across the fibers and tension along the fibers

$$\lambda_{\text{кр}} = \frac{\pi^2 \sqrt{D_1 D_2}}{b^2} \cdot 3{,}72; \quad m = 2, \ n = 1. \tag{101.7}$$

In the first case the plate's stability is higher; the critical value λ_{kr} is higher than this value for the second case by a factor of about 5.3. In both cases the loss of stability is accompanied by the formation of two semiwaves in the direction of the compressive forces and one semiwave in the direction of the tensile forces.

For a supported isotropic plate with arbitrary side ratio c, compressed by forces λ which are distributed uniformly along the four sides we obtain*

$$\lambda_{\text{кр}} = \frac{\pi^2 D}{b^2}\left(1 + \frac{1}{c^2}\right); \quad m = 1, \ n = 1. \tag{101.8}$$

2) Compressive forces λ are distributed uniformly on the sides $x = 0$ and $x = a$, and tensile forces p are distributed uniformly along the sides $y = 0$ and $y = b$; the compressive force may vary in its amount, the tensile forces remain unchanged. We have to estimate the influence of additional tensile forces on the value of the critical compressive load.

In Eq. (101.3) we must substitute $p_x = \lambda$, $p_y = -p$; then

$$\lambda = \frac{\pi^2 \sqrt{D_1 D_2}}{b^2}\left[\sqrt{\frac{D_1}{D_2}}\left(\frac{m}{c}\right)^2 + \frac{2D_3}{\sqrt{D_1 D_2}} n^2 + \sqrt{\frac{D_2}{D_1}}\left(\frac{c}{m}\right)^2 n^4 + \right.$$
$$\left. + \frac{pb^2}{\pi^2 \sqrt{D_1 D_2}}\left(\frac{c}{m}\right)^2 n^2\right]. \tag{101.9}$$

The lowest value of λ is obtained with $n = 1$, where in the direction of the tensile forces one semiwave is formed. The results of a further investigation of Eq. (101.9) may be formulated in the following way:

a) When the side ratio c satisfies the condition

$$c = \frac{m'}{\sqrt[4]{\frac{D_2}{D_1} + \frac{pb^2}{\pi^2 D_1}}}, \tag{101.10}$$

where m' is an integer, the critical value of λ is equal to

$$\lambda_{\text{кр}} = \frac{\pi^3 \sqrt{D_1 D_2}}{b^2} \cdot 2\left(\sqrt{1 + \frac{pb^3}{\pi^2 D_2}} + \frac{D_3}{\sqrt{D_1 D_2}}\right). \tag{101.11}$$

This value will be the smallest of all values determined from Eq. (101.9) and the side ratios (101.10) will not be the most favorable ones. Additional tensile forces will raise the critical load, i.e., increase the stability of the plate, as this was to be expected.

b) The limiting ratios of the sides at which the transition from *m* semiwaves in the direction of the compressive forces to *m* + 1 semiwaves takes place is equal to

$$c_{\text{пр}} = \frac{\sqrt{m(m+1)}}{\sqrt[4]{\frac{D_2}{D_1} + \frac{pb^3}{\pi^3 D_1}}}. \tag{101.12}$$

With these side ratios the plate has maximum stability.

§102. THE STABILITY OF A RECTANGULAR PLATE LOADED BY FORCES WHOSE DISTRIBUTION IS GOVERNED BY A LINEAR LAW

Let us consider a rectangular orthotropic plate whose principal directions are parallel to the sides and all four sides are supported. Assume normal forces distributed on two sides according to a linear law (Fig. 194)

$$p = \lambda\left(1 - \alpha\frac{y}{b}\right) \tag{102.1}$$

which, in the general case, can be reduced to a tensile or compressive force and a bending moment. It is required to determine the critical value λ_{kr} at which the plate loses stability.

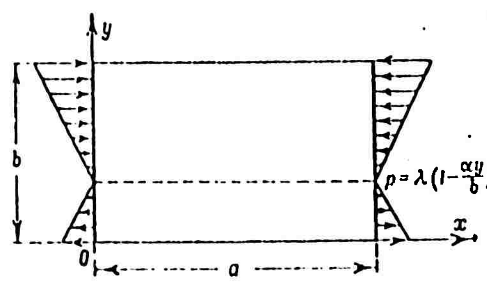

Fig. 194

In the case given the deflection equation (97.2) will have a variable coefficient, a fact which encumbers its integration. We shall therefore use the energy method.*

When the plate receives slight deflections, the forces perform work equal to

$$A = \frac{\lambda}{2} \int_0^a \int_0^b \left(1 - \alpha \frac{y}{b}\right) \left(\frac{\partial w}{\partial x}\right)^2 dx\, dy. \tag{102.2}$$

From the equation

$$V_{\text{пот}} = A \tag{102.3}$$

(see §97) we obtain

$$\lambda = \frac{1}{\int_0^a \int_0^b \left(1 - \alpha \frac{y}{b}\right) \left(\frac{\partial w}{\partial x}\right)^2 dx\, dy} \cdot \int_0^a \int_0^b \left[D_1 \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \right. \tag{102.4}$$
$$\left. + 2D_1 \nu_2 \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} + D_2 \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_k \left(\frac{\partial^2 w}{\partial x\, \partial y}\right)^2\right] dx\, dy.$$

The expression for the deflection which satisfies the boundary conditions can be chosen in the form of a sum

$$w = \sum_m \sum_n A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \tag{102.5}$$

Substituting this expression in Eq. (102.4) and integrating yields a result in the form of a fraction whose numerator and denominator are homogeneous square functions of A_{mn}. With the sum (102.5) consisting of a given number of terms we have as our next task to solve the problem of the minimum of the fraction mentioned above.

In a first approximation we put

$$w = A_{m1} \sin \frac{m\pi x}{a} \sin \frac{\pi y}{b} \tag{102.6}$$

and then we obtain from (102.4)

$$\lambda_{\text{кр}} = \frac{\pi^2 \sqrt{D_1 D_2}}{b^2 (1 - 0{,}5\alpha)} \left[\sqrt{\frac{D_1}{D_2}} \left(\frac{m}{c}\right)^2 + \frac{2D_3}{\sqrt{D_1 D_2}} + \sqrt{\frac{D_2}{D_1}} \left(\frac{c}{m}\right)^2\right] \tag{102.7}$$
$$(c = a/b).$$

Comparing this expression with (99.7) it can be noticed that in a first approximation the critical value λ_{kr} agrees with the critical load for a plate whose sides are equal to $a\sqrt{1 - 0{,}5\alpha}$ and $b\sqrt{1 - 0{,}5\alpha}$, compressed by forces which are distributed uniformly on two sides. Equation (102.7) can only be used in the case of small α where the distribution of the load is almost uniformly; in the case of pure bending $\alpha = 2$ the formula becomes senseless.

The second approximation is obtained when the sum (102.5) consists of two terms, i.e., when

$$w = \left(A_{m1} \sin \frac{\pi y}{b} + A_{m2} \sin \frac{2\pi y}{b}\right) \sin \frac{m\pi x}{a}. \tag{102.8}$$

The problem of determining the minimum of the fraction (102.4)

proves to be equivalent to the problem of the minimum of a square function of the form*

$$U = \frac{\pi^4 c}{4} \cdot \frac{m^2 \sqrt{D_1 D_2}}{a^3} \left\{ A_{m_1}^2 \left[a_{m1} - \lambda' (1 - 0{,}5\alpha) \right] - \right.$$
$$\left. - 2 A_{m1} A_{m2} \frac{16\alpha\lambda'}{9\pi^2} + A_{m2}^2 \left[a_{m2} - \lambda' (1 - 0{,}5\alpha) \right] \right\}. \tag{102.9}$$

Here we introduced the abbreviations

$$\left.\begin{aligned} a_{m1} &= \sqrt{\frac{D_1}{D_2}} \left(\frac{m}{c}\right)^2 + \frac{2D_3}{\sqrt{D_1 D_2}} + \sqrt{\frac{D_2}{D_1}} \left(\frac{c}{m}\right)^2, \\ a_{m2} &= \sqrt{\frac{D_1}{D_2}} \left(\frac{m}{c}\right)^2 + \frac{8D_3}{\sqrt{D_1 D_2}} + 16 \sqrt{\frac{D_2}{D_1}} \left(\frac{c}{m}\right)^2, \\ \lambda' &= \frac{\lambda b^2}{\pi^2 \sqrt{D_1 D_2}}. \end{aligned}\right\} \tag{102.10}$$

Setting up arbitrary functions U with respect to A_{m1} and A_{m2} and setting them equal to zero we obtain the equations

$$\left.\begin{aligned} A_{m1} \left[a_{m1} - \lambda' (1 - 0{,}5\,\alpha) \right] - A_{m2} \frac{16\alpha\lambda'}{9\pi^2} &= 0, \\ - A_{m1} \frac{16\,\alpha\lambda'}{9\pi^2} + A_{m2} \left[a_{m2} - \lambda' (1 - 0{,}5\alpha) \right] &= 0. \end{aligned}\right\} \tag{102.11}$$

When we set the determinant of this system equal to zero we obtain two series of values of λ:

$$\lambda = \frac{\pi^2 \sqrt{D_1 D_2}}{b^2} \left[\frac{(1 - 0{,}5\,\alpha) \cdot 0{,}5\,(a_{m1} + a_{m2})}{(1 - 0{,}5\alpha)^2 - \left(\frac{16\alpha}{9\pi^2}\right)^2} \pm \right.$$
$$\left. \pm \frac{\sqrt{(1 - 0{,}5\alpha)^2 \cdot 0{,}25\,(a_{m1} - a_{m2})^2 + \left(\frac{16\alpha}{9\pi^2}\right) a_{m1} a_{m2}}}{(1 - 0{,}5\alpha)^2 - \left(\frac{16\alpha}{9\pi^2}\right)^2} \right]. \tag{102.12}$$

In the following we have to determine the number m which, with the given side ratio, corresponds to the smallest λ, and we have to determine this λ which will be the critical value in the second approximation.

Let us enter into details of the case of pure bending where $\alpha = 2$ and the forces can be reduced to the moments M. The formula for the critical value of the moment can be represented in the form of

$$M_{\text{кр}} = \frac{\pi^2 \sqrt{D_1 D_2}}{6} k, \tag{102.13}$$

where

$$k = \frac{9\pi^2}{32} \sqrt{a_{m1} a_{m2}} = 2{,}78 \sqrt{a_{m1} a_{m2}}. \tag{102.14}$$

The quantity of the critical stress in the outermost fibers of the plate, $y = 0$ and $y = b$, is determined from the formula

$$\sigma_{\text{кр}} = \frac{\pi^3 \sqrt{D_1 D_2}}{b^2 h} k. \tag{102.15}$$

An investigation of the coefficient k corresponding to various side ratios is carried out similarly as this was done in §99 in the case of uniform compression and yields the following results.

1) If

$$c = 0{,}707\, m' \sqrt[4]{\frac{D_1}{D_2}}, \tag{102.16}$$

where m' is an integral number, we must, when k is determined from Eq. (102.14), set $m = m'$ and then we obtain

$$k = 11{,}1\left(1{,}25 + \frac{D_3}{\sqrt{D_1 D_2}}\right). \tag{102.17}$$

This value will also be the lowest of all possible values for the plate with given rigidities and the ratio (102.16) will therefore be the smallest favorable one.

2) The limiting ratios c_{pr} with which it becomes possible that two forms of equilibrium exist at the same time, namely those with m and $m + 1$ semiwaves, are obtained from the formula

$$c_{\text{пр}} = \sqrt{\frac{m(m+1)}{2}} \sqrt[4]{\frac{D_1}{D_2}} \quad (m = 1,\ 2,\ 3,\ \ldots). \tag{102.18}$$

Taking this formula into account we arrive at the following result:

$$\left.\begin{array}{llll} \text{if} & 0 < c < \sqrt[4]{\frac{D_1}{D_2}}, & \text{then} & m = 1; \\ \text{if} & \sqrt[4]{\frac{D_1}{D_2}} < c < 1{,}73 \sqrt[4]{\frac{D_1}{D_2}}, & \text{then} & m = 2; \\ \text{if} & 1{,}73 \sqrt[4]{\frac{D_1}{D_2}} < c < 2{,}45 \sqrt[4]{\frac{D_1}{D_2}}, & \text{then} & m = 3 \end{array}\right\} \tag{102.19}$$

etc. With a given ratio c we first determine the number m and then, on the basis of Eq. (102.14), the coefficient of the critical moment.

In particular, for the veneer plate considered previously, where the sides perpendicular to the fibers are loaded ($D_1 > D_2$) the limiting ratios are equal to

$$c_{\text{пр}} = 1{,}32 \sqrt{m(m+1)}. \tag{102.20}$$

The lowest value of the coefficient k is equal to

$$k = 18{,}1 \tag{102.21}$$

and is obtained if $c = 1.32\, m$ $(m = 1,\ 2,\ 3,\ \ldots)$.

For such a plate loaded on the sides parallel to the fibers ($D_1 < D_2$)

$$c_{np} = 0{,}382 \sqrt{m(m+1)}. \tag{102.22}$$

The lowest value of $\tilde{k}$ is equal to 18.1, as before, but now it is obtained with $c = 0.382\ m$. Just as in the case of compression, the plate whose sides parallel to the fibers are loaded will, when it has lost its stability, form a greater number of semiwaves than a plate where the sides perpendicular to the fibers of the sheet are loaded.

In Table 21 we give the values of k for a veneer plate, for several values of c and given values of the number of semiwaves m.

TABLE 21

Values of the Coefficients k for a Veneer Plate Bent by a Moment

c	$D_1 > D_2$		$D_1 < D_2$	
	k	m	k	m
0,5	15,5	1	19,8	1
1	19,8	1	18,5	3
2	19,8	2	18,2	5
3	18,5	2	18,2	8
∞	18,1	—	18,1	—

A further improvement of the coefficient k is obtained when we use a third approximation

$$w = \left(A_{m1} \sin\frac{\pi y}{b} + A_{m2} \sin\frac{2\pi y}{b} + A_{m3} \sin\frac{3\pi y}{b}\right) \sin\frac{m\pi x}{a}. \tag{102.23}$$

For an isotropic plate bent by moments, the critical value of the moment is determined in a second approximation according to the formula

$$M_{\kappa p} = \frac{\pi^3 D}{6} k, \tag{102.24}$$

where

$$k = 2{,}78\left(\frac{m}{c} + \frac{c}{m}\right)\left(\frac{m}{c} + \frac{4c}{m}\right). \tag{102.25}$$

The minimum value of this coefficient is obtained for ratios which satisfy the condition $c = m/\sqrt{2}$ $(m = 1, 2, 3, \ldots)$; it is (in a second approximation) equal to 25.0.*

§103. THE STABILITY OF A RECTANGULAR PLATE DEFORMED BY TANGENTIAL FORCES

Consider a rectangular orthotropic plate deformed by tangential forces t, which are distributed uniformly on all four sides (Fig. 195). The principal directions are parallel to the sides, all sides are resting on supports. We have to determine the critical forces t_{kr} at which the plate loses its stability. This

problem is solved approximately by the energy method of Ya.I. Sekerzh-Zen'kovich.*

The work of the tangential forces corresponding to slight deflections is equal to**

$$A = t\int_0^a \int_0^b \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}\,dx\,dy. \tag{103.1}$$

When we set this work equal to the potential energy of bending, we obtain

$$t = \frac{\int_0^a \int_0^b \Phi\,dx\,dy}{2\int_0^a \int_0^b \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}\,dx\,dy}, \tag{103.2}$$

where

$$\Phi = D_1\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_1\nu_2\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} + D_2\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_k\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2.$$

The expression for the deflection w satisfying all conditions on the sides can be chosen in the form of a sum

$$w = \sum_m \sum_n A_{mn}\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}. \tag{103.3}$$

The further progress of the solution of this problem is generally the same as in the case of normal load distributed according to a linear law (see §102).

The problem is solved such that numerical results and calculation formulas are obtained for a plate of birch veneer. Ya.I. Sekerzh-Zen'kovich used the following values for the reduced moduli and coefficients in the bending of such a veneer:

$$\begin{gathered} E_1' = 1{,}4\cdot 10^5\ \mathrm{kg/cm^2}, \\ E_2' = \frac{1}{12}E_1',\ \nu_1' = 0{,}46,\ \nu_2' = \frac{1}{12}\nu_1', \\ G' = 0{,}12\cdot 10^5\ \mathrm{kg/cm^2} \end{gathered} \tag{103.4}$$

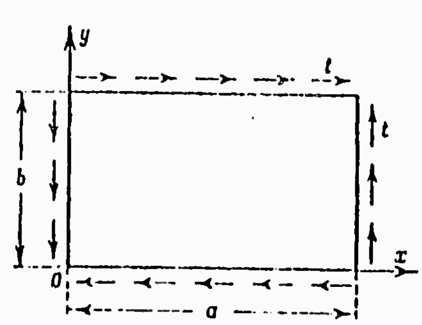

Fig. 195

The fundamental results consist of the following. When the fibers of the sheet are parallel to the long sides, it is sufficient for practice to be satisfied with an accuracy yielded by a deflection expression in the form of the sum of five terms, i.e., we put

$$w=\left(A_{11}\sin\frac{\pi x}{a}+A_{31}\sin\frac{3\pi x}{a}\right)\sin\frac{\pi y}{b}+A_{22}\sin\frac{2\pi x}{a}\sin\frac{2\pi y}{b}+ \\ +\left(A_{13}\sin\frac{\pi x}{a}+A_{33}\sin\frac{3\pi x}{a}\right)\sin\frac{3\pi y}{b}. \tag{103.5}$$

For the constants A_{ij} we obtain a system of five homogeneous equations whose determinant depends on t and must be set equal to zero.

Considering a plate in which the sheet fibers are parallel to the short sides, the author mentioned above, intending to obtain the same accuracy, took another number of terms in the sum of (103.3) according to the side ratio. For a square plate the expression for w was taken in the form of (103.5). For a plate with the side ratio $c = 2$ it is suggested to use the formula

$$w=\left(A_{21}\sin\frac{2\pi x}{a}+A_{41}\sin\frac{4\pi x}{a}\right)\sin\frac{\pi y}{b}+ \\ +\left(A_{32}\sin\frac{3\pi x}{a}+A_{52}\sin\frac{5\pi x}{a}\right)\sin\frac{2\pi y}{b}. \tag{103.6}$$

For a plate with a side ratio $c = 3$

$$w=\left(A_{31}\sin\frac{3\pi x}{a}+A_{51}\sin\frac{5\pi x}{a}\right)\sin\frac{\pi y}{b}+ \\ +\left(A_{42}\sin\frac{4\pi x}{a}+A_{62}\sin\frac{6\pi x}{a}\right)\sin\frac{2\pi y}{b}. \tag{103.7}$$

The author gives the following formula for the critical load:

$$t_{\kappa p}=10^4\frac{h^3}{b^2}k, \tag{103.8}$$

where k is a coefficient depending on the side ratio.

The critical tangential stress is obtained by dividing the load by the thickness h:

$$\tau_{\kappa p}=10^4\left(\frac{h}{b}\right)^2k. \tag{103.9}$$

In Table 22 we have compiled the numerical values of the coefficient k for several ratios c.

For a veneer plate with the elastic constants (103.4) the following approximate formulas have been suggested:

a) the fibers of the sheet are parallel to the long sides:

$$t_{\kappa p}=\frac{E'h^3}{b^2}\left(0{,}664+\frac{2{,}08}{\sqrt{c^3}}\right); \tag{103.10}$$

b) the fibers of the sheet are parallel to the short sides:

$$t_{\kappa p} = \frac{E'h^3}{b^2}\left(2,3 + \frac{0,446}{c}\right). \tag{103.11}$$

TABLE 22

Values of the Coefficient *k* for a Veneer Plate. Tangential Forces

	$c = 1$	1,2	1,4	1,6	1,8	2	2,5	3	∞
1. Волокна рубашки параллельны коротким сторонам	38,39	—	—	—	—	35,06	—	33,66	32,15
2. Волокна рубашки параллельны длинным сторонам	38,39	27,39	21,28	17,63	15,35	13,87	11,98	11,29	9,28

1) The fibers of the sheet are parallel to the short sides; 2) the fibers of the sheet are parallel to the long sides.

In these formulas $c = a/b$, $a > b$; E' is the reduced Young's modulus in the direction parallel to the fibers of the sheet.

The first formula yields resul s of errors not exceeding 4%, the second formula such of an error up to 2%.

§104. THE STABILITY OF AN INFINITELY LONG STRIP LOADED BY TANGENTIAL FORCES

In the case of a high side ratio ($c > 4$) the influence of the short sides can be neglected and a rectangular plate may be considered as an infinite strip.

Let us consider an infinite band on the sides of which tangential forces t are distributed uniformly per unit length (Fig. 196). A strict solution of the problem of the stability was obtained by Seydel for an orthotropic strip whose principal directions are parallel and perpendicular to the sides, in the case of supported and fixed sides; the solution was obtained with the help of the static method.*

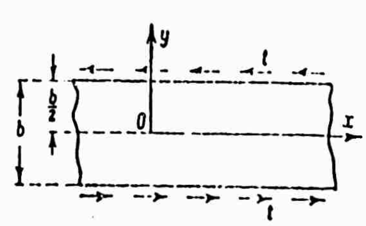

Fig. 196

Let us indicate briefly the way of derivation of this solution and give the basic results.

The deflection equation reads

$$D_1 \frac{\partial^4 w}{\partial x^4} + 2D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} + 2t \frac{\partial^2 w}{\partial x \partial y} = 0. \qquad (104.1)$$

It is required to find a nonzero solution to this equation which satisfies the boundary conditions:

a) on the supported sides

$$w = \frac{\partial^2 w}{\partial y^2} + \nu_1 \frac{\partial^2 w}{\partial x^2} = 0; \qquad (104.2)$$

b) on fixed sides

$$w = \frac{\partial w}{\partial y} = 0. \qquad (104.3)$$

The solution is sought in the form

$$w = f(y) e^{\frac{2\pi}{b} ix} \quad (i = \sqrt{-1}). \qquad (104.4)$$

For the function f we obtain the equation

$$D_2 f^{IV} - 2D_3 \left(\frac{2\pi}{b}\right)^2 f'' + it \frac{2\pi}{b} f' + D_1 \left(\frac{2\pi}{b}\right)^4 f = 0, \qquad (104.5)$$

whose general integral depends on the roots of the characteristic equation

$$D_2 s^4 - 2D_3 \left(\frac{2\pi}{b}\right)^2 s^2 + it \frac{2\pi}{b} s + D_1 \left(\frac{2\pi}{b}\right)^4 = 0. \qquad (104.6)$$

Introducing the new quantity $\beta = bs/2i$ we obtain the following equation for it:

$$\beta^4 + \frac{2D_3}{D_2} \varkappa^2 \beta^2 - \left(\frac{b}{2}\right)^2 \frac{t\varkappa}{D_2} \beta + \frac{D_1}{D_2} \varkappa^4 = 0. \qquad (104.7)$$

This equation has the solutions

$$\left.\begin{aligned} \beta_1 &= \alpha(1+n), & \beta_2 &= \alpha(1-n), \\ \beta_3 &= \alpha(-1+m), & \beta_4 &= \alpha(-1-m), \end{aligned}\right\} \qquad (104.8)$$

where α is a real number, m and n are real or complex numbers.

An expression for the bending is obtained in the following form:

$$w = e^{\frac{2\pi}{b} ix} \left(C_1 e^{\frac{2\beta_1}{b} iy} + C_2 e^{\frac{2\beta_2}{b} iy} + C_3 e^{\frac{2\beta_3}{b} iy} + C_4 e^{\frac{2\beta_4}{b} iy} \right). \qquad (104.9)$$

On the basis of the boundary conditions (104.2) or (104.3)

we obtain four homogeneous equations for the constants C_1, C_2, C_3, C_4, and when the determinant of the system is set equal to zero we obtain an equation for the determination of the critical force t_{kr}.

The load t with which bending becomes possible is obtained as a function of κ. We then have to determine the value of κ corresponding to the minimum value of t and precisely this minimum value will be equal to the critical value t_{kr}.

Having lost the stability, the plate forms a series of waves which make with the sides a certain angle; the wavelength is characterized by the parameter $\varkappa$.

Seydel introduces the quantity $\vartheta = \frac{\sqrt{D_1 D_2}}{D_3}$ and gives the results of calculations of the critical load as functions of ϑ.

For plates where ϑ is between 1 and ∞ we obtain

$$t_{\kappa p} = 4C_a \frac{\sqrt[4]{D_1 D_2^3}}{b^2}, \tag{104.10}$$

$$\varkappa = 0{,}5 C_a' b \sqrt[4]{\frac{D_1}{D_2}}; \tag{104.11}$$

the values of the coefficients C_a and C_a' are compiled in Table 23.

TABLE 23

Values of the Coefficients C_a and C_a'

	1 Стороны $y=\pm\frac{b}{2}$ 2 оперты		1 Стороны $y=\pm\frac{b}{2}$ 3 заделаны	
ϑ	C_a	C_a'	C_a	C_a'
1	13,17	2,49	22,15	1,66
2	10,8	2,28	18,75	1,54
3	9,95	2,16	17,55	1,48
5	9,25	2,13	16,6	1,44
10	8,7	2,08	15,85	1,41
20	8,4	—	15,45	—
30	—	2,05	—	1,38
40	8,25	—	15,25	—
∞	8,125	—	15,071	—

1) Sides; 2) supported; 3) fixed.

For plates where ϑ lies within the limits of zero and 1,

$$t_{\kappa p} = 4C_b \frac{\sqrt{D_2 D_3}}{b^2}; \tag{104.12}$$

$$\varkappa = 0{,}5C_b' b\sqrt{\frac{D_3}{D_2}}. \qquad (104.13)$$

The values of the coefficients C_b and C_b' are compiled in Table 24.

For a strip with supported sides, which may also be nonorthotropic, we can obtain an approximate solution by means of S.P. Timoshenko's method.

Let us first consider an orthotropic plate whose principal directions are parallel and perpendicular to the sides.

TABLE 24

The Values of the Coefficients C_b and C_b'

	1 Стороны $y=\pm\frac{b}{2}$ 2 оперты		1 Стороны $y=\pm\frac{b}{2}$ 3 заделаны	
ϑ	C_b	C_b'	C_b	C_b'
0	11,71	—	18,59	—
0,05	—	1,92	—	1,16
0,2	11,8	1,91	18,85	1,20
0,5	12,2	2,07	19,9	1,36
1	13,17	2,49	22,15	1,66

1) Sides; 2) supported; 3) fixed.

The directions of the axes are not the same as in Fig. 196, but a little other: we take an arbitrary point on the side of the strip as the origin of coordinates and the x-axis coincident with the side. We shall seek an approximate solution to Eq. (104.1) in the form

$$w = A\sin\frac{\pi y}{b}\sin\frac{\pi}{s}(x - y\operatorname{tg}\psi). \qquad (104.14)$$

This expression satisfies the necessary boundary conditions and represents the equation of a surface with oblique waves, the length of which in the direction of the x-axis is equal to s and the angle of slope with respect to the y-axis is equal to ψ. An exact solution does not exist in this form but we can choose the quantities s, t and ψ such that Expression (104.14) becomes an approximate solution to Eq. (104.1). We shall use the following method: we substitute Eq. (104.14) in the left-hand side of Eq. (104.1), multiply the result by

$$\sin\frac{\pi y}{b}\sin\frac{\pi}{s}(x - y\operatorname{tg}\psi)\,dx\,dy$$

and integrate with respect to y from zero to b and with respect to x from zero to s.* Note that

$$\int_0^b \int_0^a \sin\frac{\pi y}{b}\cos\frac{\pi y}{b}\sin\frac{\pi}{s}(x-y\,\mathrm{tg}\,\psi)\cos\frac{\pi}{s}(x-y\,\mathrm{tg}\,\psi)\,dx\,dy=0,$$

whereas

$$\int_0^b \int_0^a \sin^2\frac{\pi y}{b}\sin^2\frac{\pi}{s}(x-y\,\mathrm{tg}\,\psi)\,dx\,dy\neq 0,$$

where the coefficient of the latter integral is set equal to zero. We then obtain

$$t=\frac{\pi^2\sqrt{D_1D_2}}{2b^2}\left[\sqrt{\frac{D_1}{D_2}}\cdot\frac{\gamma}{\alpha}+\frac{2D_3}{\sqrt{D_1D_2}}\left(\alpha\gamma+\frac{1}{\alpha}\right)+\sqrt{\frac{D_2}{D_1}}\left(\alpha^3\gamma+6\alpha+\frac{1}{\alpha\gamma}\right)\right]. \tag{104.15}$$

where

$$\gamma=\left(\frac{b}{s}\right)^2,\quad \alpha=\mathrm{tg}\,\psi. \tag{104.16}$$

We determine the minimum of Expression (104.15) which is a function of the two variables α and γ according to the usual rules, i.e., we solve the equation

$$\frac{\partial t}{\partial \alpha}=0,\quad \frac{\partial t}{\partial \gamma}=0. \tag{104.17}$$

When the α and γ obtained in this way are substituted in Eq. (104.15) we obtain the sought critical strain.

The complex formula according to which the critical tangential forces are determined, can be simplified in the case of a plate with markedly expressed anisotropy where the rigidity in one direction is several times higher than the rigidity in the other direction (as, e.g., veneer plates). L.I. Balabukh derived approximate formulas for the critical tangential forces for such plates. These formulas have the form:

for a strip with supported sides

$$t_{\text{кр}}=\frac{2\pi^2\sqrt{D_1D_2}}{b^2}\sqrt{1+3\frac{\sqrt{D_1D_2}}{D_3}+\frac{D_3}{\sqrt{D_1D_2}}}; \tag{104.18}$$

for a strip with fixed sides*

$$t_{\text{кр}}=\frac{3.45\pi^2\sqrt{D_1D_2}}{b^2}\sqrt{1+3\frac{\sqrt{D_1D_2}}{D_3}+\frac{D_3}{\sqrt{D_1D_2}}}. \tag{104.19}$$

Considering a plate of three-layer birch veneer, L.I. Balabukh used for the reduced elastic constants in bending the same numerical values as Ya.I. Sekerzh-Zeh'kovich in his paper referred to in §103, i.e.,

$$\left.\begin{array}{l} E_1' = 1{,}4\cdot 10^5\,\mathrm{kg/cm^2},\ E_2' = \frac{1}{12}E_1',\quad \nu_1' = 0{,}46, \\ \nu_2' = \frac{\nu_1'}{12},\quad G' = 0{,}12\cdot 10^5\,\mathrm{kg/cm^2}. \end{array}\right\} \tag{104.20}$$

In the way shown above we can obtain approximate solutions also for orthotropic plates where the principal directions of elasticity are not parallel to the sides. For this purpose one must first of all calculate the rigidity D_{ij} for the directions x and y, parallel and perpendicular to the sides, using Eqs. (69.6) and (69.7). Then the equation of deflection of a plate deformed by tangential forces takes the form

$$\begin{gathered} D_{11}\frac{\partial^4 w}{\partial x^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3\partial y} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2} + \\ + 4D_{26}\frac{\partial^4 w}{\partial x\partial y^3} + D_{22}\frac{\partial^4 w}{\partial y^4} + 2t\frac{\partial^2 w}{\partial x\partial y} = 0. \end{gathered} \tag{104.21}$$

Let us give the results of calculations for veneer strips [with the elastic constants (104.20)] where the fibers make an angle of 45° with the sides.*

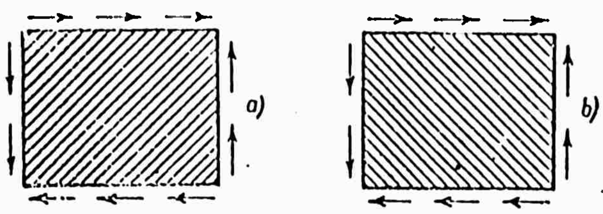

Fig. 197

When the action of the forces on the fibers is such as shown in Fig. 197a ("shear at an angle of 45°"), we have

$$t_{\text{кр}} = 9{,}21\cdot 10^4\frac{h^3}{b^2}. \tag{104.22}$$

When, however, the forces act in such a way as shown in Fig. 197b, ("shear at an angle of 135°") in this case

$$t_{\text{кр}} = 50{,}66\cdot 10^4\frac{h^3}{b^2}. \tag{104.23}$$

In the latter case the strip proves to be much more stable than in the case of "shear at an angle of 45°," the critical load is higher by a factor of almost 5.5. Equations (104.22) and (104.23) apply to a plate whose side ratio is not smaller than 4.

§105. THE STABILITY OF PLATES UNDER THE SIMULTANEOUS ACTION OF NORMAL AND TANGENTIAL LOADS

Let us consider a rectangular plate with supported sides which is deformed by normal forces p (per unit length) which are distributed uniformly on two sides, and by tangential forces t distributed uniformly on all four sides (Fig. 198). We must de-

termine the critical values of the forces.

At present we know an approximate solution for an infinitely long strip compressed along the sides and also deformed by tangential forces ($c = \infty$). This solution was found by L.I. Balabukh [by the same method as in the case of the sole action of tangential forces].*

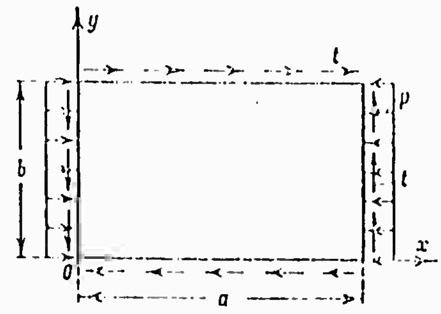

Fig. 198

In the case of an orthotropic strip whose principal directions are parallel and perpendicular to the sides, the equation of the bent surface has the form

$$D_1 \frac{\partial^4 w}{\partial x^4} + 2D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} + p \frac{\partial^2 w}{\partial x^2} - 2t \frac{\partial^2 w}{\partial x \partial y} = 0. \qquad (105.1)$$

An approximate solution to this equation can be sought in the form

$$w = A \sin \frac{\pi y}{b} \sin \frac{\pi}{s} (x - y \operatorname{tg} \psi). \qquad (105.2)$$

We substitute Eq. (105.2) in the left-hand side of Eq. (105.1 , multiply the results by

$$\sin \frac{\pi y}{b} \sin \frac{\pi}{s} (x - y \operatorname{tg} \psi)\, dx\, dy,$$

integrate with respect to y from zero to b and with respect to x from zero to s and finally require that the result must be equal to zero. We then obtain the relation

$$p = \frac{\pi^2 \sqrt{D_1 D_2}}{b^2} \left[\sqrt{\frac{D_1}{D_2}} \gamma + \frac{2D_3}{\sqrt{D_1 D_2}} (\alpha^2 \gamma + 1) + \sqrt{\frac{D_2}{D_1}} \left(\alpha^4 \gamma + 6\alpha^2 + \frac{1}{\gamma} \right) \right] + 2\alpha t, \qquad (105.3)$$

where

$$\gamma = \left(\frac{b}{s} \right)^2, \quad \alpha = \operatorname{tg} \psi. \qquad (105.4)$$

Assuming (temporarily) the force t constant, we shall seek the minimum of p as a function of the two variables α and γ. From the equations

$$\frac{\partial p}{\partial \alpha} = 0, \quad \frac{\partial p}{\partial \gamma} = 0 \tag{105.5}$$

we obtain the relation

$$\left.\begin{aligned} t &= -\frac{2\pi^2 \sqrt{D_1 D_2}}{b^2}\alpha\left[\frac{D_3}{\sqrt{D_1 D_2}}\gamma + \sqrt{\frac{D_2}{D_1}}(3 + \alpha^2\gamma)\right], \\ \gamma &= \sqrt{\frac{D_2}{D_1 + 2D_3\alpha^2 + D_2\alpha^4}}. \end{aligned}\right\} \tag{105.6}$$

Substituting the value of γ in Eq. (105.3) and the first formula of (105.6) we obtain

$$\left.\begin{aligned} p &= \frac{\pi^2 \sqrt{D_1 D_2}}{b^2} \cdot 2\left(\frac{D_3}{\sqrt{D_1 D_2}} + 3\alpha^2\sqrt{\frac{D_2}{D_1}} + \right. \\ &\qquad \left. + \sqrt{1 + \frac{2D_3}{D_1}\alpha^2 + \frac{D_2}{D_1}\alpha^4}\right) + 2\alpha t, \\ t &= -\frac{\pi^2 \sqrt{D_1 D_2}}{b^2} \cdot 2\alpha\left(3\sqrt{\frac{D_2}{D_1}} + \frac{D_3 + \alpha^2 D_2}{\sqrt{1 + \frac{2D_3}{D_1}\alpha^2 + \frac{D_2}{D_1}\alpha^4}}\right). \end{aligned}\right\} \tag{105.7}$$

These equations represent in a parametrical form the relationship between p and t with which stability is lost; they become more illustrative when we interpret them geometrically in the following way.

When the parameter α is allowed to vary from zero to infinity and plot the corresponding values of p on the abscissa and the values of t on the ordinate axis we obtain a curve which resembles a quadratic parabola (Fig. 199). The length of the abscissa section is equal to the critical compressive force p'_{kr} in the absence of tangential forces and the ordinate section cut off by the curve is equal to the critical tangential force t'_{kr} in the absence of normal forces.* When the curve is considered to be a quadratic parabola, the relation linking p and t can be given the form

$$\frac{p}{p'_{kp}} + \frac{t^2}{t'^2_{kp}} = 1. \tag{105.8}$$

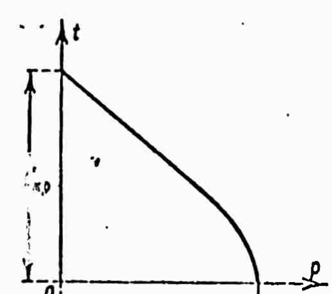

Fig. 199

As we know how to determine p'_{kr} and t'_{kr} (see §§99 and 104), we can always determine the critical normal load from Eq. (105.8) when the tangential forces are given, or the critical tangential forces if the normal forces are given, or, finally, the critical value λ_{kr} if the normal and tangential forces are given, with an accuracy to within a factor λ (i.e., if the ratio of their values is given).

The critical normal load p'_{kr} for an orthotropic strip is equal to

$$p'_{\text{кр}} = \frac{\pi^2 \sqrt{D_1 D_3}}{b^3} \cdot 2\left(1 + \frac{D_1}{\sqrt{D_1 D_3}}\right). \tag{105.9}$$

The critical tangential load t'_{kr} was dealt with in the last section, 104. For such plates where the rigidities in the principal directions differ essentially from one another (e.g., a veneer) one can use the approximate formula

$$\sigma_{\text{кр}} + \frac{\tau^2}{\frac{2\pi^2}{b^2 h}\left(3D_2 + D_3\sqrt{\frac{D_2}{D_1}}\right)} = \frac{2\pi^2}{b^2 h}\left(\sqrt{D_1 D_2} + D_3\right), \tag{105.10}$$

where σ_{kr} and τ are the normal and the tangential critical stresses.

Equation (105.8) can also be used in the case of a strip whose sides are fixed. In this case, for a veneer, we shall have the equation*

$$\sigma_{\text{кр}} + \frac{\tau^2}{\frac{3{,}4\pi^2}{b^2 h}\left(3D_2 + D_3\sqrt{\frac{D_2}{D_1}}\right)} = \frac{3{,}5\pi^2}{b^2 h}\left(\sqrt{D_1 D_2} + D_3\right). \tag{105.11}$$

instead of (105.10). The problem becomes more complicated when the principal directions make an angle with the sides of the plate. L. I. Balabukh arrived at numerical results when solving the problem for a veneer plate [with the elastic constants (104.20)] resting on a support, where the fibers of the sheet crossed the sides at an angle of 45°.

These results are the following.

Let us agree in calling tangential forces which result in a "shear at an angle of 45°" (Fig. 197a) positive and tangential forces producing a "shear at an angle of 135°" (Fig. 197b) negative, and let us denote by t'_{kr} and t''_{kr} the quantities of the critical tangential forces producing a shear at an angle of 45° and 135° in the absence of normal loads. With concrete values of p and t of the normal and tangential loads acting simultaneously, one obtains a parametrical representation of the same type as (105.6), only with more complex expressions depending on α. Analyzing this relation the following may be remarked. When various values are attributed to the parameter α and the corresponding p-values are plotted on the abscissa and the t-values on the

ordinate axis, we obtain a curve resembling a parabola. This curve cuts off on the abscissa a section p'_{kr} whose length corresponds to the value of the critical compressive load for compression at an angle of 45° to the fibers of the sheet. When the curve is considered to be a parabola we can represent (approximately) the dependence between p and t in the form of the equation

$$\frac{p}{p'_{\text{кр}}} + \left(\frac{t}{t'_{\text{кр}}} - 1\right)\left(\frac{t}{t''_{\text{кр}}} + 1\right) = 0. \tag{105.12}$$

For a veneer plate with the elastic constants (104.20) the values of t'_{kr} and t''_{kr} are given in §104. The critical compressive load p'_{kr} in the case of compression at an angle of 45° with respect to the fibers of the sheet is, according to L.I. Balabukh approximately equal to the critical load in the case of compression along and across the fibers of the sheet.

Equation (105.12) permits the determination of the approximate value of the critical compressive load with given tangential forces, the critical tangential forces with given tensile or compressive forces and the like.

It is interesting to note that with positive tangential force the amount of the critical compressive load decreases and with negative tangential forces it increases relative to the forces p_{kr} in the absence of tangential loads. In this way it proves to be possible to increase the stability of a compressed strip by means of distributing tangential forces along its edges.*

Manuscript Page No.	[Footnotes]
405	This problem was solved for the first time (by means of the energy method) by Huber (see M.T. Huber, Probleme der Statik technisch wichtiger orthotroper Platten [Problems of the Statics of Technically Important Orthotropic Plates], Warsaw 1929).
406	This formula was derived by Yamana (M. Yamana, On the elastic stability of aeroplane structures, J. Fac. Eng. Tokyo Univ. Vol. 20, No. 8, 1933).
407	See, e.g., the book by S.P. Timoshenko, Plastinki i obolochki [Plates and Shells] Gostekhizdat, 1948, page 296.
410*	L.I. Balabukh, Ustoychivost' fanernykh plastinok [Stability of Veneer Plates] Tekhnika Vozdushnogo flota [Aviation Engineering] 1937, No. 9.
410**	See, e.g., S.P. Timoshenko's book "Plastinki i oboloch-

ki" [Plates and Shells], page 306.

* The only exception is the case of a plate with four supported sides.

411 See (98.11).

413 See our paper "Ustoychivost' anizotropnykh plastinok" [Stability of Anisotropic Plates], Gostekhizdat, 1943, §17.

415 Timoshenko, S.P., Ustoychivost' uprugikh sistem [Stability of Elastic Systems] Gostekhizdat, 1946, §64, page 299.

416 See our paper "Ustoychivost' anizotropnykh plastinok" [Stability of Anisotropic Plates], Gostekhizdat, 1943, §19.

418 See Eqs. (97.6) and (97.7).

420 Timoshenko, S.P., Ustouchivost' uprugikh sistem [Stability of Elastic Systems] §66, pages 312-318. According to S.P. Timoshenko, the third approximation yields $k_{min} = 23.9$.

421* Sekerzh-Zen'kovich, Ya.I., K raschetu na ustoychivost' lista fanery, kak anizotropnoy plastinki [To the Calculation of Instability of a Veneer Sheet as an Anisotropic Plate] Trudy TsAGI, No. 76, 1931.

421** See Eq. (98.11).

423 Seydel, E., 1) Beitrag zur Frage des Ausbeulens von versteiften Platten bei Schubbeanspruchung [Contribution to Buckling of Stiffened Plates Under Shearing Stress] DVL Bericht, Luftfahrtforschung [German Soc. of Aviation Report, Aviation Research] Vol. 8, No. 3, 1930. 2) Über das Ausbeulen von rechteckigen isotropen oder orthogonal-isotropen Platten bei Schubbeanspruchung [On the Buckling of Rectangular Isotropic or Orthotropic Plates Under Shearing Stress] Ing. Archiv, Vol. 4, No. 2, 1933; 3) Ausbeul-Schublast rechteckiger Platten [Buckling Shearing Load of Rectangular Plates] Z. Flugtech. Motorluftsch. [Journal of Aviation Eng., Motor Aviation] Vol. 24, No. 33, 1933.

426 See the paper by Ya.I. Sekerzh-Zen'kovich, mentioned in §103.

427 See the paper by L.I. Balabukh, Ustoychivost' fanernykh plastinok [Stability of Veneer Plates], Tekhnika vozdushnogo flota, 1937, No. 9. In this paper other denotations are used.

428 See the paper mentioned previously, by L.I. Balabukh and "Spravochnik aviakonstruktora" [Handbook of the

Airplane Designer], Vol. III, Prochnost' samoleta [Airplane Stability] Izd. TsAGI, 1939, page 247.

429 See the paper by L.I. Balabukh mentioned in §104.

430 The graph was taken from the book by S.N. Kan and I.A. Sverdlov "Raschet samoleta na prochnost'" [Strength Calculation of Airplanes] Oborongiz, Moscow, 1940, §93, page 263.

431 See the paper by L.I. Balabukh mentioned in the preceding section, pages 31 and 33. We use somewhat different notations.

432 See the paper by L.I. Balabukh, pages 35, 36.

[Transliterated Symbols]

406 кр = kr = kriticheskiy = critical

407 пр = pr = predel'nyy = limiting

411 изг = izg = izgib = bending

Chapter 15

THE STABILITY OF PLATES LOADED BY CONCENTRATED FORCES

§106. THE STABILITY OF A RECTANGULAR ORTHOTROPIC PLATE LOADED BY CONCENTRATED FORCES

In this chapter a number of problems concerning the stability of anisotropic plates where the load is given in the form of concentrated forces are considered.

To start with, let us consider an anisotropic homogeneous rectangular plate two opposite sides of which are fixed in an arbitrary manner or free, whereas the two other ones are supported and under the action of normal concentrated forces λ, λP_2, ..., λP_N grouped by pairs (Fig. 200). The forces are given except for the factor λ; the critical value λ_{kr} must be determined. We shall restrict ourselves to the investigation of the stability of an orthotropic plate with principal directions parallel to the directions of the sides.

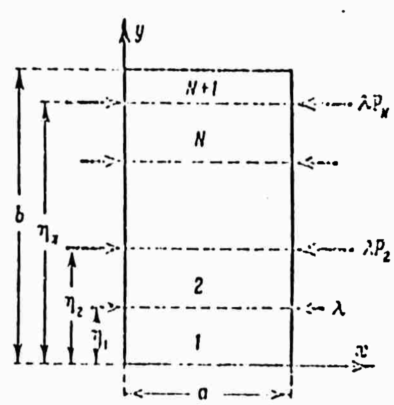

Fig. 200

We shall place the axes along the plate sides and designate by a and b, as usually, the side lengths, and by η_1, η_2, ..., η_N the ordinates of the points of application of the forces; we shall leave the designations for the principal elastic constants and rigidities as before.

Let us solve the problem by the energetic method.* We shall assume that small deflections are imparted to the plate. The lines of action of the forces divide the plate into rectangular

regions whose number is $N+1$; we shall designate the deflections of these regions by $w_1, w_2, \ldots, w_{N+1}$. If the midsurface is distorted the forces perform a work equal to*

$$A = \frac{\lambda}{2} \sum_{k=1}^{N} P_k \int_0^a \left(\frac{\partial w_k}{\partial x}\right)^2_{y=\eta_k} dx. \tag{106.1}$$

Putting this work equal to the potential energy of the bending we obtain:

$$\lambda = \frac{\sum\limits_{k=1}^{N+1} \int\limits_{\eta_{k-1}}^{\eta_k} \int\limits_0^a \Phi_k \, dx \, dy}{\sum\limits_{k=1}^{N} P_k \int\limits_0^a \left(\frac{\partial w_k}{\partial x}\right)^2_{y=\eta_k} dx}, \tag{106.2}$$

where

$$\Phi_k = D_1 \left(\frac{\partial^2 w_k}{\partial x^2}\right)^2 + 2D_1\nu_2 \frac{\partial^2 w_k}{\partial x^2} \cdot \frac{\partial^2 w_k}{\partial y^2} + D_2 \left(\frac{\partial^2 w_k}{\partial y^2}\right)^2 + 4D_k \left(\frac{\partial^2 w_k}{\partial x \, \partial y}\right)^2$$

(in this equation $\eta_0 = 0$, $\eta_{N+1} = b$, $P_1 = 1$).

We shall seek the expressions for the deflections in a form which ensures the satisfaction of the conditions on the supported (loaded) sides:

$$w_k = f_k(y) \sin \beta x, \tag{106.3}$$

where $\beta = \frac{m\pi}{a}$, and m is an integer. The functions $f_1(y)$ and $f_{N+1}(y)$ must fulfill the conditions on the sides $y = 0$ and $y = b$, according to the way in which they are fixed; e.g.,

$$f = f' = 0 \tag{106.4}$$

on the fixed side,

$$f = f'' = 0 \tag{106.5}$$

on the supported side, etc. Besides, on the line of action of the forces the functions f_k must fulfill the conditions due to the quite natural requirement that the bent surface should be continuous and have no fissures:

$$\left.\begin{array}{c} f_k(\eta_k) = f_{k+1}(\eta_k), \\ f'_k(\eta_k) = f'_{k+1}(\eta_k) \\ (k = 1, 2, \ldots, N). \end{array}\right\} \tag{106.6}$$

Aiming at the determination of the smallest value of λ not equal to zero we shall pose the problem as follows: to find functions $f_1, f_2, \ldots, f_{N+1}$ fulfilling Conditions (106.6) and the conditions on the sides $y = 0$ and $y = b$, for which Expression (106.2) becomes minimum.

The problem of finding the minimum of the fraction (106.2) is equivalent to the problem of finding the minimum of the expression*

$$U = \sum_{k=1}^{N+1} \int_{\eta_{k-1}}^{\eta_k} \left(D_2 f_k''^2 - 2D_1\nu_2\beta^2 f_k f_k'' + \right.$$
$$\left. + D_1\beta^4 f_k^2 + 4D_k\beta^2 f_k'^2\right) dy - \lambda\beta^2 \sum_{k=1}^{N} P_k f_k^2(\eta_k). \tag{106.7}$$

We solve this problem according to the rules of variational calculus: we set up the variation δU and set it equal to zero, taking account of Conditions (106.6) and the conditions on the a sides.

After a number of intermediate calculations we obtain the following result.

The functions f_k satisfy the equation

$$D_2 f_k^{IV} - 2D_3\beta^2 f_k'' + D_1\beta^4 f_k = 0 \tag{106.8}$$

and the following conditions:

a) on the sides $y = 0$, $y = b$ the conditions corresponding to the way in which the sides are fixed;

b) on the line of action of the forces $y = \eta_k$.

$$\left.\begin{array}{c} f_{k+1} = f_k, \\ f'_{k+1} = f'_k, \\ f''_{k+1} = f''_k, \\ D_2 f'''_{k+1} - D_2 f'''_k = \lambda\beta^2 P_k f_k \\ (\kappa = 1,\ 2,\ \ldots,\ N,\ P_1 = 1). \end{array}\right\} \tag{106.9}$$

Each function $f_k(y)$ contains four arbitrary constants; the overall number of constants is $4N + 4$. The same number of conditions for the functions f_k is obtained (4 conditions on the sides a and $4N$ conditions on the lines $y = \eta_k$). Fulfilling these conditions we shall obtain a system of $4N + 4$ homogeneous equations with the same number of unknowns, and setting the determinant of this system equal to zero we obtain an equation (an algebraic one of Nth degree) to determine λ.

Equation (106.8) and Conditions (106.9) show that the functions f_k may be regarded as the deflections of the zones of a beam of length b. This beam has a rigidity of D_2, lies on a massive elastic base with a coefficient of elasticity $D_1\beta^4$, is stretched by a force $T = 2D_3\beta^2$ and bent by the forces Q_1, Q_2, ..., Q_N (Fig. 201); its ends are fixed according to the fixation of the sides $y = 0$ and $y = b$. The fourth conditions (106.9) show that the forces Q_k are proportional to the deflections f_k at

their points of application.

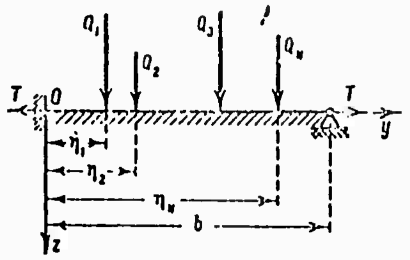

Fig. 201

Designating the beam deflection at an arbitrary point by $f(y)$ without subscript we may write the latter conditions in the following form:

$$\left.\begin{aligned} Q_1 &= \lambda\beta^2 f(\eta_1), \\ Q_2 &= \lambda\beta^2 P_2 f(\eta_2), \\ &\cdots\cdots\cdots \\ Q_N &= \lambda\beta^2 P_N f(\eta_N). \end{aligned}\right\} \qquad (106.10)$$

The analogy with a beam which we have established permits a considerable simplification of the problem. We may further leave the plate out of consideration and solve the problem of a beam acted upon by forces proportional to the deflections. Designating by $\delta(\eta, y)$ the force influence function introduced in §77 we shall write the expression for the beam deflection at an arbitrary point in the following form:

$$f(y) = \sum_{k=1}^{N} Q_k \delta(\eta_k, y). \qquad (106.11)$$

Substituting the value of the deflection at the points $y = \eta_1$, $y = \eta_2$, ..., $y = \eta_N$ into Conditions (106.10) we obtain a system of homogeneous equations for Q_k:

$$\left.\begin{aligned} Q_1(1 - \lambda\beta^2\delta_{11}) - Q_2\lambda\beta^2\delta_{21} - \ldots - Q_N\lambda\beta^2\delta_{N1} &= 0, \\ -Q_1\lambda\beta^2 P_2\delta_{12} + Q_2(1 - \lambda\beta^2\delta_{22}P_2) - \ldots - Q_N\lambda\beta^2 P_2\delta_{N2} &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ -Q_1\lambda\beta^2 P_N\delta_{1N} - Q_2\lambda\beta^2 P_N\delta_{2N} - \ldots + Q_N(1 - \lambda\beta^2 P_N\delta_{NN}) &= 0 \\ [\delta_{ij} = \delta(\eta_i, \eta_j)]. & \end{aligned}\right\} \qquad (106.12)$$

Putting the determinant of this system equal to zero we obtain an equation for λ:

$$\begin{vmatrix} \delta_{11} - \frac{1}{\lambda\beta^2}, & \delta_{21}, & \ldots, & \delta_{N1}, \\ \delta_{12}, & \delta_{22} - \frac{1}{\lambda\beta^2 P_2}, & \ldots, & \delta_{N2} \\ \cdot & \cdot & \cdot & \cdot \\ \delta_{1N}, & \delta_{2N}, & \ldots, & \delta_{NN} - \frac{1}{\lambda\beta^2 P_N} \end{vmatrix} = 0. \tag{106.13}$$

Hence we find, in general, N different values of λ each of which will depend on m. The minimum of all values of λ which is not equal to zero will also be the critical value λ_{kr}.

If there are stretching forces among the forces deforming the plate they must be regarded as negative in Eq. (106.13).

The problem of determining the critical value λ_{kr} and the corresponding number of semiwaves m in the direction of the force action is considerably simplified if there are tables of the influence functions for a beam with the necessary values of D_1, D_2, D_3.

§107. THE STABILITY OF A RECTANGULAR ORTHOTROPIC PLATE COMPRESSED BY TWO FORCES

Let a rectangular orthotropic plate be compressed by two forces applied at a distance η from one of the sides; the sides where the forces are applied are assumed to be supported, and the other ones fixed in an arbitrary manner or free* (Fig. 202).

In this case

$$w = Q\delta(\eta, y)\sin\beta x. \tag{107.1}$$

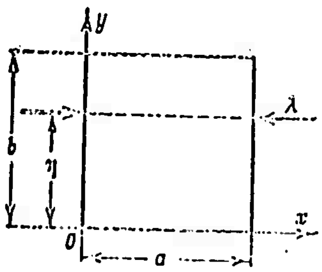

Fig. 202

Instead of the determinant (106.13) the equation

$$\delta_{11} - \frac{1}{\lambda\beta^2} = 0. \tag{107.2}$$

is obtained. Hence

$$\lambda = \frac{1}{\xi^2 \delta_{11}} = \frac{4\pi \sqrt{D_1 D_2}}{a} \cdot \frac{m}{g(\eta, \eta)}. \qquad (107.3)$$

g is here a function proportional to the influence function δ introduced in §78 [see Formulas (78.1)].

We shall represent the critical value of the force λ_{kr} as follows:

$$\lambda_{\text{кр}} = \frac{4\pi \sqrt{D_1 D_2}}{a} k. \qquad (107.4)$$

The coefficient k is equal to the minimum of the ratio m/g_{11} which is easily established by using the tables of the g functions (if, of course, there are such tables for the given material and the end fixing in question).

Table 25 presents the values of k for the plate considered in §67* with four supported sides, made of isotropic material and plywood. The corresponding values of m — the number of semiwaves (in the direction of the force action) into which the plate is divided after having lost its stability — is also indicated there. Four cases are considered where the distances of the points of force application from the side a are equal to b/S, $b/4$, $3b/8$ and $b/2$ and four ratios $d = b/a$.

TABLE 25

The Values of the Coefficient k for a Supported Plate Compressed by Two Forces

	d	$\eta = \frac{b}{8}$		$\eta = \frac{b}{4}$		$\eta = \frac{3b}{8}$		$\eta = \frac{b}{2}$	
		k	m	k	m	k	m	k	m
1 Изотропный материал	0,5	8,61	5	4,41	3	3,28	2	3,03	2
	1	4,31	2	2,28	1	1,61	1	1,49	1
	2	2,16	1	1,22	1	1,06	1	1,03	1
	3	1,48	1	1,06	1	1,01	1	1,01	1
2 Фанера. Сжатие вдоль волокон ($D_1 > D_2$)	0,5	6,45	3	3,53	2	2,31	1	2,11	1
	1	3,25	1	1,77	1	1,55	1	1,52	1
	2	1,77	1	1,51	1	1,55	1	1,55	1
	3	1,56	1	1,56	1	1,55	1	1,55	1
3 Фанера. Сжатие поперек волокон ($D_1 < D_2$)	0,5	6,73	6	3,17	5	2,28	4	2,08	4
	1	3,36	3	1,60	2	1,14	2	1,04	2
	2	2,00	1	0,801	1	0,571	1	0,521	1
	3	1,12	1	0,352	1	0,454	1	0,437	1

1) Isotropic material; 2) plywood. Compression along the fibers; 3) plywood. Compression across the fibers.

Let us pay our attention to the case where the forces compressing the plate act along the axis of symmetry ($\eta = b/2$), and give the formulas for the coefficient k.* The expressions for this coefficient depend on the roots of the equation

$$
D_2 u^4 - 2D_3 u^2 + D_1 = 0 \tag{107.5}
$$

and are different according to which of the three possible cases of the roots is given.

For a plate with four sides supported we obtain:*

Case 1 [the roots of Eq. (105.7) are real and unequal],

$$
k = \frac{u_1^2 - u_2^2}{2} \min \left(\frac{m}{u_1 \operatorname{th} \frac{m\pi u_2 d}{2} - u_2 \operatorname{th} \frac{m\pi u_1 d}{2}} \right). \tag{107.6}
$$

Case 2 (the roots are real and equal),

$$
k = u \min \left(\frac{m}{\operatorname{th} \frac{m\pi u d}{2} + \frac{m\pi u d}{2 \operatorname{ch}^2 \frac{m\pi u d}{2}}} \right). \tag{107.7}
$$

Case 3 (the roots are complex)

$$
k = uv \min \left(m \frac{\operatorname{ch} m\pi u d + \cos m\pi v d}{v \operatorname{sh} m\pi u d + u \sin m\pi v d} \right). \tag{107.8}
$$

Here and in the following $d = b/a$.

From Formula (107.7) for $u = 1$ we obtain the value of the coefficient k found by S.P. Timoshenko for an isotropic plate with supported sides, compressed along the axis of symmetry.**

For a plate whose sides acted upon by the forces are supported, and the two other sides fixed we obtain the following formulas for k.

Case 1:

$$
k = \frac{u_1^2 - u_2^2}{2} \min \times
$$

$$
\times \left[m \frac{u_1 \operatorname{th} \frac{m\pi u_1 d}{2} - u_2 \operatorname{th} \frac{m\pi u_2 d}{2}}{-\frac{2u_1 u_2}{\operatorname{ch} \frac{m\pi u_1 d}{2} \operatorname{ch} \frac{m\pi u_2 d}{2}} + 2u_1 u_2 + (u_1^2 + u_2^2) \operatorname{th} \frac{m\pi u_1 d}{2} \operatorname{th} \frac{m\pi u_2 d}{2}} \right]. \tag{107.9}
$$

Case 2:

$$
k = \frac{u}{2} \min \left[m \frac{\operatorname{sh} m\pi u d + m\pi u d}{\operatorname{sh}^2 \frac{m\pi u d}{2} - \left(\frac{m\pi u d}{2} \right)^2} \right]. \tag{107.10}
$$

Case 3:

$$
k = \frac{1}{2} uv \min \left(m \frac{v \operatorname{sh} m\pi u d + u \sin m\pi v d}{v^2 \operatorname{sh}^2 \frac{m\pi u d}{2} + u^2 \sin^2 \frac{m\pi v d}{2}} \right). \tag{107.11}
$$

The solution of the problem of the stability of an isotropic plate with two supported and two fixed sides if it is compressed by concentrated forces along the axis of symmetry was obtained by A.P. Filippov*** and, by another method, by A.I. Lur'ye.**** The results of A.P. Filippov and A.I. Lur'ye are obtained from Formula (107.10) for $u = 1$.

The problem for which m the expressions in the parentheses of Formulas (107.6)-(107.11) assume the minimum values for a given ratio d of the sides is solved by calculations. Apparently, it is impossible to determine beforehand the number of semiwaves for a plate with an arbitrary side ratio (excluding an infini·· strip).

Table 26 shows the values of the coefficient k and the corresponding values of m for a plywood plate with two supported and two fixed sides, compressed along the axis of symmetry $\eta = b/2$).

TABLE 26

The Values of the Coefficient k for a Plywood Plate with Two Supported and Two Fixed Sides Compressed Along the Axis of Symmetry

d	1 Сжатие вдоль волокон		2 Сжатие поперек волокон	
	k	m	k	m
0,5	3,37	2	3,22	6
1	1,69	1	1,61	3
2	1,55	1	[illegible]	2
3	1,55	1	0,537	1

1) Compression along the fibers;
2) compression across the fibers.

Making the value of d in Formulas (107.6)-(107.11) infinitely large we obtain in the limit the coefficient k for an infinite strip with supported sides, compressed by two forces. The formula from which the critical force for an infinite band will be determined (Fig. 203) has the form:

$$\lambda_{\text{кр}} = \frac{4\pi\sqrt{D_1D_2}}{a}\cdot\frac{u_1+u_2}{2}, \quad m=1 \tag{107.12}$$

[in the case of unequal real roots of Eq. (107.5)] or

$$\lambda_{\text{кр}} = \frac{4\pi\sqrt{D_1D_2}}{a}u, \quad m=1 \tag{107.13}$$

(for equal or complex roots).

The critical for an isotropic strip ($D_1 = D_2 = D$, $u = 1$) was ·· nd by Sommerfeld.*

The tables presented show that a plywood plate compressed across the fibers, i.e., in a direction for which the reduced Young's modulus is minimum will tend, after having lost its stability, to form a number of waves in the direction of the line of action of the forces; the number of waves is the greater the

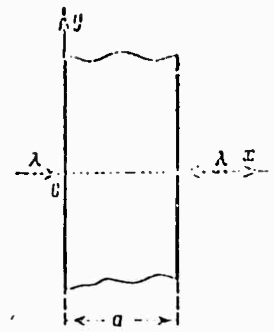

Fig. 203

longer the plate, i.e., the smaller *d*. A plate which is compressed along the fibers (in the direction of maximum reduced Young's modulus) with the same size forms a smaller numbers of waves, and for $d \geq 0.5$ one or two semiwaves.

§108. THE CASE OF FOUR FORCES

Let us consider a rectangular orthotropic plate which is deformed by four forces applied to two sides, as shown in Fig. 204. The loaded sides are assumed to be supported, the sidew without load may be fixed in an arbitrary manner or free.*

Let the values of the acting forces be equal to λ and λ*P*. The critical value of λ will be determined from the equation

$$\begin{vmatrix} \delta_{11} - \frac{1}{\lambda\gamma^2}, & \delta_{21} \\ \delta_{12}, & \delta_{22} - \frac{1}{\lambda P\gamma^2} \end{vmatrix} = 0, \qquad (108.1)$$

from which two series of values are obtained, after some straightforward transformations:

$$\lambda' = \frac{4\pi\sqrt{D_1D_2}}{a} \cdot \frac{m}{\frac{g_{11}+Pg_{22}}{2} + \sqrt{\left(\frac{g_{11}-Pg_{22}}{2}\right)^2 + Pg_{12}g_{21}}}; \qquad (108.2)$$

$$\lambda'' = \frac{4\pi\sqrt{D_1D_2}}{a} \cdot \frac{m}{\frac{g_{11}+Pg_{22}}{2} - \sqrt{\left(\frac{g_{11}-Pg_{22}}{2}\right)^2 + Pg_{12}g_{21}}} \qquad (108.3)$$

$$(m = 1, 2, 3, \ldots).$$

The smallest nonvanishing value of these two series must be selected; the work becomes considerably simpler if there are tables of the influence functions for the given material and the given fixing.

We shall pay our attention to two special cases.

Case I. All forces are compressive, equal and applied at symmetric points with ordinates $y = n - \eta$ and $y = \eta$ (Fig. 205), all four sides are supported.

Formulas (108.2) and (108.3) are simplified and assume the form:

$$\lambda' = \frac{4\pi\sqrt{D_1D_2}}{a}\cdot\frac{m}{g_{11}+g_{12}}; \tag{108.4}$$

$$\lambda'' = \frac{4\pi\sqrt{D_1D_2}}{a}\cdot\frac{m}{g_{11}-g_{12}}. \tag{108.5}$$

To the values of λ' corresponds a symmetric shape of the bent midsurface of the plate (Fig. 206a, where the approximate form of the surface for m = 1 is shown). An antisymmetric or obliquely symmetric shape of the bent surface corresponds to the values of λ" (see Fig. 206b, where also m = 1 is assumed).

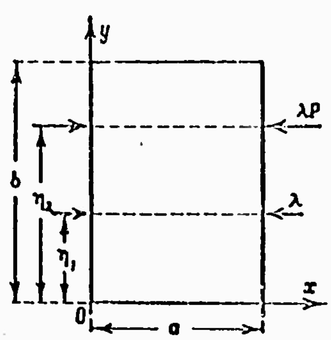

Fig. 204

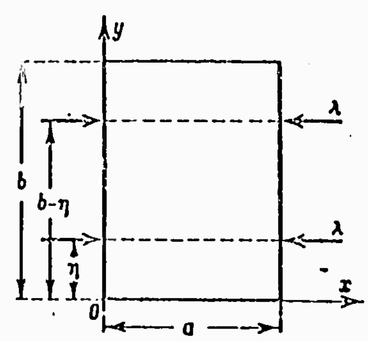

Fig. 205

It is impossible to tell beforehand to which of the two series of λ the critical value belongs and which m corresponds to it; the question is answered by calculations.

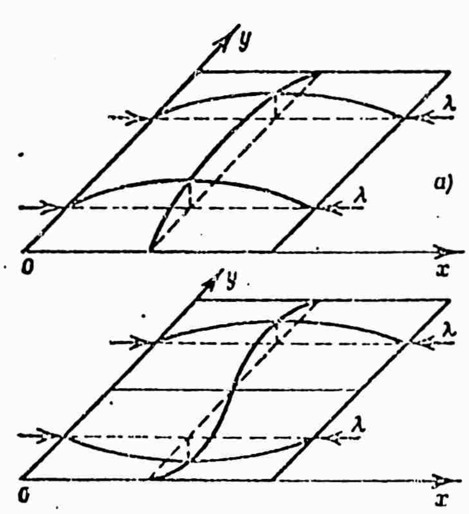

Fig. 206

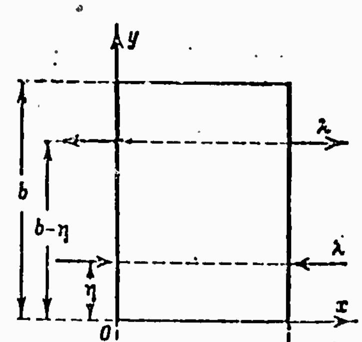

Fig. 207

The formula for the critical value λ_{kr} will be represented in the following form, as in the case of two forces,

$$\lambda_{kr} = \frac{4\pi \sqrt{D_1 D_2}}{a} k. \tag{108.6}$$

Table 27 gives the values of the coefficient k for an isotropic plate with four sides supported and for the plywood plate considered in §67, also with four sides supported. The coefficient k and the corresponding m are found with the help of Tables 15-17 (see §78). The calculations show that in all cases considered, except for two, the bent plate surface after the plate has lost its stability must be symmetric with respect to the line $y = b/2$. Exceptions to this rule which are labeled by asterisks in Table 27 may exist, by insufficient accuracy of the tables for the g functions or by an inaccuracy in the determination of the elastic constants of plywood.

TABLE 27

The Values of the Coefficient k for a Supported Plate Compressed by Four Forces

	d	$\eta = \frac{b}{8}$		$\eta = \frac{b}{4}$		$\eta = \frac{3b}{8}$	
		k	m	k	m	k	m
1 Изотропный материал	0,5 1 2 3	7,96 4,07 2,01 1,47	3 2 1 1	2,99 1,50 1,03 1,01	2 1 1 1	1,82 0,911 0,689 0,767	2 1 1 1
2 Фанера. Сжатие вдоль волокон	0,5 1 2 3	6,27 3,13 1,77* 1,56	2 1 1 1	2,11 1,52 1,52* 1,56	1 1 1 1	1,27 1,00 1,38 1,55	1 1 1 1
3 Фанера. Сжатие поперек волокон	0,5 1 2 3	5,96 2,99 1,50 1,02	5 2 1 1	2,08 1,01 0,521 0,437	4 2 1 1	1,26 0,631 0,315 0,278	4 2 1 1

1) Isotropic material; 2) plywood. Compression along the fibers; 3) plywood. Compression across the fibers.

Case 2. All forces acting on the plate are equal and applied at symmetric points; two forces are compressive and two stretching (Fig. 207).

The critical value λ_{kr} will be determined from the Formula (108.6) where

$$k = \min\left(\frac{m}{\sqrt{g_{11}^2 - g_{12}^2}}\right). \tag{108.7}$$

The formula for the critical value λ_{kr} may also be written in another form:

$$k = \min \sqrt{\lambda'\lambda''}, \tag{108.8}$$

where λ' and λ" are determined by Formulas (108.4) and (108.5) of the preceding case.

Manu- script Page No.	[Footnotes]
435	The solution is to be found in our book "Ustoychivost' anizotropnykh plastinok" [The Stability of Anisotropic Plates], Gostekhizdat [State Publishing House of Theoretical and Technical Literature], 1943. §23, pages 60-64.
436	See Formula (98.11).
437	See (97.6)-(97.8).
439	The solution of this problem is found in our mentioned work "Ustoychivost' anizotropnykh plastinok," §26, pages 22-75.
440*	I.e., of that plate for which the reduced Young's moduli and the Poisson coefficients in bending have numerical values (67.9) or (67.13) (rather than (67.16) as was assumed in §§103-105).
440**	See our work "K raschetu na ustoychivost' ortotropnoy plastinki" [On the Calculation of the Stability of an Orthotropic Plate], Vestnik inzhenerov i tekhnikov [Herald of Engineers and Technicians], 1941, No. 1.
441*	Minimum values of the expressions not equal to zero and corresponding to the integer *m* are implied.
441**	Timoshenko, S.P., Ustoychivost' uprugikh sistem [The Stability of Elastic Systems], Gostekhizdat, 1946, §68, pages 326-327.
441***	Filippov, A.P., Ustoychivost' pryamougol'nykh plastinok szhatykh sosredotochennymi silami [The Stability of Rectangular Plates Compressed by Concentrated Forces], Izv. AN SSSR [Bulletin of the Academy of Sciences of the USSR], OMYeN [Department of Mathematics and Natural Sciences] 1933, No. 7.
441****	Lur'ye, A.I., Ustoychivost' plastinki szhatoy sosredotochennymi silami [The Stability of a Plate Compressed by Concentrated Forces]. Trudy Leningr. industrial'nogo in-ta [Transactions of the Leningrad Industrial Institute], 1939, No. 3, Department of Physical and Mathematical Sciences, No. 1. In another work A.I. Lur'ye gave a method of solving also the more general problem of the instability of an

isotropic plate with one axis of symmetry in the compression along the axis of symmetry.

442 Sommerfeld, Ueber die Knicksicherheit der Stege von Walzenprofilen [On the Buckling Strength of Webs in Roller Profiles], Zeitschr. f. Math. u. Physik [Journal of Mathematics and Physics], Vol. 54, 1906.

443 As to the solution of this problem, see our work "Ustoychivost' anizotropnykh plastinok," §26, pages 75-77.

[Transliterated Symbols]

435 кр = kr = kriticheskiy = critical

Chapter 16

THE STABILITY OF PLATES REINFORCED BY STIFFENING RIBS

§109. THE STABILITY OF A RECTANGULAR ORTHOTROPIC PLATE WITH LONGITUDINAL RIBS IN THE COMPRESSION IN THE MAIN DIRECTION

A plate with parallel stiffening ribs placed close to each other may be considered orthotropic and homogeneous; it was shown in §65 how the rigidities of such plate are determined. If, however, the plate is reinforced by a small number of ribs, as, e.g., one or two, the consideration as a homogeneous and orthotropic plate is not well founded, but the combined effects of the plate and the ribs must be taken into account. In this chapter we consider problems of the stability of rectangular orthotropic plates with longitudinal and transverse ribs, compressed by a normal load which is uniformly distributed along the sides, or by concentrated forces applied to the rib ends.

Let us consider first a homogeneous orthotropic rectangular plate with principal directions parallel to the directions of the sides, which is reinforced by several parallel stiffening ribs. Let a compressive load p which is distributed along the sides perpendicular to the ribs act on the plate. It is assumed that the sides on which the rib ends are fastened are supported, the other sides are fastened in an arbitrary manner, the rib ends are also supported and fixed with respect to revolution about their axes. Moreover, we assume that the ribs are rigidly fastened on the plate, and their cross sections have symmetry axes perpendicular to the undeformed midsurface of the plate. The critical value of the compressive load p_{kr} at which the plate which is reinforced by ribs loses its stability must be determined.

Let us direct the coordinate axes along the plate sides (the x-axis is parallel to the ribs, Fig. 208) and introduce the denotations: $D_1, D_2, D_k, \nu_1, \nu_2$ are the rigidities and Poisson coefficients of the plate; E_1 is Young's modulus of the plate for directions parallel to the ribs; a, b are the side lengths of the plate; N is the number of ribs; $E^{(1)}, E^{(2)}, \ldots, E^{(N)}$ are Young's moduli of the ribs; $EJ_1, EJ_2, \ldots, EJ_N$ are the bending strengths of the ribs in planes perpendicular to the plate plane; $C_1, C_2, \ldots, C_N$ are the torsional rigidities of the ribs; $F_1, F_2, \ldots, F_N$ are the cross-sectional areas of the ribs; $\eta_1, \eta_2, \ldots \eta_N$ are the distances of the ribs from the lower edge of the plate;

$$
\left.\begin{array}{l}
\delta_k = \dfrac{E^{(k)}F_k}{bE_1}, \quad \gamma_k = \dfrac{EJ_k}{b\sqrt{D_1D_2}}, \quad \chi_k = \dfrac{C_k}{b\sqrt{D_1D_2}}; \\
c = \dfrac{a}{b}, \quad \beta = \dfrac{m\pi}{a} \ (m = 1,\ 2,\ 3,\ \ldots).
\end{array}\right\} \tag{109.1}
$$

An approximate solution of the problem of the stability of an isotropic plate with longitudinal ribs was for the first time obtained by S.P. Timoshenko with the help of the energetic method.

Using the energetic method it is easy to obtain an approximate solution of the problem also for an anisotropic plate in an analogous manner; we shall pay our attention to this approximate solution in this and the following sections.

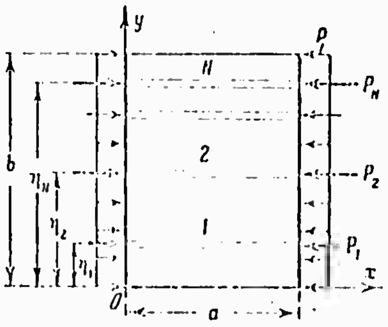

Fig. 208

Until it loses its stability the plate is in a plane stressed state, and the ribs are in the state of simple compression along their axes. We shall easily determine the compressive forces P_k applied to the rib ends if we assume that the relative compression of the plate ε_x is equal to the relative compression ε_{xk} of any rib (we neglect the rib width).

Obviously,

$$
\varepsilon_x = \frac{p}{E_1}, \quad \varepsilon_{xk} = \frac{P_k}{E^{(k)}F_k}, \tag{109.2}
$$

but with $\varepsilon_x = \varepsilon_{xk}$ we have,

$$
P_k = p\frac{E^{(k)}F_k}{E_1}. \tag{109.3}
$$

Approaching the solution of the problem we assume that the whole system has undergone small deviations from the plane shape: the plate has been bent, and the ribs have been bent and twisted. Designating $w(x, y)$, $W_k(x)$, the deflections of the plate and the ribs and the angles of rib torsion we have by virtue of the rigid connection of the plate with the ribs:

$$W_k(x) = w(x, \eta_k), \quad \theta_k(x) = \left(\frac{\partial w}{\partial y}\right)_{y=\eta_k}. \tag{109.4}$$

In this case the potential energy of the system is increased by the quantity

$$\begin{aligned} V_{\text{изг}} = \frac{1}{2}\int_0^a\int_0^b \Big[& D_1\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_1\nu_2 \frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} + D_2\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \\ & + 4D_k\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\Big] dx\,dy + \\ & + \frac{1}{2}\sum_{k=1}^{N}\left[EJ_k\int_0^a (W_k'')^2\,dx + C_k\int_0^a (\theta_k')^2\,dx\right] \end{aligned} \tag{109.5}$$

and is decreased on account of the work performed by the external forces:

$$A = \frac{p}{2}\int_0^a\int_0^b\left(\frac{\partial w}{\partial x}\right)^2 dx\,dy + \frac{1}{2}\sum_{k=1}^{N} P_k\int_0^a (W_k')^2\,dx. \tag{109.6}$$

The conditions on the supported (loaded) sides of the plate and at the rib ends are satisfied if we put:

$$\left.\begin{aligned} w &= f(y)\sin\beta x, \\ W_k = f(\eta_k)\sin\beta x, &\quad \theta_k = f'(\eta_k)\sin\beta x. \end{aligned}\right\} \tag{109.7}$$

From equation

$$V_{\text{изг}} = A \tag{109.8}$$

we obtain an expression for the load p which may give rise to a distortion of the system:

$$p = \sqrt{D_1D_2}\,\frac{\int_0^b \Phi\,dy + b\sum_{k=1}^{N}[\gamma_k\beta^4 f^2(\eta_k) + \varkappa_k\beta^2 f'^2(\eta_k)}{\beta^2\left[\int_0^b f^2\,dy + b\sum_{k=1}^{N}\delta_k f^2(\eta_k)\right]}, \tag{109.9}$$

where

$$\Phi = \sqrt{\frac{D_2}{D_1}}f''^2 - 2\beta^2\nu_2\sqrt{\frac{D_1}{D_2}}ff'' + \beta^4\sqrt{\frac{D_1}{D_2}}f^2 + \frac{4D_k}{\sqrt{D_1D_2}}\beta^2 f'^2.$$

The further procedure may be as follows: we may choose an expression for the function $f(y)$ in the form of a sum with undetermined coefficients

$$f = \sum_n A_{mn} f_n(y), \tag{109.10}$$

where f_n are continuous functions fulfilling the conditions on the sides $y=0$, $y = b$; substitute this expression into Formula (109.9) and determine the minimum of p as a function of the coefficients A_{mn}. The problem of finding the minimum of the fraction (109.9) is equivalent to the problem of finding the minimum of

the expression*

$$U = \sqrt{D_1 D_2}\left\{\int_0^b \left(\sqrt{\frac{D_2}{D_1}} f''^2 - 2\beta^2\nu_2 \sqrt{\frac{D_1}{D_2}} f f'' + \beta^4 \sqrt{\frac{D_1}{D_2}} f^2 + \right.\right.$$
$$\left.\left. + \frac{4D_k}{\sqrt{D_1 D_2}} \beta^2 f'^2\right) dy + b \sum_{k=1}^{N} [\gamma_k \beta^4 f^2(\eta_k) + \varkappa_k \beta^2 f'^2(\eta_k)]\right\} - \tag{109.11}$$
$$- p\beta^2 \left[\int_0^b f^2\, dy + b \sum_{k=1}^{N} \delta_k f^2(\eta_k)\right].$$

Substituting here Expression (109.10) and carrying out the integration we obtain U as a homogeneous quadratic function of the coefficients A_{mn}. Further we seek the minimum of this function for which purpose we differentiate U with respect to the A_{mn} and put the derivatives equal to zero:

$$\frac{\partial U}{\partial A_{m1}} = 0, \quad \frac{\partial U}{\partial A_{m2}} = 0, \ \ldots \tag{109.12}$$

Putting the determinant of the homogeneous system (109.12) equal to zero we obtain the equation

$$\Delta(p) = 0, \tag{109.13}$$

whose minimum root will also yield the value of the critical load.

Considering a plate with ribs which are symmetrically distributed with respect to the line $y = b/2$ we may assume in first approximation:

a) for a plate with supported sides a

$$f = A_{mn} \sin\frac{n\pi y}{b}; \tag{109.14}$$

b) for a plate with fixed sides a

$$f = A_{mn}\left(1 - \cos\frac{2n\pi y}{b}\right); \tag{109.15}$$

c) for a plate with free sides a

$$f = A_{mn}\left(\nu_1 m^2 + 4n^2c^2 - \nu_1 m^2 \cos\frac{2n\pi y}{b}\right). \tag{109.16}$$

If, however, the ribs are asymmetrically distributed the bent surface will, in general, be asymmetric with respect to the line $y = b/2$, after the stability has been lost, for which reason the first approximation may yield a value of the critical load whose accuracy is not sufficient for practical purposes. In these cases, the expression for f must be assumed in the form of a sum of several terms, in accordance with the number of ribs.

As is shown by S.P. Timoshenko, the first approximation yields quite satisfactory results for an isotropic plate which is reinforced by some equidistant ribs of equal rigidity, in other cases the first approximation yields, according to S.P. Timoshenko,

quite satisfactory results for long isotropic plates, for which $c > 2$.*

§110. THE CASE OF A SINGLE LONGITUDINAL RIB

A plate which is reinforced by a stiffening rib in the direction of the axis of symmetry is compressed by a normal load which is uniformly distributed along the sides perpendicular to the rib (Fig. 209).

We shall introduce the designations: *E*, *EJ*, *C*, *F* are Young's modulus, the bending strength, the torsional rigidity and the area of the rib cross section;

$$\delta = \frac{EF}{bE_1}, \quad \gamma = \frac{EJ}{b\sqrt{D_1 D_2}}, \quad \varkappa = \frac{C}{b\sqrt{D_1 D_2}}; \tag{110.1}$$

the designations of the quantities referring to the plate will be left as before.

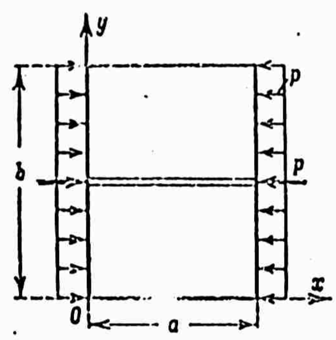

Fig. 209

Let us consider two cases of side fixing.

1. The sides $y = 0$ and $y = b$ are supported.

Putting

$$f = A_{mn} \sin \frac{n\pi y}{b}, \quad w = A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \tag{110.2}$$

we obtain on the basis of the general formula (109.9):

$$p = \frac{\pi^2 \sqrt{D_1 D_2}}{b^2 (1 + 2\delta)} \left[\left(\sqrt{\frac{D_1}{D_2}} + 2\gamma \right) \left(\frac{m}{c} \right)^2 + \frac{2D_3}{\sqrt{D_1 D_2}} n^2 + \sqrt{\frac{D_2}{D_1}} n^4 \left(\frac{c}{m} \right)^2 \right] \tag{110.3}$$

(for $n = 1, 3, 5, \ldots$) and

$$p = \frac{\pi^2 \sqrt{D_1 D_2}}{b^2} \left[\sqrt{\frac{D_1}{D_2}} \left(\frac{m}{c} \right)^2 + 2 \left(\frac{D_3}{\sqrt{D_1 D_2}} + \varkappa \right) n^2 + \sqrt{\frac{D_2}{D_1}} n^4 \left(\frac{c}{m} \right)^2 \right] \tag{110.4}$$

(for $n = 2, 4, 6, \ldots$).

The critical load p_{kr} will be obtained as the minimum of all values determined by Formulas (110.3) and (110.4). Obviously, the minimum value of the series (110.3) will be obtained for $n = 1$ to which bending in one semiwave in the direction of the y-axis corresponds, and the minimum value of the series (110.4) will be obtained for $n = 2$ to which corresponds bending in two semiwaves in the direction of the y-axis. Which of the two formulas must be used cannot be told beforehand; that depends on the value of the ratios D_1/D_2, $D_3/\sqrt{D_1 D_2}$, $\varkappa$, δ, γ.

Let us introduce the brief denotations:

$$\left.\begin{aligned}
&r_1 = \sqrt[4]{\frac{D_1}{D_2} + 2\gamma\sqrt{\frac{D_1}{D_3}}}, \quad r_2 = \sqrt[4]{\frac{D_1}{D_2}},\\
&k_{m1} = \frac{1}{1+2\delta}\left[\left(\sqrt{\frac{D_1}{D_3}} + 2\gamma\right)\left(\frac{m}{c}\right)^2 + \frac{2D_3}{\sqrt{D_1 D_2}} + \sqrt{\frac{D_2}{D_1}}\left(\frac{c}{m}\right)^2\right],\\
&k_{m2} = \sqrt{\frac{D_1}{D_2}}\left(\frac{m}{c}\right)^2 + 8\left(\frac{D_3}{\sqrt{D_1 D_2}} + \varkappa\right) + 16\sqrt{\frac{D_2}{D_1}}\left(\frac{c}{m}\right)^2,\\
&\bar{k}_1 = \frac{2}{1+2\delta}\left(\sqrt{1 + 2\gamma\sqrt{\frac{D_2}{D_1}}} + \frac{D_3}{\sqrt{D_1 D_2}}\right),\\
&\bar{k}_2 = 8\left(1 + \varkappa + \frac{D_3}{\sqrt{D_1 D_2}}\right).
\end{aligned}\right\} \qquad (110.5)$$

We shall represent the formula for the critical load in the form

$$p_{\text{кр}} = \frac{\pi^2\sqrt{D_1 D_2}}{b^2}\,k. \qquad (110.6)$$

The results of the rather elementary investigation may boil down to the following statements:

1) if the ratios of the plate sides satisfy the condition

$$c = m' r_1, \qquad (110.7)$$

where m' is an integer, then $m = m'$ and

$$k = \bar{k}_1; \qquad (110.8)$$

2) if the side ratio satisfies the condition

$$c = 0{,}5\, m' r_2, \qquad (110.9)$$

then $m = m'$ and

$$k = \bar{k}_2; \qquad (110.10)$$

3) for large side ratios $c \geq 3$

$$\left.\begin{aligned}
&k = \bar{k}_1, \text{ if } \bar{k}_1 < \bar{k}_2,\\
&k = \bar{k}_2, \text{ if } \bar{k}_1 > \bar{k}_2.
\end{aligned}\right\} \qquad (110.11)$$

It is convenient to arrange these results in a table in which the values of c and the corresponding values of k and m (for $n = 1$

and n = 2) are given.

Survey Table of the Coefficients k for the Case of Supported Sides y = 0 and y = b

n	c	m	k	n	c	m	k
1	$0<c<r_1$	1	k_{11}	2	$0<c<\frac{1}{2}r_2$	1	k_{12}
	$c=r_1$	1	$\bar{k}_1$		$c=\frac{1}{2}r_2$	1	$\bar{k}_2$
	$r_1<c<\sqrt{2}r_1$	1	k_{11}		$\frac{1}{2}r_2<c<\frac{\sqrt{2}}{2}r_2$	1	k_{12}
	$\sqrt{2}r_1<c<2r_1$	2	k_{21}		$\frac{\sqrt{2}}{2}r_2<c<r_2$	2	k_{22}
	$c=2r_1$	2	$\bar{k}_1$		$c=r_2$	2	$\bar{k}_2$
	$2r_1<c<\sqrt{6}r_1$	2	k_{21}		$r_2<c<\frac{\sqrt{6}}{2}r_2$	2	k_{22}
	$\sqrt{6}r_1<c<3r_1$	3	k_{31}		$\frac{\sqrt{6}}{2}r_2<c<\frac{3}{2}r_2$	3	k_{32}
	$c=3r_1$	3	$\bar{k}_1$		$c=\frac{3}{2}r_2$	3	$\bar{k}_2$
	[1] и т. д.				[1] и т. д.		

1) etc.

Let us assume that we know κ, γ and the ratio of the rigidities D_1/D_2 and the critical load for a plate with given side ratio c must be found. In order to determine the corresponding k we establish at which place of the table there is the given ratio c for n = 1 and n = 2. Having the lines in which c is to be found we choose the corresponding value of this coefficient from the column for k; the lower of the two values found must be taken (which of the two values will be lower depends on the ratio of the plate parameters).

Let, e.g., the values $r_1 = 1{,}5$, $r_2 = 1{,}2$ be found for a square plate (c = 1) and be determined k. The given ratio of the sides is found in the 1st line of the table, corresponding to n = 1 and in the 4th line of the table for n = 2. Consequently, one of the values may be chosen: $k = k_{11}$ or $k = k_{22}$, i.e., that which is smaller. According to the ratio of the plate parameters one of the two possibilities may be realized: 1) the plate forms one semiwave in the direction of the load (x-axis) and one direction in the y-axis; $k = k_{11}$; 2) the plate forms two semiwaves in the direction of the x-axis and two semiwaves in the direction of the y-axis; $k = k_{22}$.*

2. The sides y = 0 and y = b are fixed.

Putting

$$
\left.\begin{aligned}
f &= A_{mn}\left(1 - \cos\frac{2n\pi y}{b}\right), \\
w &= A_{mn}\sin\frac{m\pi x}{a}\left(1 - \cos\frac{2n\pi y}{b}\right),
\end{aligned}\right\} \tag{110.12}
$$

we obtain on the basis of the general formula (109.9):

$$
\begin{aligned}
p = \frac{\pi^2\sqrt{D_1D_2}}{b^3(1+2{,}667\delta)}\Bigg[\left(\sqrt{\frac{D_1}{D_2}} + 2{,}667\gamma\right)\left(\frac{m}{c}\right)^2 &+ 2{,}667\,\frac{D_3}{\sqrt{D_1D_2}}\,n^2 + \\
&+ 5{,}333\sqrt{\frac{D_2}{D_1}}\,n^4\left(\frac{c}{m}\right)^2\Bigg]
\end{aligned} \tag{110.13}
$$

(for n = 1, 3, 5, ...);

$$
\begin{aligned}
p = \frac{\pi^2\sqrt{D_1D_2}}{b^2}\Bigg[\sqrt{\frac{D_1}{D_2}}\left(\frac{m}{c}\right)^2 &+ 2{,}667\,\frac{D_3}{\sqrt{D_1D_2}}\,n^2 + \\
&+ 5{,}333\sqrt{\frac{D_2}{D_1}}\,n^4\left(\frac{c}{m}\right)^2\Bigg]
\end{aligned} \tag{110.14}
$$

(for n = 2, 4, 6, ...).

Obviously, the minimum value of p of the first formula will be realized at $n = 1$, and of the second formula, at $n = 2$.

The formula for the critical load will be represented in the form (110.6). We shall introduce the brief denotations:

$$
\left.\begin{aligned}
s_1 &= \sqrt[4]{\frac{3D_1}{D_2} + 8\gamma\sqrt{\frac{D_1}{D_2}}}, \quad s_2 = \sqrt[4]{\frac{3D_1}{D_3}}, \\
k'_{m1} &= \frac{1}{1+2{,}667\delta}\Bigg[\left(\sqrt{\frac{D_1}{D_2}} + 2{,}667\gamma\right)\left(\frac{m}{c}\right)^2 + \\
&\qquad + 2{,}667\,\frac{D_3}{\sqrt{D_1D_2}} + 5{,}333\sqrt{\frac{D_2}{D_1}}\left(\frac{c}{m}\right)^2\Bigg], \\
k'_{m2} &= \sqrt{\frac{D_1}{D_2}}\left(\frac{m}{c}\right)^2 + 10{,}667\,\frac{D_3}{\sqrt{D_1D_2}} + 85{,}333\sqrt{\frac{D_2}{D_1}}\left(\frac{c}{m}\right)^2, \\
\bar{k}'_1 &= \frac{2{,}667}{1+2{,}667\delta}\left(\sqrt{3 + 8\gamma\sqrt{\frac{D_2}{D_1}}} + \frac{D_3}{\sqrt{D_1D_2}}\right), \\
\bar{k}'_2 &= 10{,}667\left(\sqrt{3} + \frac{D_3}{\sqrt{D_1D_2}}\right).
\end{aligned}\right\} \tag{110.15}
$$

The results boil down to the following:

1) if the side ratio c satisfies the condition

$$
c = \frac{1}{2}m's_1, \tag{110.16}
$$

where m' is an integer, then $m = m'$,

$$
k = \bar{k}'_1; \tag{110.17}
$$

2) if the side ratio is equal to

$$
c = \frac{1}{4}m's_2, \tag{110.18}
$$

then $m = m'$,

$$k = \overline{k}'_2; \tag{110.19}$$

3) for great values of the side ratios c we may choose:

$$\left.\begin{array}{l} k = \overline{k}'_1, \text{ if } \overline{k}'_1 < \overline{k}'_2, \\ k = \overline{k}'_2, \text{ if } \overline{k}'_1 > \overline{k}'_2. \end{array}\right\} \tag{110.20}$$

Survey Table of the Coefficient k for the Case of Fixed Sides $y = 0$ and $y = b$

n	c	m	k	n	c	m	k
1	$0 < c < \frac{1}{2} s_1$	1	k'_{11}	2	$0 < c < \frac{1}{4} s_2$	1	k'_{12}
	$c = \frac{1}{2} s_1$	1	$\overline{k}'_1$		$c = \frac{1}{4} s_2$	1	$\overline{k}'_2$
	$\frac{1}{2} s_1 < c < \frac{\sqrt{2}}{2} s_1$	1	k'_{11}		$\frac{1}{4} s_2 < c < \frac{\sqrt{2}}{4} s_2$	1	k'_{12}
	$\frac{\sqrt{2}}{2} s_1 < c < s_1$	2	k'_{21}		$\frac{\sqrt{2}}{4} s_2 < c < \frac{1}{2} s_2$	2	k'_{22}
	$c = s_1$	2	$\overline{k}'_1$		$c = \frac{1}{2} s_2$	2	$\overline{k}'_2$
	$s_1 < c < \frac{\sqrt{6}}{2} s_1$	2	k'_{21}		$\frac{1}{2} s_2 < c < \frac{\sqrt{6}}{4} s_2$	2	k'_{22}
	$\frac{\sqrt{6}}{2} s_1 < c < \frac{3}{2} s_1$	3	k'_{31}		$\frac{\sqrt{6}}{4} s_2 < c < \frac{3}{4} s_2$	3	k'_{32}
	$c = \frac{3}{2} s_1$	3	$\overline{k}'_1$		$c = \frac{3}{4} s_2$	3	$\overline{k}'_2$
	1 и т. д.				1 и т. д.		

1) etc.

This table must be used as in the case of supported sides.

§111. THE STABILITY OF A PLATE WITH TRANSVERSE RIBS COMPRESSED BY A UNIFORMLY DISTRIBUTED LOAD

Let us consider a rectangular homogeneous orthotropic plate reinforced by parallel stiffening ribs, compressed by a load p which is distributed uniformly along the sides parallel to the ribs. It is assumed that the sides along which the compressive forces are distributed are supported, and the other ones are fixed in an arbitrary manner. The ribs are rigidly fixed to the plate, their cross sections have axes of symmetry perpendicular to the original midsurface of the plate.

The critical value of the compressive is to be determined.

Let us direct the x and y axes along the plate sides (Fig. 210). We shall retain the denotations adopted in §109, and introduce new ones: ξ_1, ξ_2, ..., ξ_N – the distances of the ribs from

one of the loaded sides,

$$\left.\begin{aligned}\gamma_k' &= \frac{EJ_k}{a\sqrt{D_1D_2}},\\ \varkappa_k' &= \frac{C_k}{a\sqrt{D_1D_2}}.\end{aligned}\right\} \tag{111.1}$$

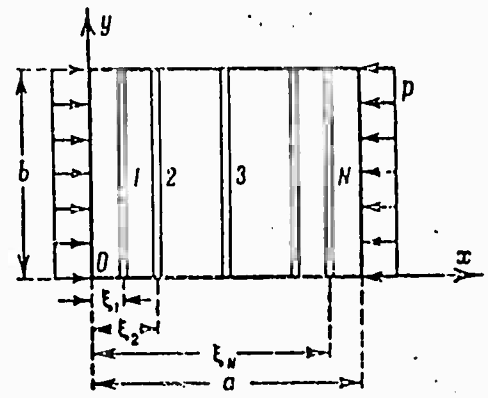

Fig. 210

The problem of the stability of an isotropic plate with transverse ribs was solved approximately with the help of the energetic method by S.P. Timoshenko.* An approximate solution of the same problem for an orthotropic plate is obtained analogously, also making use of the energetic method.

The potential energy of the bending of a system which has experienced small deviations from the plane shape is equal to

$$\begin{aligned}V_{\text{изг}} = \frac{1}{2}\int_0^a\int_0^b\Big[D_1\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_1\nu_2\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} + D_2\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \\ + 4D_k\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\Big]dx\,dy + \\ + \frac{1}{2}\sum_{k=1}^{N}\left[EJ_k\int_0^b (W_k'')^2\,dy + C_k\int_0^b(\vartheta_k')^2\,dy\right],\end{aligned} \tag{111.2}$$

where

$$W_k = w(\xi_k,\ y),\quad \vartheta_k = \left(\frac{\partial w}{\partial x}\right)_{x=\xi_k}.$$

As in the case of longitudinal ribs we neglect the rib width.

The forces p in the distortion of the system will perform the work

$$A=\frac{p}{2}\int_0^a\int_0^b\left(\frac{\partial w}{\partial x}\right)^2 dx\,dy. \qquad (111.3)$$

For a plate with arbitrarily distributed ribs the expression for the deflection may be assumed in the form

$$w=\sum_m f_m(y)\sin\beta x \qquad (111.4)$$

and, moreover, determining p from the equation $V_{izg} = A$ the functions f_m must be chosen such that they satisfy the conditions on the sides $y = 0$ and $y = b$ and yield the minimum value for p.

Let us consider the case where the ribs are distributed symmetrically with respect to the axis of symmetry of the plate $x = a/2$, in which case the bending strengths and the torsional rigidities of the ribs which are at equal distances from the loaded sides are the same, i.e.,

$$EJ_N=EJ_1,\quad C_N=C_1;\quad EJ_{N-1}=EJ_2,\quad C_{N-1}=C_2 \text{ etc.}$$

In this case

$$\xi_N=a-\xi_1,\quad \xi_{N-1}=a-\xi_2,\ \ldots$$

We shall assume in first approximation that the section of the midsurface of the plate with a plane normal to the loaded sides is a sinusoidal curve with a certain number of semiwaves. According to S.P. Timoshenko, this assumption usually yields satisfactory accuracy in the case of an isotropic plate.*

Putting

$$w=f(y)\sin\beta x, \qquad (111.5)$$

we find

$$p=\sqrt{D_1D_2}\,\frac{\int_0^b\Phi\,dy+2\sum_{k=1}^{N}\int_0^b\Phi_{1k}\,dy}{\beta^2\int_0^b f^2\,dy}, \qquad (111.6)$$

where

$$\Phi=\sqrt{\frac{D_2}{D_1}}f''^2-2\beta^2\nu_2\sqrt{\frac{D_1}{D_2}}ff''+\beta^4\sqrt{\frac{D_1}{D_2}}f^2+\frac{4D_k}{\sqrt{D_1D_2}}\beta^2f'^2,$$

$$\Phi_{1k}=\gamma_k'f''^2\sin^2\frac{m\pi\xi_k}{a}+\varkappa_k'f'^2\beta^2\cos^2\frac{m\pi\xi_k}{a}.$$

Let us consider the cases of supported and fixed sides.

1. The sides $y = 0$ and $y = b$ are supported; the rib ends are supported and cannot be twisted.

Let us put

$$f = A_{mn} \sin \frac{n\pi y}{b}. \tag{111.7}$$

Substituting this function into Formula (111.6) we find:

$$p = \frac{\pi^2 \sqrt{D_1 D_2}}{b^2}\left[\sqrt{\frac{D_1}{D_2}}\left(\frac{m}{c}\right)^2 + 2n^2\left(\frac{D_3}{\sqrt{D_1 D_2}} + \sum_{k=1}^{N} \varkappa'_k \cos^2 \frac{m\pi\xi_k}{a}\right) + \right.$$
$$\left. + n^4\left(\frac{c}{m}\right)^2\left(\sqrt{\frac{D_2}{D_1}} + 2\sum_{k=1}^{N} \gamma'_k \sin^2 \frac{m\pi\xi_k}{a}\right)\right]. \tag{111.8}$$

Obviously, the minimum value of p will be obtained for $n = 1$; consequently,

$$p = \frac{\pi^2 \sqrt{D_1 D_2}}{b^2}\left[\sqrt{\frac{D_1}{D_2}}\left(\frac{m}{c}\right)^2 + 2\left(\frac{D_3}{\sqrt{D_1 D_2}} + \sum_{k=1}^{N} \varkappa'_k \cos^2 \frac{m\pi\xi_k}{a}\right) + \right.$$
$$\left. + \left(\frac{c}{m}\right)^2\left(\sqrt{\frac{D_2}{D_1}} + 2\sum_{k=1}^{N} \gamma'_k \sin^2 \frac{m\pi\xi_k}{a}\right)\right]. \tag{111.9}$$

To these conditions corresponds a bent surface of the form

$$w = A_{m1} \sin \frac{m\pi x}{a} \sin \frac{\pi y}{b}. \tag{111.10}$$

For a plate with given ratio c the critical load will be determined as the minimum of all loads determined by Formula (111.9). In order to finish this problem Expression (111.9) must be studied: the side ratios $c = c_{pr}$ for which the transition from m semiwaves to $m + 1$ semiwaves takes place, must be determined, the minimum favorable ratio c and the corresponding critical load must be found, etc. This investigation may be carried out in general form for an arbitrary number of ribs N in the same way as in the case of one longitudinal rib. As a result rather cumbersome formulas we shall not present here will be determined.

2. The sides $y = 0$ and $y = b$ are fixed; the rib ends are also fixed and cannot be twisted about the rib axes.

In this case we obtain in first approximation

$$p = \frac{\pi^2 \sqrt{D_1 D_2}}{b^2}\left[\sqrt{\frac{D_1}{D_2}}\left(\frac{m}{c}\right)^2 + 2{,}667\left(\frac{D_3}{\sqrt{D_1 D_2}} + \sum_{k=1}^{N} \varkappa'_k \cos^2 \frac{m\pi\xi_k}{a}\right) + \right.$$
$$\left. + 5{,}333\left(\frac{c}{m}\right)^2\left(\sqrt{\frac{D_2}{D_1}} + 2\sum_{k=1}^{N} \gamma'_k \sin^2 \frac{m\pi\xi_k}{a}\right)\right]; \tag{111.11}$$

$$w = A_{m1} \sin \frac{m\pi x}{a}\left(1 - \cos \frac{2\pi y}{b}\right). \tag{111.12}$$

instead of Formulas (111.9)-(111.10).

§112. THE CASE OF A SINGLE TRANSVERSE RIB

A plate which is reinforced by one rib along the axis of symmetry is compressed in the direction perpendicular to the rib (Fig. 211).

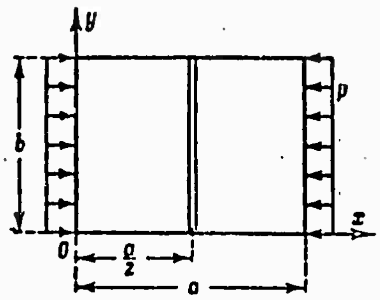

Fig. 211

Let us introduce the designations:

$$\left.\begin{aligned}\gamma' &= \frac{EJ}{a\sqrt{D_1D_2}};\\ \varkappa' &= \frac{C}{a\sqrt{D_1D_2}},\\ \alpha' &= \frac{EJ}{aD_2}.\end{aligned}\right\} \qquad (112.1)$$

Different formulas are obtained for odd and even m for the critical load on the basis of Formulas (111.9) or (111.11).

We shall choose two cases of fixing of the sides on which the rib ends are fastened (the loaded sides are assumed to be supported, as had been said already).

1. The sides $y = 0$ and $y = b$ are supported.

For an odd number of semiwaves m we obtain:

$$p = \frac{\pi^2\sqrt{D_1D_2}}{b^2}\left[\sqrt{\frac{D_1}{D_2}}\left(\frac{m}{c}\right)^2 + \frac{2D_3}{\sqrt{D_1D_2}} + \left(\frac{c}{m}\right)^2\left(\sqrt{\frac{D_2}{D_1}} + 2\gamma'\right)\right]. \qquad (112.2)$$

for an even number m

$$p = \frac{\pi^2\sqrt{D_1D_2}}{b^2}\left[\sqrt{\frac{D_1}{D_2}}\left(\frac{m}{c}\right)^2 + 2\left(\frac{D_3}{\sqrt{D_1D_2}} + \varkappa'\right) + \left(\frac{c}{m}\right)^2\sqrt{\frac{D_2}{D_1}}\right]. \qquad (112.3)$$

We shall represent the formula for the critical load, as before, in the form:

$$p_{\text{кр}} = \frac{\pi^2\sqrt{D_1D_2}}{b^2}k. \qquad (112.4)$$

The results of studying Expressions (112.2) and (112.3) boil down to the following.

If the side ratios $c = a/b$ satisfy the condition

$$c = \frac{m'}{\sqrt[4]{\frac{D_2}{D_1}+2\gamma'\sqrt{\frac{D_2}{D_1}}}}, \tag{112.5}$$

where m' is an odd number, then $m = m'$ and

$$k = 2\left(\sqrt{1+2\alpha'} + \frac{D_3}{\sqrt{D_1 D_2}}\right). \tag{112.6}$$

For ratios c fulfilling the condition

$$c = m' \sqrt[4]{\frac{D_1}{D_2}}, \tag{112.7}$$

where m' is an even number, we have* $m = m'$ and

$$k = 2\left(1 + \varkappa' + \frac{D_3}{\sqrt{D_1 D_2}}\right). \tag{112.8}$$

The limiting ratios c_{pr} for which a simultaneous existence of two forms of bent surface is possible – with m semiwaves and with $m + 1$ semiwaves – will be found in the form

$$c_{\text{пр}} = \sqrt{m(M+1)} \cdot r_m, \tag{112.9}$$

where we have used the denotations:

$$r_m = \sqrt[4]{\frac{D_1}{D_2}} \sqrt{\frac{\sqrt{m^2(m+1)^2\varkappa'^2 + (2m+1)^2 + 2(2m+1)(m+1)^2\alpha'} + m(m+1)\varkappa'}{2m+1+2(m+1)^2\alpha'}} \tag{112.10}$$

for $m = 1, 3, 5, \ldots$;

$$r_m = \sqrt[4]{\frac{D_1}{D_2}} \sqrt{\frac{\sqrt{m^2(m+1)^2\varkappa'^2 + (2m+1)^2 - 2(2m+1)m^2\alpha'} - m(m+1)\varkappa'}{2m+1-2m^2\alpha'}} \tag{112.11}$$

for $m = 2, 4, 6, \ldots$

The number semiwaves m is determined for different values of c with the help of the following inequalities:

$$\left.\begin{array}{l} \text{if} \quad 0 < c < 1{,}41 r_1, \text{ then } m = 1; \\ \text{if} \quad 1{,}41 r_1 < c < 2{,}45 r_2, \text{ then } m = 2; \\ \text{if} \quad 2{,}45 r_2 < c < 3{,}46 r_3, \text{ then } m = 3 \end{array}\right\} \tag{112.12}$$

etc.

If the quantities D_1/D_2, γ' and κ' are known it is easy to determine the number m corresponding to the given side ratio. If an odd m will be obtained Formula (112.2) must be used to determine the critical load; if, however, even m is obtained the critical load will be determined from Formula (112.3). For large ratios c $(c > 4)$ the coefficient will be determined either from Formula (112.6) or from Formula (112.8), according to which of the two formulas will yield the minimum value.

2. The sides $y = 0$ and $y = b$ are fixed.

For odd m

$$p=\frac{\pi^2\sqrt{D_1D_2}}{b^2}\left[\sqrt{\frac{D_1}{D_2}}\left(\frac{m}{c}\right)^2+2,667\frac{D_3}{\sqrt{D_1D_2}}+\right.$$
$$\left.+5,333\left(\frac{c}{m}\right)^2\left(\sqrt{\frac{D_2}{D_1}}+2\gamma'\right)\right]; \tag{112.13}$$

for even m

$$p=\frac{\pi^2\sqrt{D_1D_2}}{b^2}\left[\sqrt{\frac{D_1}{D_2}}\left(\frac{m}{c}\right)^2+\right.$$
$$\left.+2,667\left(\frac{D_1}{\sqrt{D_1D_2}}+\varkappa'\right)+5,333\left(\frac{c}{m}\right)^2\sqrt{\frac{D_2}{D_1}}\right]. \tag{112.14}$$

For ratios c satisfying the condition

$$c=\frac{m'}{2}\sqrt[4]{\frac{3}{\frac{D_2}{D_1}+2\gamma'\sqrt{\frac{D_2}{D_1}}}}, \tag{112.15}$$

where the number m' is odd the coefficient of the critical load will be obtained from the formula

$$k=2,667\left(1,73\sqrt{1+2\alpha'}+\frac{D_3}{\sqrt{D_1D_2}}\right). \tag{112.16}$$

If

$$c=\frac{m'}{2}\sqrt[4]{\frac{3D_1}{D_2}}, \tag{112.17}$$

where m' is an even number, then $m = m'$ and

$$p_{кр}=\frac{\pi^2\sqrt{D_1D_2}}{b^2}2,667\left(1,73+\varkappa'+\frac{D_3}{\sqrt{D_1D_2}}\right). \tag{112.18}$$

The limiting ratios of the sides are equal to

$$c_{пр}=0,5\sqrt{m(m+1)}\,s_m, \tag{112.19}$$

where

$$s_m=\sqrt[4]{\frac{D_1}{D_2}}\sqrt{\frac{\sqrt{m^2(m+1)^2\varkappa'^2+3(2m+1)^2+6(2m+1)(m+1)^2\alpha'}+m(m+1)\varkappa'}{2m+1+2(m+1)^2\alpha'}} \tag{112.20}$$

for m = 1, 3, 5, ... and

$$s_m=\sqrt[4]{\frac{D_1}{D_2}}\sqrt{\frac{\sqrt{m^2(m+1)^2\varkappa'^2+3(2m+1)^2-6(2m+1)m^2\alpha'}-m(m+1)\varkappa'}{2m+1-2m^2\alpha'}} \tag{112.21}$$

for m = 2, 4, 6, ...

Instead of (112.12) we have the following inequalities with the help of which the number of semiwaves for given ratios c is determined, in this case:

$$\left.\begin{array}{l} \text{if} \quad 0 < c < 0{,}707 s_1, \text{ then } m = 1; \\ \text{if } 0{,}707 s_1 < c < 1{,}22\ s_2, \text{ then } m = 2; \\ \text{if } 1{,}22\ s_2 < c < 1{,}73\ s_3, \text{ then } m = 3 \end{array}\right\} \quad (112.22)$$

etc.

§113. THE STABILITY OF A PLATE WITH STIFFENING RIBS DEFORMED BY FORCES APPLIED TO THE RIB ENDS

In all cases considered in this chapter it was assumed that the load acting on the reinforced plate was uniformly distributed along the two sides. We shall now pass over to cases of deformation of a plate with ribs under the action of concentrated forces applied to the rib ends.

Let us pose the problem as follows. Let be given a rectangular orthotropic plate with principal directions of elasticity parallel to the side directions, reinforced by stiffening ribs parallel to two sides. The ribs are rigidly connected with the plate; the sides on which the rib ends are fixed are supported, the other other sides fastened in an arbitrary manner; the rib ends are supported and fixed in order to prevent twisting. At the rib ends act axial forces given except for a factor λ; the critical value λ_{kr} must be determined.

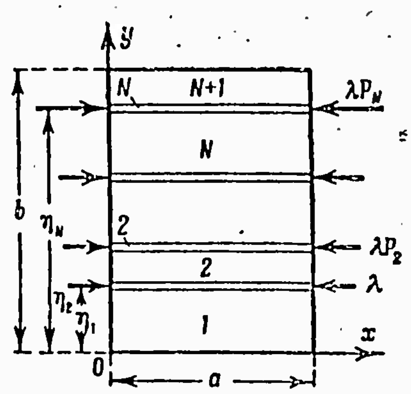

Fig. 212

Let us direct the x and y axes along the plate sides (Fig. 212) and introduce the denotations: a, b are the lengths of the plate sides; $D_1, D_2, D_k, \nu_1, \nu_2$ are the principal rigidities and the Poisson coefficients of the plate; N is the number of ribs; $\lambda, \lambda P_2, \ldots, \lambda P_N$ are forces deforming the system; EJ_k, C_k are the bending strength and the torsional rigidity of the ribs; $\beta = \frac{m\pi}{a}, m = 1, 2, 3, \ldots$; $\eta_1, \eta_2, \ldots, \eta_N$ are the distances of the ribs from the lower edge of the plate (from the x-axis).

We shall solve the problem starting from the energetic method as well as the problem of the stability of nonreinforced plates compressed by concentrated forces.

Let us assume that the system has experienced small deviations from the plane form; the plate has been deflected, and the ribs have been deflected and twisted. The lines of the ribs divide the whole plate into a number of sections whose number is equal to $N + 1$. We shall denote the deflections of the plate by $w_k(x, y)$; the rib deflections will then be $w_k(x, \eta_k)$, by virtue of the rigid connection of plate and ribs, and the angles of their twisting* will be $\theta_k = \left(\frac{\partial w_k}{\partial y}\right)_{y=\eta_k}$.

In the distortion the forces will carry out a work equal to

$$A = \frac{\lambda}{2} \sum_{k=1}^{N} P_k \int_0^a \left(\frac{\partial w_k}{\partial x}\right)^2_{y=\eta_k} dx. \tag{113.1}$$

to the potential energy of the system the energy of bending V_{izg} added which consists of the energy of bending of the plate sections and the energy of rib bending and distortion. From the equation

$$V_{изг} = A \tag{113.2}$$

we find:

$$\lambda = \left\{ \sum_{k=1}^{N+1} \int_{\eta_{k-1}}^{\eta_k} \int_0^a \left[D_1 \left(\frac{\partial^2 w_k}{\partial x^2}\right)^2 + 2D_1\nu_2 \frac{\partial^2 w_k}{\partial x^2} \cdot \frac{\partial^2 w_k}{\partial y^2} + D_2 \left(\frac{\partial^2 w_k}{\partial y^2}\right)^2 + \right.\right.$$
$$\left. + 4D_k \left(\frac{\partial^2 w_k}{\partial x \partial y}\right)^2 \right] dx\, dy + \sum_{k=1}^{N} \int_0^a \left[EJ_k \left(\frac{\partial^2 w_k}{\partial x^2}\right)^2_{y=\eta_k} + \right. \tag{113.3}$$
$$\left.\left. + C_k \left(\frac{\partial^2 w_k}{\partial x \partial y}\right)^2_{y=\eta_k} \right] dx \right\} : \sum_{k=1}^{N} P_k \int_0^a \left(\frac{\partial w_k}{\partial x}\right)^2_{y=\eta_k} dx$$
$$(P_1 = 1).$$

We seek expressionf ror the plate deflections satisfying the conditions on the sides $x = 0$, $x = a$, in the form

$$w_k = f_k(y) \sin \beta x; \tag{113.4}$$

the deflections and angles of rib twisting will be, respectively,

$$f_k(\eta_k) \sin \beta x, \quad f'_k(\eta_k) \sin \beta x,$$

The problem boils down to the determination of the functions $f_k(y)$ minimizing (113.3), and this is equivalent to the problem of seeking the minimum of the expression**

$$U=\sum_{k=1}^{N+1}\int_{\eta_{k-1}}^{\eta_k}(D_2f_k''^2-2D_1\nu_2\beta^2f_kf_k''+D_1\beta^4f_k^2+4D_k\beta^2f_k'^2)dy+$$
$$+\sum_{k=1}^{N}[(EJ_k\beta^4-\lambda P_k\beta^2)f_k^2(\eta_k)+C_k\beta^2f_k'^2(\eta_k)]. \tag{113.5}$$

Solving this problem by the methods of variational calculus we obtain the following results.

The functicns f_k satisfy the equation

$$D_2f_k^{IV}-2D_3\beta^2f_k''+D_1\beta^4f_k=0. \tag{113.6}$$

and the conditions: a) on the sides $y = 0$ and $y = b$ the conditions of fixing; b) at the boundaries between the sections $y=\eta_k$ $(\kappa=1,\ 2,\ \ldots,\ N)$, i.e., on the rib lines

$$\left.\begin{aligned} f_{k+1}&=f_k,\\ f'_{k+1}&=f'_k,\\ D_2f''_{k+1}-D_2f''_k&=C_k\beta^2f'_k,\\ D_2f'''_{k+1}-D_2f'''_k&=(\lambda P_k-EJ_k\beta^2)\beta^2f_k\\ (\kappa=1,\ 2,\ &\ldots,\ N).\end{aligned}\right\} \tag{113.7}$$

Each of the functions f_k to be determined from Eq. (113.6) contains four arbitrary constants. The conditions for f_k yield a system of $4N + 4$ homogeneous equations with the same number of unknown. As in the case of the compression of a plate without ribs the problem can be simplified considerably by using the analogy with beams and the influence functions.

Equation (113.6) and Conditions (113.7) may be interpreted as the equation and the conditions for the sections of a beam (whose length and rigidity are, respectively, equal to b and D_2), which lies on a solid elastic base (the coefficient of elasticity is $k = D_1\beta^4$), is stretched by an axial force $T = 2D_3\beta^2$ and bent by forces Q_k and moments M_k applied at the points $y = \eta_k$) (see Fig. 213, where the forces and moments shown are considered positive). With such a beam the deflection $f(y)$ and its first derivative are continuous whereas the second and third derivatives at the points of application of forces and moments are discontinuous in the following way:

$$\left.\begin{aligned} D_2f''(\eta_k+0)-D_2f''(\eta_k-0)&=M_k,\\ D_2f'''(\eta_k+0)-D_2f'''(\eta_k-0)&=Q_k.\end{aligned}\right\} \tag{113.8}$$

The third and fourth conditions (113.7) show that the forces acting on the beam are proportional to the deflections, and the moments are proportional to the first derivatives of the deflection at the points of their application. Adopting the designation $f(y)$ for the beam deflection at an arbitrary point we have

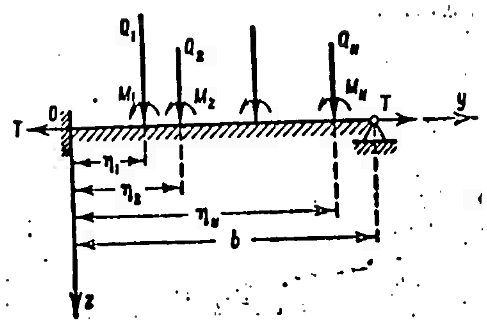

Fig. 213

$$
\left.\begin{array}{ll}
M_1 = C_1\beta^2 f'(\eta_1), & Q_1 = (\lambda - EJ_1\beta^2)\beta^2 f(\eta_1), \\
M_2 = C_2\beta^2 f'(\eta_2), & Q_2 = (\lambda P_2 - EJ_2\beta^2)\beta^2 f(\eta_2), \\
\cdots\cdots\cdots & \cdots\cdots\cdots \\
M_N = C_N\beta^2 f'(\eta_N), & Q_N = (\lambda P_N - EJ_N\beta^2)\beta^2 f(\eta_N).
\end{array}\right\} \qquad (113.9)
$$

The beam deflection f can be represented in the form

$$
f = \sum_{k=1}^{N} [Q_k \delta(\eta_k, y) + M_k \Delta(\eta_k, y)], \qquad (113.10)
$$

where $\delta(\eta, y)$ is the influence function of the force, and $\Delta(\eta, y)$ is the influence function of the moment, which were introduced in §77.

Substituting the values of f and f' into (113.9) we obtain a system of $2N$ equations for Q_k and M_k:*

$$
\left.\begin{aligned}
& M_1\left(\Delta'_{11} - \frac{1}{C_1\beta^2}\right) + M_2\Delta'_{21} + \ldots + M_N\Delta'_{N1} + \\
& \qquad + Q_1\delta'_{11} + Q_2\delta'_{21} + \ldots + Q_N\delta'_{N1} = 0, \\
& M_1\Delta'_{12} + M_2\left(\Delta'_{22} - \frac{1}{C_2\beta^2}\right) + \ldots + M_N\Delta'_{N2} + \\
& \qquad + Q_1\delta'_{12} + Q_2\delta'_{22} + \ldots + Q_N\delta'_{N2} = 0, \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& M_1\Delta'_{1N} + M_2\Delta'_{2N} + \ldots + M_N\left(\Delta'_{NN} - \frac{1}{C_N\beta^2}\right) + \\
& \qquad + Q_1\delta'_{1N} + Q_2\delta'_{2N} + \ldots + Q_N\delta'_{NN} = 0, \\
& M_1\Delta_{11} + M_2\Delta_{21} + \ldots + M_N\Delta_{N1} + \\
& \qquad + Q_1\left(\delta_{11} - \frac{1}{\lambda\beta^2 - EJ_1\beta^4}\right) + Q_2\delta_{21} + \ldots + Q_N\delta_{N1} = 0, \\
& M_1\Delta_{12} + M_2\Delta_{22} + \ldots + M_N\Delta_{N2} + \\
& \qquad + Q_1\delta_{12} + Q_2\left(\delta_{22} - \frac{1}{\lambda P_2\beta^2 - EJ_2\beta^4}\right) + \ldots + Q_N\delta_{N2} = 0, \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& M_1\Delta_{1N} + M_2\Delta_{2N} + \ldots + M_N\Delta_{NN} + \\
& \qquad + Q_1\delta_{1N} + Q_2\delta_{2N} + \ldots + Q_N\left(\delta_{NN} - \frac{1}{\lambda P_N\beta^2 - EJ_N\beta^4}\right) = 0.
\end{aligned}\right\} \qquad (113.11)
$$

Having solved the first N of these equations for the moments we obtain expressions of the form:

$$M_1 = \sum_{k=1}^{N} Q_k q_{k1}, \quad M_2 = \sum_{k=1}^{N} Q_k q_{k2}, \quad \ldots, \quad M_N = \sum_{k=1}^{N} Q_k q_{kN}. \tag{113.12}$$

Moreover, we shall substitute the moment values found into the rest of equations and obtain N equations which will only contain Q_k, besides λ:

$$\left.\begin{aligned} Q_1\left(k_{11} - \frac{1}{\lambda\beta^2 - EJ_1\beta^4}\right) + Q_2 k_{21} + \ldots + Q_N k_{N1} = 0, \\ Q_1 k_{12} + Q_2\left(k_{22} - \frac{1}{\lambda P_2\beta^2 - EJ_2\beta^4}\right) + \ldots + Q_N k_{N2} = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Q_1 k_{1N} + Q_2 k_{2N} + \ldots + Q_N\left(k_{NN} - \frac{1}{\lambda P_N\beta^2 - EJ_N\beta^4}\right) = 0. \end{aligned}\right\} \tag{113.13}$$

We have used the brief denotation here

$$k_{ij} = \delta_{ij} + \sum_{n=1}^{N} q_{in}\Delta_{ni}. \tag{113.14}$$

Putting the determinant of the system equal to zero we obtain an equation for the determination of λ:

$$\begin{vmatrix} k_{11} - \frac{1}{\lambda\beta^2 - EJ_1\beta^4}, & k_{21} & \ldots, & k_{N1} \\ k_{12}, & k_{22} - \frac{1}{\lambda P_2\beta^2 - EJ_2\beta^4}, & \ldots, & k_{N2} \\ \ldots & \ldots & \ldots & \ldots \\ k_{1N}, & k_{2N}, & \ldots, & k_{NN} - \frac{1}{\lambda P_N\beta^2 - EJ_N\beta^4} \end{vmatrix} = 0. \tag{113.15}$$

This is an Nth-degree equation; generally, it will determine N values λ each of which will depend on m:

$$\lambda_1(m), \lambda_2(m), \ldots, \lambda_N(m).$$

The minimum of these N series of values not equal to zero must be selected; it will be the critical one.

§114. THE STABILITY OF A PLATE WITH ONE RIB

A rectangular plate which is reinforced by one rib is compressed by axial forces λ applied to the rib ends (Fig. 214). In the general case we assume that the rib axis does not coincide with the axis of symmetry of the plate.

Let us designate by EJ and C the bending strengths (in the plane perpendicular to xy) and the torsional rigidities of the rib and

$$c = \frac{C\pi}{4a\sqrt{D_1 D_2}}.$$

In the case considered $N = 1$, $\eta_1 = \eta$,

$$f = Q\delta(\eta, y) + M\Delta(\eta, y). \tag{114.1}$$

Equations (113.11) assume the form:

$$\left.\begin{aligned} M\left(\Delta'_{11} - \frac{1}{C\beta^2}\right) + Q\delta'_{11} &= 0, \\ M\Delta_{11} + Q\left(\delta_{11} - \frac{1}{\lambda\beta^2 - EJ\beta^4}\right) &= 0. \end{aligned}\right\} \tag{114.2}$$

Hence

$$M = Q\frac{\delta'_{11}C\beta^2}{1 - \Delta'_{11}C\beta^2}, \tag{114.3}$$

$$\lambda = \frac{EJ\pi^2 m^2}{a^2} + \frac{4\pi\sqrt{D_1D_2}}{a} \cdot \frac{m}{g_{11}} \cdot \frac{1}{1 - \dfrac{ch_{11}g'_{11}}{cmh'_{11} - 1} \cdot \dfrac{m}{g_{11}}}. \tag{114.4}$$

The number of semiwaves will also here remain undetermined, for the present. The determination of m and λ_{kr} is simplified if there are tables of the functions g, h and h'.

In a special case, when the torsional rigidity is negligible and the rib resists only bending and not twisting

$$\lambda = \frac{EJ\pi^2 m^2}{a^2} + \frac{4\pi\sqrt{D_1D_2}}{a} \cdot \frac{m}{g_{11}}, \tag{114.5}$$

where

$$g_{11} = g(\eta, \eta).$$

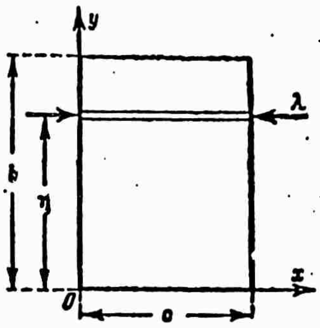

Fig. 214

In the case of infinite torsional rigidity, however, when the rib cannot be twisted, but may only be bent

$$\lambda = \frac{EJ\pi^2 m^2}{a^2} + \frac{4\pi\sqrt{D_1D_2}}{a} \cdot \frac{m}{g_{11}} \cdot \frac{1}{1 - \dfrac{h_{11}\, g'_{11}}{h'_{11}\, g_{11}}}. \tag{114.6}$$

Let us briefly consider two cases where we do not make any additional assumptions with respect to the rib rigidities.

1. The symmetric rib. If the rib is directed along the axis of symmetry of the plate when $\delta'_{11} = g'_{11} = 0$,

$$\lambda = \frac{EJ\pi^2 m^2}{a^2} + \frac{4\pi\sqrt{D_1 D_2}}{a} \cdot \frac{m}{g_{11}}, \tag{114.7}$$

where $g_{11} = g(b/2, b/2)$. The number of semiwaves will be determined by calculations or with the help of the tables of influence functions (in those cases where there are such tables). It is evident from Formula (114.7) that the critical force for a plate with a rib directed along the axis of symmetry does not depend on the torsional rigidity of the rib, but only on its bending strength and on the rigidities of the plate.

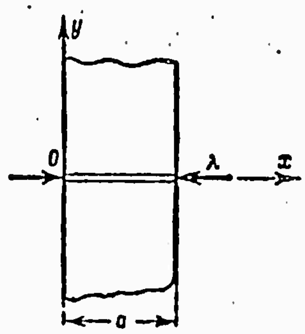

Fig. 215

2. The band with one rib. For an infinite band with one rib (Fig. 215)

$$\left.\begin{aligned} g'_{11} &= h_{11} = 0, \\ g_{11} &= g(\eta, \eta) = \frac{1}{2}(u_1 + u_2), \end{aligned}\right\} \tag{114.8}$$

where u_1, u_2 are roots of the equation

$$D_2 u^4 - 2D_3 u^2 + D_1 = 0. \tag{114.9}$$

From Formula (114.6) we obtain:

$$\lambda = \frac{EJ\pi^2 m^2}{a^2} + \frac{2\pi\sqrt{D_1 D_2}}{a} m (u_1 + u_2). \tag{114.10}$$

Obviously, the minimum value of λ will be obtained for $m = 1$, and, finally,

$$\lambda_{кр} = \frac{EJ\pi^2}{a^2} + \frac{2\pi\sqrt{D_1 D_2}}{a}(u_1 + u_2). \tag{114.11}$$

The formula apply to any of the three possible cases of roots 1, 2, 3 (see §78).

It is interesting to note that the critical force for a band with rib is obtained in the form of a sum of the Eulerian critical force for the rib (i.e., for a rod with supported ends compressed

by axial forces) and of the critical force for a band compressed by two forces of opposite directions (see §107).

§115. THE STABILITY OF A RECTANGULAR PLATE REINFORCED BY RIBS ALONG TWO SIDES AND COMPRESSED BY CONCENTRATED FORCES

Let us consider a rectangular orthotropic plate with principal directions parallel to the side directions two sides of which are reinforced by stiffening ribs. The system is deformed by two equal forces with opposite directions λ applied at the ends of one rib, and two forces λP applied to the ends of the other rib; the critical value λ_{kr} is to be determined. The sides on which the rib ends are fixed are assumed to be supported; the rib ends are assumed to be supported and fixed with respect to twisting.

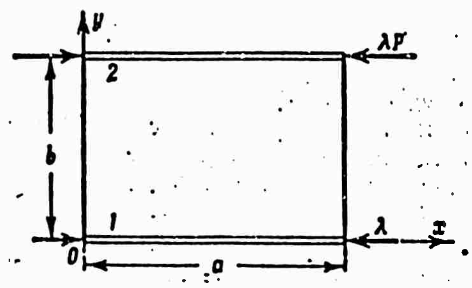

Fig. 216

Let us direct the x-axis along the reinforced side (Fig. 216) and introduce the designations: EJ_1, EJ_2 are the bending strengths of the ribs in the planes perpendicular to the original plate plane xy; C_1, C_2 are the torsional rigidities of the ribs; a, b are the lengths of the plate sides; $d = b/a$, $\beta = m\pi/a$ (m = 1, 2, 3, ...).

We shall solve the problem by the energetic method.* Let us assume that the system has undergone a small distortion: the plate has been bent, and the ribs have been bent and twisted.

Let us designate by $w(x, y)$ the plate deflection; $w_1 = w(x, 0)$, $w_2 = w(x, b)$ are the rib deflections; $\theta_1 = \left(\frac{\partial w}{\partial y}\right)_{y=0}$ and $\theta_2 = \left(\frac{\partial w}{\partial y}\right)_{y=b}$ are the angles of twisting of the ribs. The potential energy of bending V_{izg} will be composed of the potential energy of the plate [see (61.22)] and the potential energy of the ribs

$$\frac{1}{2}\int_0^a [EJ_1(w_1'')^2 + EJ_2(w_2'')^2 + C_1(\theta_1')^2 + C_2(\theta_2')^2]\,dx. \tag{115.1}$$

In the distortion of the system the forces will perform a work of:

$$A = \frac{\lambda}{2}\int_0^a [(w_1')^2 + P(w_2')^2]\,dx. \tag{115.2}$$

From equation $V_{izg} = A$ we find:

$$\lambda = \left\{ \int_0^a \int_0^b \left[D_1 \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \ldots + 4D_k \left(\frac{\partial^2 w}{\partial x\, \partial y} \right)^2 \right] dx\, dy + \right.$$

$$\left. + \int_0^a [EJ_1 (w_1'')^2 + EJ_2 (w_2'')^2 + C_1 (\theta_1')^2 + C_2 (\theta_2')^2]\, dx \right\} : \tag{115.3}$$

$$: \int_0^a [(w_1')^2 + P (w_2')^2]\, dx.$$

We focus our attention on the expression for w satisfying the conditions on the supported sides $x = 0$ and $x = a$, in the form:

$$w = f(y) \sin \beta x; \tag{115.4}$$

then

$$\left. \begin{array}{ll} w_1 = f(0) \sin \beta x, & w_2 = f(b) \sin \beta x, \\ \theta_1 = f'(0) \sin \beta x, & \theta_2 = f'(b) \sin \beta x. \end{array} \right\} \tag{115.5}$$

Substituting w into (115.3) we shall seek the function $f(y)$ minimizing λ. We reduce this problem to the problem of determining the minimum of the expression*

$$U = \int_0^b (D_2 f''^2 - 2D_1 \nu_2 \beta^2 f f'' + D_1 \beta^4 f^2 + 4D_k \beta^2 f'^2)\, dy +$$

$$+ (EJ_1 \beta^2 - \lambda) \beta^2 f^2 (0) + (EJ_2 \beta^2 - \lambda P) \beta^2 f^2 (b) + \tag{115.6}$$

$$+ [C_1 f'^2 (0) + C_2 f'^2 (b)] \beta^2.$$

Solving it according to the rules of variational calculus we obtain the following results.

The function f satisfies the equation

$$D_2 f^{\mathrm{IV}} - 2D_3 \beta^2 f'' + D_1 \beta^4 f = 0 \tag{115.7}$$

and the conditions

$$\left. \begin{array}{r} f''(0) - \nu_1 \beta^2 f(0) - \frac{C_1}{D_2} \beta^2 f'(0) = 0, \\ f''(b) - \nu_1 \beta^2 f(b) + \frac{C_2}{D_2} \beta^2 f'(b) = 0, \\ f'''(0) - \left(\nu_1 + \frac{4D_k}{D_2} \right) \beta^2 f'(0) + \frac{1}{D_2} (EJ_1 \beta^2 - \lambda) \beta^2 f(0) = 0, \\ f'''(b) - \left(\nu_1 + \frac{4D_k}{D_2} \right) \beta^2 f'(b) - \frac{1}{D_2} (EJ_2 \beta^2 - \lambda P) \beta^2 f(b) = 0. \end{array} \right\} \tag{115.8}$$

The general integral of Eq. (115.7) depends on the roots of the characteristic equation which has occurred already several times

$$D_2 u^4 - 2D_3 u^2 + D_1 = 0 \tag{115.9}$$

and is written in a different form according to which of the three cases of roots is realized.

Case 1. Real unequal roots $\pm u_1, \pm u_2$:

$$f = A\,\mathrm{ch}\,\beta u_1 y + B\,\mathrm{sh}\,\beta u_1 y + C\,\mathrm{ch}\,\beta u_2 y + D\,\mathrm{sh}\,\beta u_2 y. \tag{115.10}$$

Case 2. Real equal roots $\pm u$:

$$f = (A + By)\,\mathrm{ch}\,\beta u y + (C + Dy)\,\mathrm{sh}\,\beta u y. \tag{115.11}$$

Case 3. Complex roots $u \pm iv, -u \pm iv$:

$$\begin{aligned} f = (A\cos\beta v y + B\sin\beta v y)\,\mathrm{ch}\,\beta u y + \\ + (C\cos\beta v y + D\sin\beta v y)\,\mathrm{sh}\,\beta u y. \end{aligned} \tag{115.12}$$

On the basis of Conditions (115.8) we obtain a system of four homogeneous equations for A, B, C, D, and putting the determinant of this system equal to zero we obtain an equation for λ. The latter equation is of the second degree; it will yield two series of values: $\lambda_1\,(m)$ and $\lambda_2\,(m)$. Furthermore, the number of semiwaves (in the direction of the ribs) m to which the minimum nonvanishing λ corresponds and this minimum value itself must be determined.

§116. THE PLATE REINFORCED BY IDENTICAL RIBS ALONG TWO SIDES

Let us deal in a more detailed manner with the case of a symmetric system where the ribs have identical rigidities and the forces are equal.

In this case

$$EJ_1 = EJ_2 = EJ, \quad C_1 = C_2 = C, \quad P = 1.$$

Two fundamental forms of plate distortion are here possible after the stability has been lost; the first one is characterized by the fact that the plate, having lost its stability, will bulge, thereby forming a surface symmetric with respect to the axis $y = b/2$, the second one is characterized by an antisymmetric bent surface.

1. The symmetric form.

If the roots of the characteristic equation (115.9) are real and unequal (Case 1) then the equation of the bent symmetric surface has the form:

$$f = A\,\mathrm{ch}\,\beta u_1 (y - b/2) + B\,\mathrm{ch}\,\beta u_2 (y - b/2). \tag{116.1}$$

Conditions (115.8) will be reduced to two conditions:

$$\left.\begin{aligned} f''(0) - \nu_1\beta^2 f(0) - \frac{C}{D_2}\beta^2 f'(0) = 0, \\ f'''(0) - \left(\nu_1 + \frac{4D_k}{D_2}\right)\beta^2 f'(0) + \frac{1}{D_2}(EJ\beta^2 - \lambda)\beta^2 f(0) = 0. \end{aligned}\right\} \tag{116.2}$$

Substituting the values of f, f', f'', and f''' for $y = 0$ into these conditions we obtain two homogeneous equations for the constants A and B, and putting the determinant of this system equal to zero, we obtain, after some simple transformations:

$$\lambda = \frac{EJ\pi^2 m^3}{a^3} +$$

$$+ \frac{\pi\sqrt{D_1 D_2}}{a} m \frac{\sqrt{\frac{D_1}{D_2}} - \nu_1^2 \sqrt{\frac{D_2}{D_1}} + \frac{4D_k}{D_2} e_m + \frac{\pi C}{aD_2} m s_m}{l_m + \frac{\pi C}{aD_2} m}. \tag{116.3}$$

The quantities l_m, e_m, s_m entering this formula are functions of the ratios $d = b/a$ and depend on m which is equal to the number of semiwaves in the direction of the forces.

In Case 1 (see §115) these quantities have the form:

$$\left.\begin{aligned}
l_m &= \frac{1}{H_m}(u_1^2 - u_2^2)\operatorname{ch}\frac{m\pi u_1 d}{2}\operatorname{ch}\frac{m\pi u_2 d}{2},\\
e_m &= \frac{1}{H_m}\left(u_1 \operatorname{ch}\frac{m\pi u_1 d}{2}\operatorname{sh}\frac{m\pi u_2 d}{2} - u_2 \operatorname{ch}\frac{m\pi u_2 d}{2}\operatorname{sh}\frac{m\pi u_1 d}{2}\right),\\
s_m &= \frac{1}{H_m}(u_1^2 - u_2^2)\operatorname{sh}\frac{m\pi u_1 d}{2}\operatorname{sh}\frac{m\pi u_2 d}{2},\\
H_m &= u_1 \operatorname{sh}\frac{m\pi u_1 d}{2}\operatorname{ch}\frac{m\pi u_2 d}{2} - u_2 \operatorname{sh}\frac{m\pi u_2 d}{2}\operatorname{ch}\frac{m\pi u_1 d}{2}.
\end{aligned}\right\} \tag{116.4}$$

The same formula for (116.3) is obtained in Case 2 and Case 3, but l_m, e_m, and s_m will have another form:

In Case 2:

$$\left.\begin{aligned}
l_m &= \frac{1}{H_m} 2u(\operatorname{ch} m\pi ud + 1),\\
e_m &= \frac{1}{H_m}(\operatorname{sh} m\pi ud - m\pi ud),\\
s_m &= \frac{1}{H_m} 2u(\operatorname{ch} m\pi ud - 1),\\
H_m &= \operatorname{sh} m\pi ud + m\pi ud.
\end{aligned}\right\} \tag{116.5}$$

In Case 3:

$$\left.\begin{aligned}
l_m &= \frac{1}{H_m} 2u(\operatorname{ch} m\pi ud + \cos m\pi vd),\\
e_m &= \frac{1}{H_m}\left(\operatorname{sh} m\pi ud - \frac{u}{v}\sin m\pi vd\right),\\
s_m &= \frac{1}{H_m} 2u(\operatorname{ch} m\pi ud - \cos m\pi vd),\\
H_m &= \operatorname{sh} m\pi ud + \frac{u}{v}\sin m\pi vd.
\end{aligned}\right\} \tag{116.6}$$

For large ratios d we may assume:

$$\left.\begin{aligned}
e_m &= 1,\\
l_m &= s_m = u_1 + u_2
\end{aligned}\right\} \tag{116.7}$$

(or, respectively, $l_m = s_m = 2u$).

2. The antisymmetric form.

In Case 1 we obtain the equation of the bent surface

$$f = A \operatorname{sh}\beta u_1 (y - b/2) + B \operatorname{sh}\beta u_2 (y - b/2) \tag{116.8}$$

and Conditions (116.2). Putting the determinant of the system of equations for A and B obtained from Conditions (116.2) equal to zero we find

$$\lambda = \frac{EJ\pi^2 m^2}{a^2} + \frac{\pi\sqrt{D_1 D_2}}{a} m \frac{\left(\sqrt{\frac{D_1}{D_2}} - \nu_1^2 \sqrt{\frac{D_2}{D_1}}\right) e_m + \frac{4D_k}{D_2} + \frac{\pi C}{aD_2} m l_m}{s_m + \frac{\pi C}{aD_2} m e_m}. \qquad (116.9)$$

The investigation of Formulas (116.3) and (116.9) obtained permits us to draw the following conclusions.

1) For any finite side ratios a symmetric form of stability loss with one semiwave ($m = 1$) in the direction of the compressive forces must exist, and only for large ratios d (theoretically, for $d = \infty$) the symmetric and the antisymmetric form are equally possible. The form of the symmetric and antisymmetric surfaces for $m = 1$ is shown in Fig. 217.

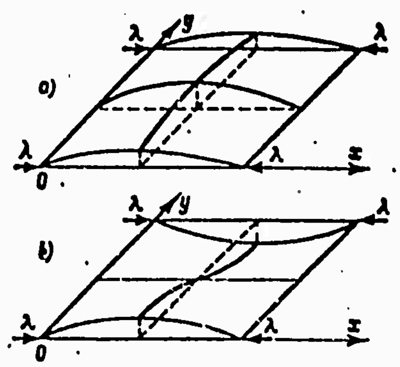

Fig. 217

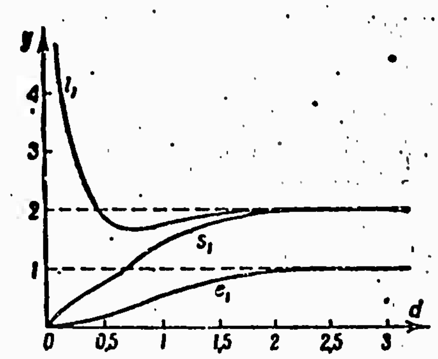

Fig. 218

2) The critical load is determined from formula

$$\lambda_{\text{кр}} = \frac{EJ\pi^2}{a^2} + \frac{\pi\sqrt{D_1 D_2}}{a} k, \qquad (116.10)$$

where

$$k = \frac{\sqrt{\frac{D_1}{D_2}} - \nu_1^2 \sqrt{\frac{D_2}{D_1}} + \frac{4D_k}{D_2}\; e_1 + \frac{\pi C}{aD_2} s_1}{l_1 + \frac{\pi C}{aD_2}}. \qquad (116.11)$$

The critical load, as can be seen from these formulas, is composed of a Eulerian critical force and of an additional force depending on the elastic constants of the plate.

If $EJ = C = 0$ we obtain the critical load for a plate without ribs with two free and two supported sides compressed by forces applied at its free boundaries:

$$\lambda_{\text{кр}} = \frac{\pi\sqrt{D_1D_2}}{al_1}\left(\sqrt{\frac{D_1}{D_2}} - \nu_1^2\sqrt{\frac{D_2}{D_1}} + \frac{4D_k}{D_2}e_1\right). \qquad (116.12)$$

3) For large side ratios d we may take

$$k = \frac{\sqrt{\frac{D_1}{D_2}} - \nu_1^2\sqrt{\frac{D_2}{D_1}} + \frac{4D_k}{D_2} + \frac{\pi C}{aD_2}(u_1 + u_2)}{u_1 + u_2 + \frac{\pi C}{aD_2}}. \qquad (116.13)$$

4) For a plate with ribs for which the torsional rigidity $C = 0$ we obtain:

$$k = \frac{1}{l_1}\left(\sqrt{\frac{D_1}{D_2}} - \nu_1^2\sqrt{\frac{D_2}{D_1}} + \frac{4D_k}{D_2}e_1\right). \qquad (116.14)$$

This case is equivalent to the case of a plate which not rigidly linked with the ribs, but freely supported by ribs because the equation $C = 0$ may be regarded as the condition of free revolution of the plate boundaries about the axes of the ribs.

5) For a plate with infinite torsional rigidity

$$k = s_1; \qquad (116.15)$$

physically, this means that the ribs can only be bent, but not twisted (owing to, e.g., special side fixing devices).

The stability calculation of plates reinforced by ribs along the edges is considerably facilitated if there are tables of the quantities l_1, e_1, s_1. These tables (brief ones) will be given below for an isotropic plate and for a plywood plate as considered in §67.

TABLE 28

The Functions l_1, e_1, s_1 for an Isotropic Plate

d	l_1	e_1	s_1
0,25	2,81	0,0503	0,393
0,5	1,81	0,189	0,661
0,75	1,67	0,379	1,14
1	1,71	0,572	1,44
2	1,96	0,954	1,95
3	2,00	0,997	2,00
∞	2,00	1,00	2,00

If we want to obtain λ_{kr} for a plate with given side ratio d we take the corresponding table and substitute the values of l_1, e_1, s_1 found from it into Formulas (116.10)-(116.11). The values of these quantities for ratios d not appearing in the table may be determined approximately by interpolation since the functions $l_1(d)$, $e_1(d)$ and $s_1(d)$ for $d > 1$ are sufficiently smooth curves. The diagrams of these functions for an isotropic plate are shown in

TABLE 29

The Functions l_1, e_1, s_1 for a Plywood Plate

d	Сжатие вдоль волокон[1] ($D_1 > D_2$)			Сжатие поперек волокон[2] ($D_1 < D_2$)		
	l_1	e_1	s_1	l_1	e_1	s_1
0,25	2,98	0,174	1,36	2,57	0,0141	0,112
0,5	2,47	0,585	2,52	1,33	0,0576	0,225
0,75	2,84	0,906	3,08	0,918	0,129	0,336
1	3,06	1,01	3,16	0,784	0,224	0,445
2	3,10	1,00	3,10	0,731	0,703	0,790
3	3,10	1,00	3,10	0,851	0,972	0,901
∞	3,10	1,00	3,10	0,888	1,000	0,888

1) Compression along the fibers; 2) compression across the fibers.

Fig. 218 which also explains the general character of the variation of l_1, e_1, s_1 in the case of a plywood plate.

Manuscript Page No. [Footnotes]

449 See his work "Ob ustoychivosti plastinok podkreplennykh zhestkimi rebrami" [On the Stability of Plates Reinforced by Rigid Ribs], izd. In-ta inzhenerov putey soobshcheniya [Institute for Communication Path Engineers], Petrograd, 1914, or his book "Ustoychivost' uprugikh sistem" [The Stability of Elastic Systems] which has been mentioned several times, §70, pages 331-343.

451 See (97.6)-(97.8).

452 See the first footnote.

454 In the latter case the rib remains rectilinear, but in this case it is twisted.

457 See the works by S.P. Timoshenko, mentioned in §§109-110.

458 See the first footnote in §109.

461 If m is an even number the rib remains rectilinear and is only twisted after having lost its stability.

464* We neglect the rib width.

464** See (97.6)-(97.8).

466	As to the designations $\delta_{ij}, \delta'_{ij}, \Delta_{ij}, \Delta'_{ij}$ see §77.
470	Lekhnitskiy, S.G., Ustoychivost' anizotropnoy plastinki usilennoy rebrami po dvum storonam [The Stability of an Anisotropic Plate Reinforced by Ribs Along Two Sides], Nauchnyy byulleten' Leningr. gos. un-ta [Scientific Bulletin of the Leningrad State University], 1945, No. 2.
471	See (97.6)-(97.8).

Manuscript Page No.	[Transliterated Symbols]
448	кр = kr = kriticheskiy = critical
450	изг = izg = izgib = bending
459	пр = pr = predel'nyy = limiting